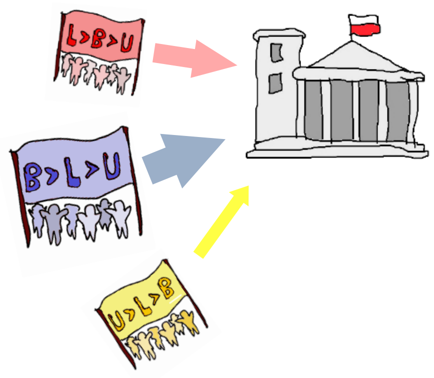
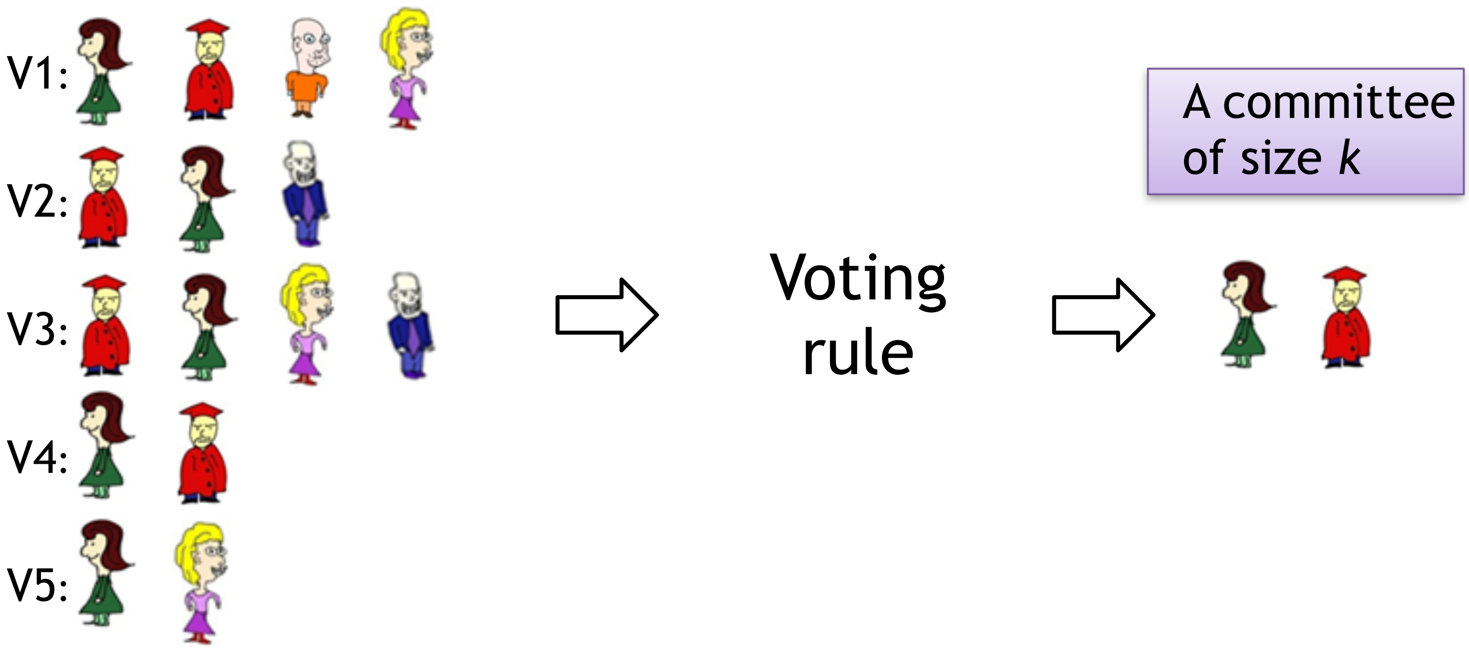
Proportionality of Approval-Based Multi-winner Rules

Piotr Skowron
University of Warsaw

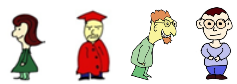


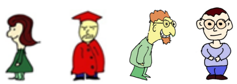
Model: Approval-Based Elections

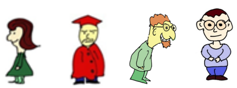


A preference profile: an example

We have n = 8 voters, m = 9 candidates.

v1: 

v2: 

v3: 

v4: 

v5: 

v6: 

v7: 

v8: 

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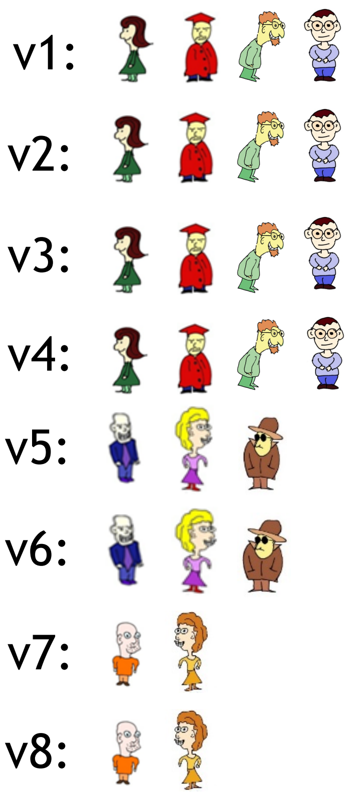

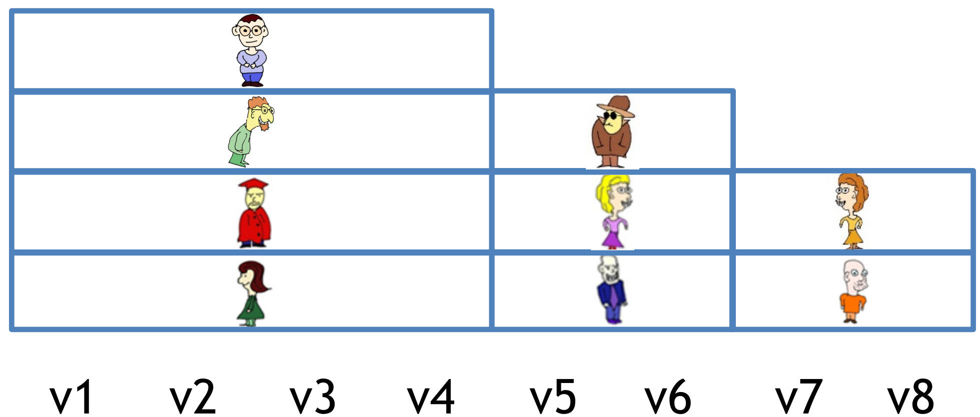

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v3: c1 c2 c3 c4

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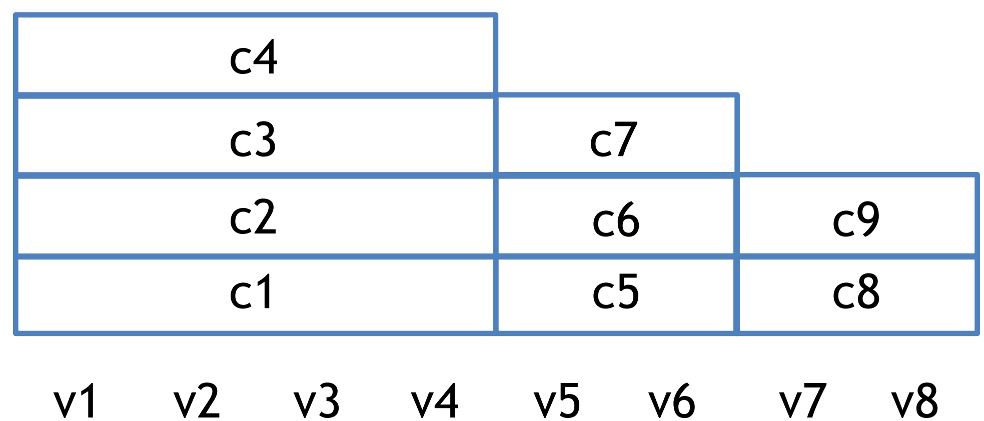
v5: c5 c6 c7

v6: c5 c6 c7

v7: c8 c9

v8: c8 c9



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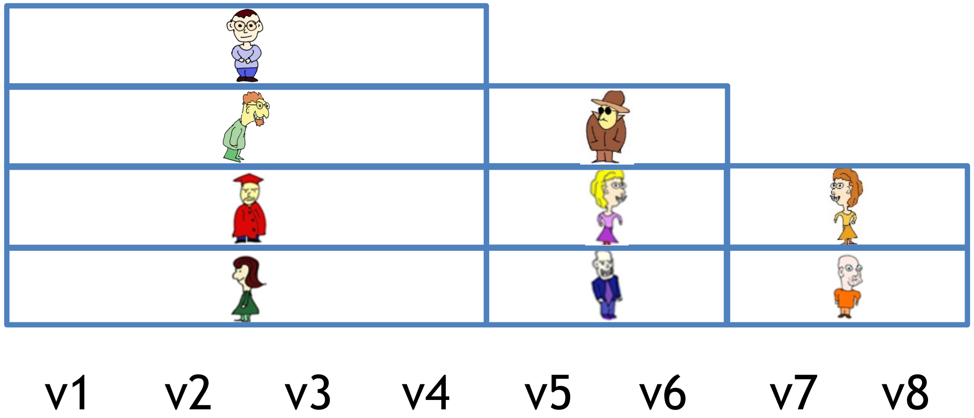

Assume the committee size to be elected is **k = 4.**

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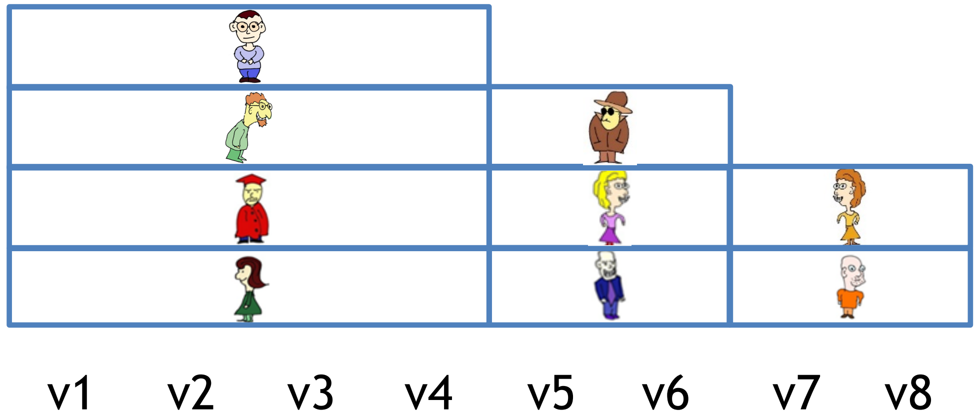

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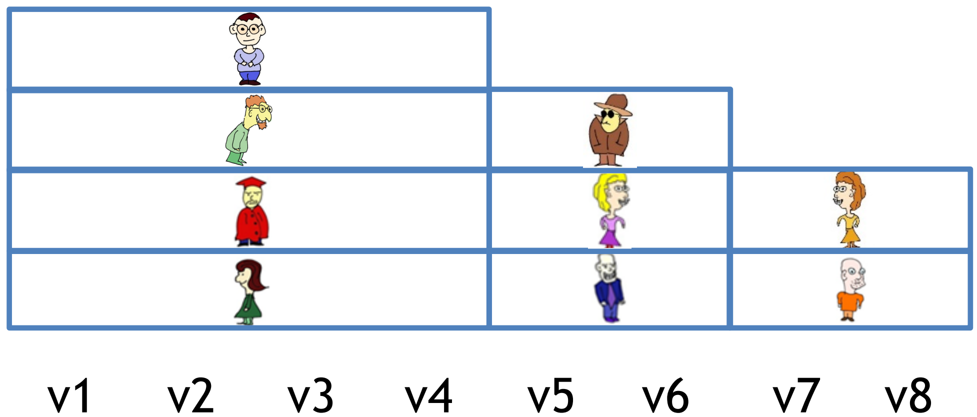

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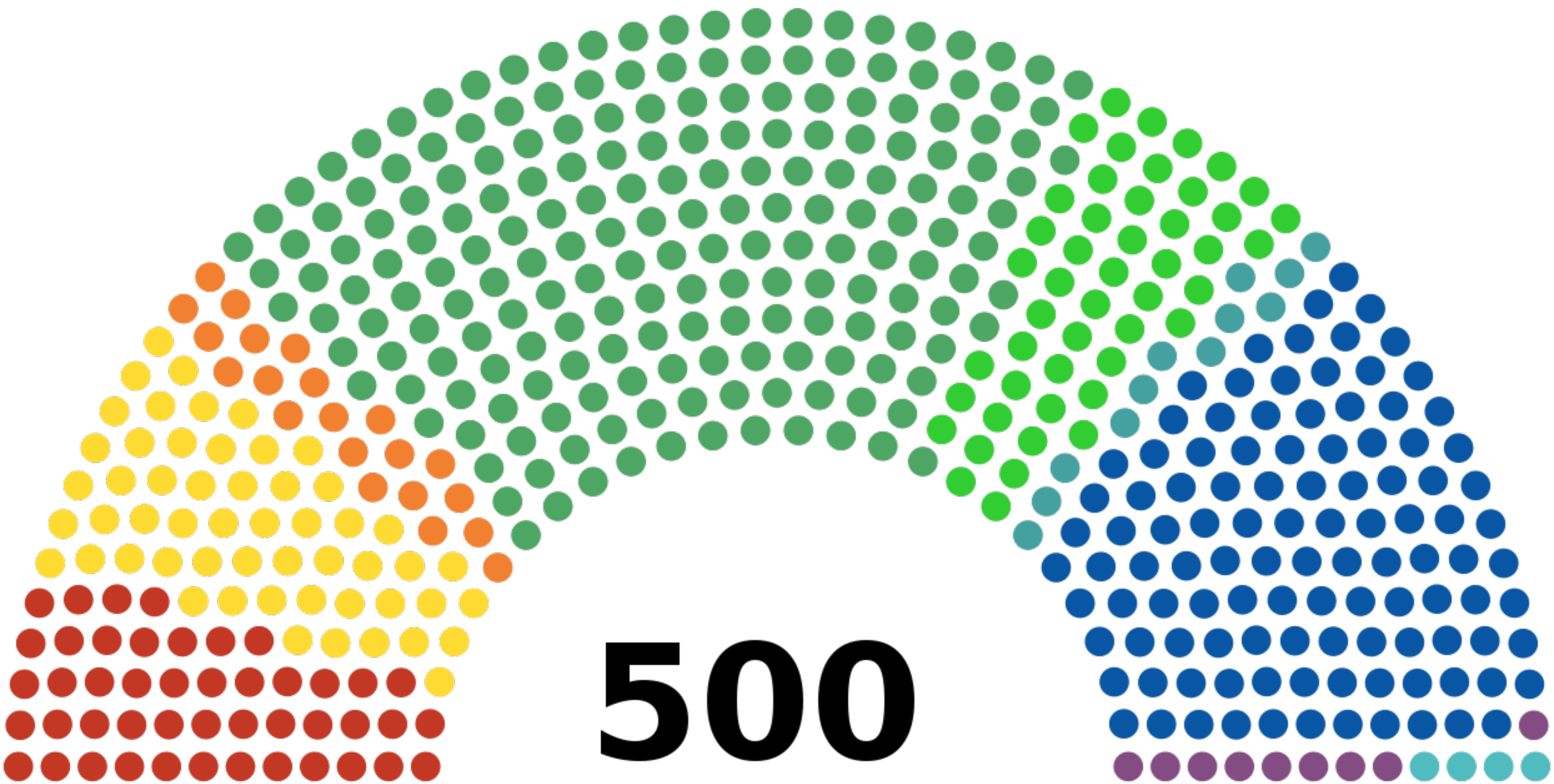
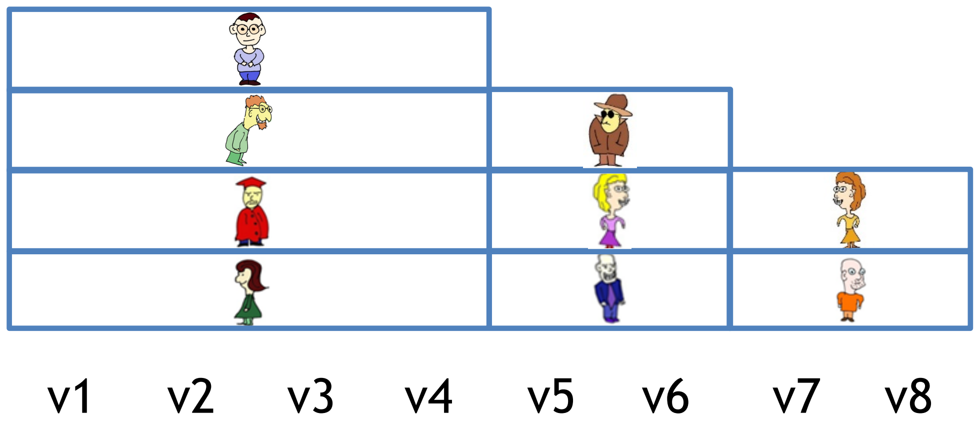
Everything depends on the context!

Context: electing a representative body



Back to the example!



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Which committee should be selected?

In this context the committee should be proportional.

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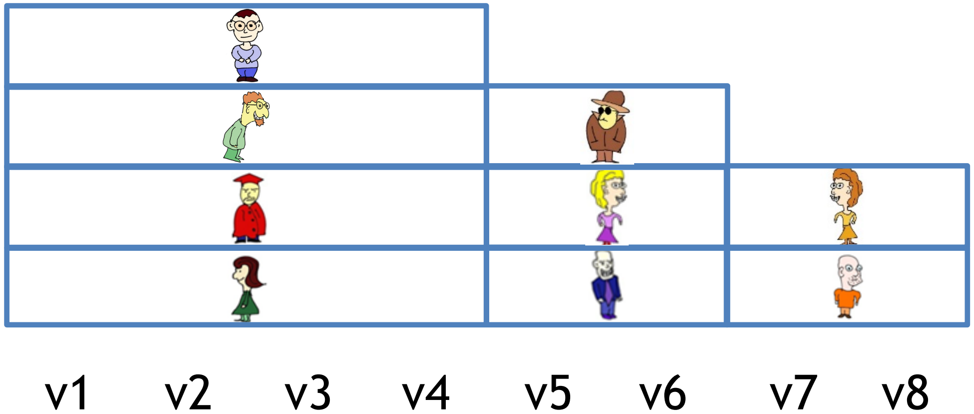

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But what does it mean and how could we achieve that?

Proportionality on the example of party-list systems.

Each voter casts one vote for a single party.
Our goal is to select a committee of size **k = 4**:

- Party 1 gets 40 votes.
- Party 2 gets 20 votes.
- Party 3 gets 20 votes.

How should the parliament look like?

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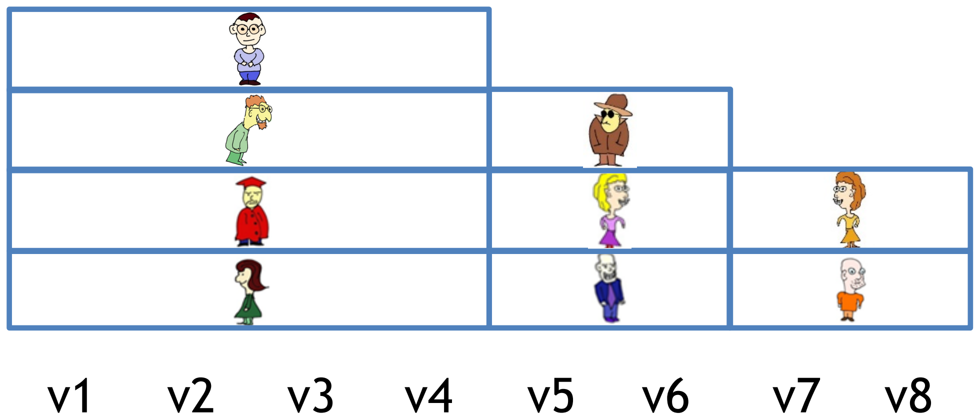
Our goal is to select a committee of size **k = 4**:

- Party 1 gets 40 votes.
- Party 2 gets 20 votes.
- Party 3 gets 20 votes.

How should the parliament look like?

- Party 1 should get 2 seats.
- Party 2 should get 1 seat.
- Party 3 should get 1 seat.

Back to the example!



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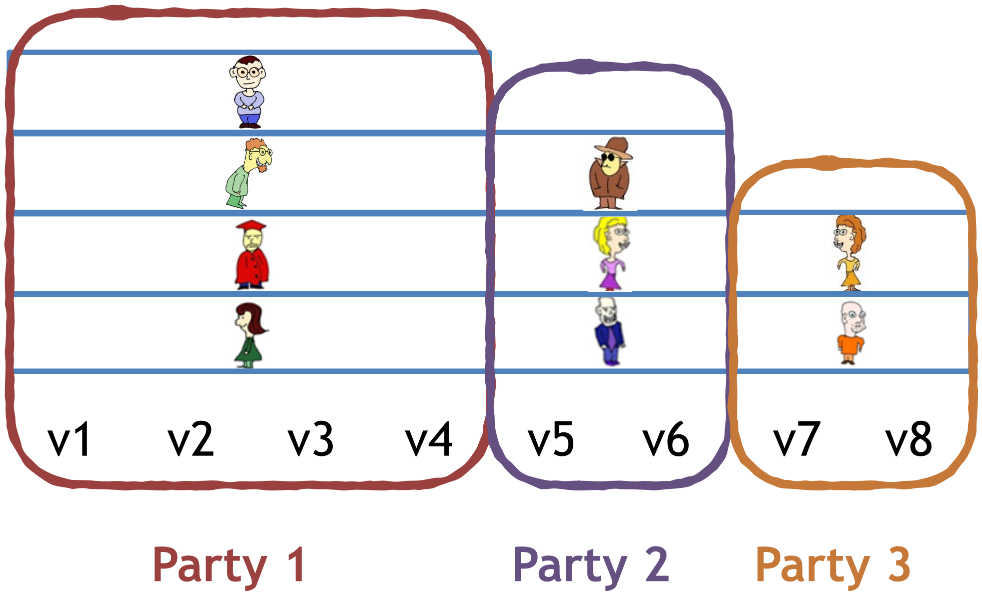

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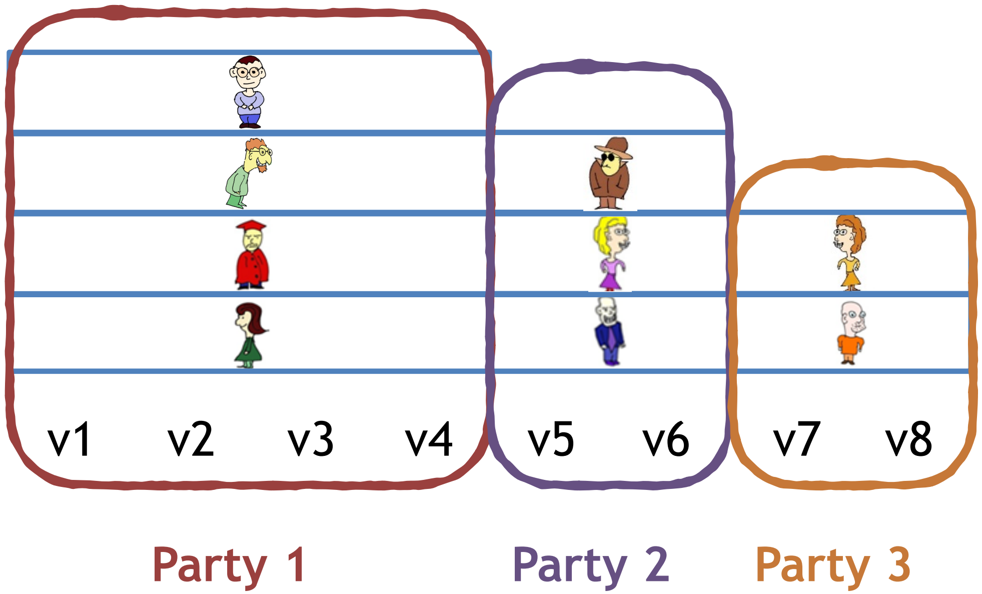

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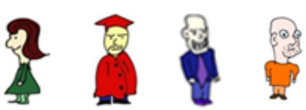
Which committee should be selected? 

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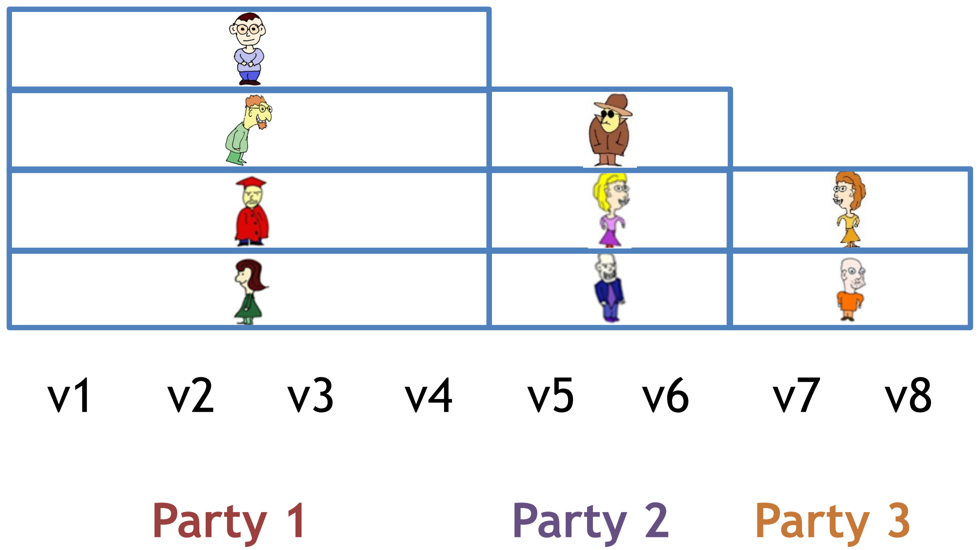
Intuition: The party P_i gets x_i votes.

If all $\frac{x_i}{n} \cdot k$ are integers, then

party P_i should get $\frac{x_i}{n} \cdot k$ seats.

Recall the first example



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How to define proportionality for more complex preferences?

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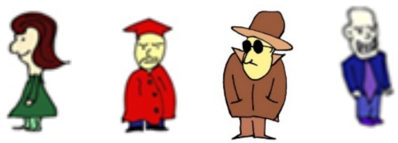
V3: 

V4: 

V5: 

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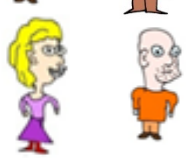
V7: 

V8: 

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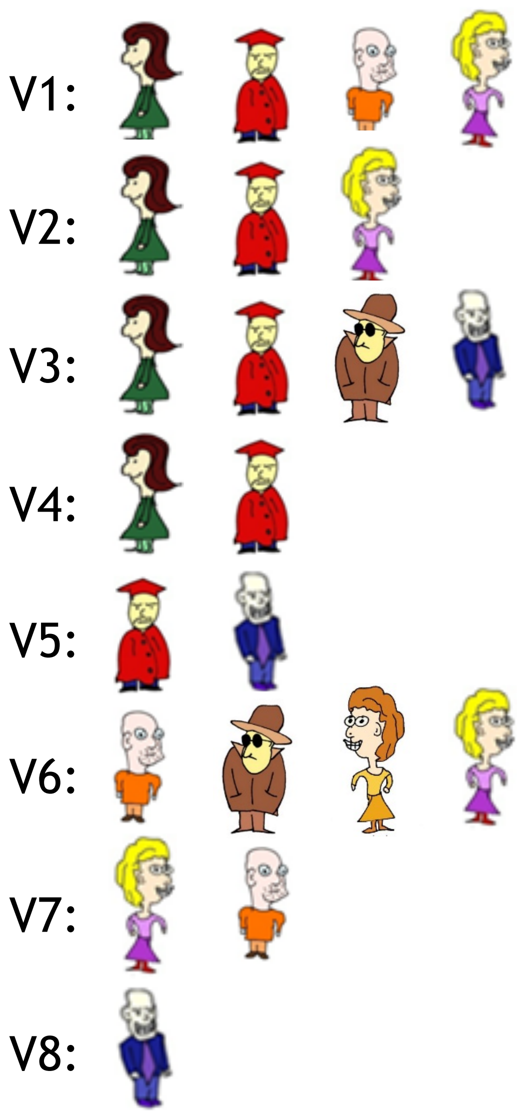

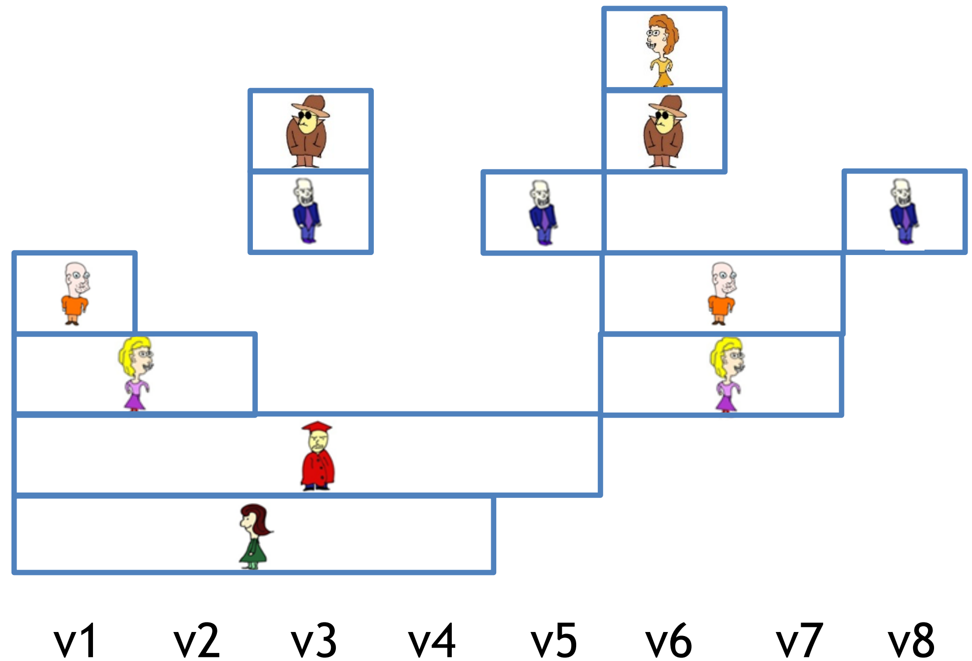

Let’s move back in time to the end of the 19th century?

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Thorvald N. Thiele

Edvard Phragmén

Proportional Approval Voting (Thiele)

Assume voter **v** approves **t** members of a committee **W**. Then **v** gives to **W** the following number of points:

$$\sum_{i=1}^{t} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{t}$$

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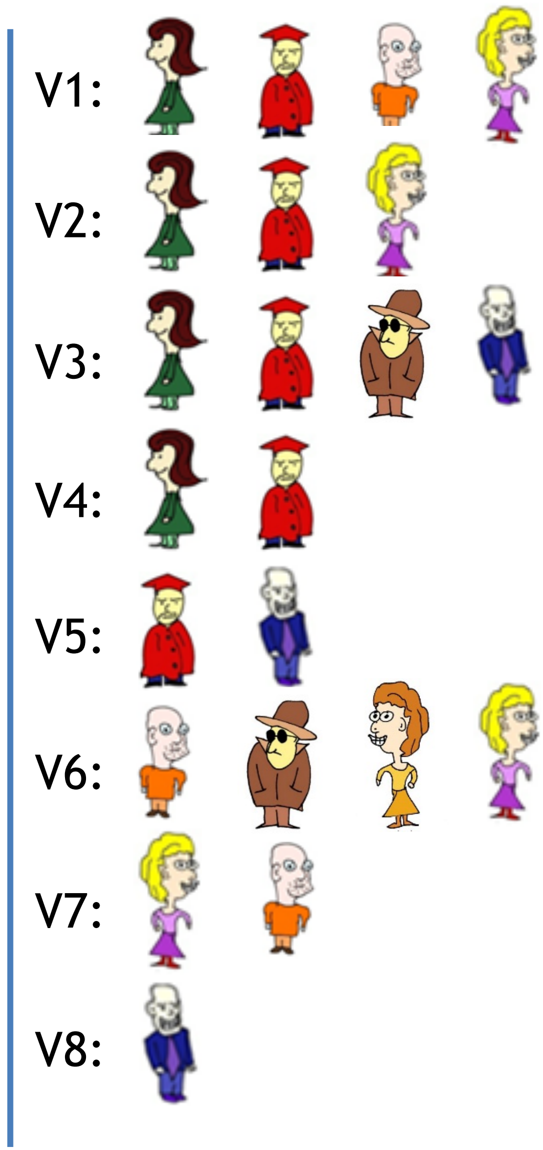

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E.g., consider a committee 

Points per voter:

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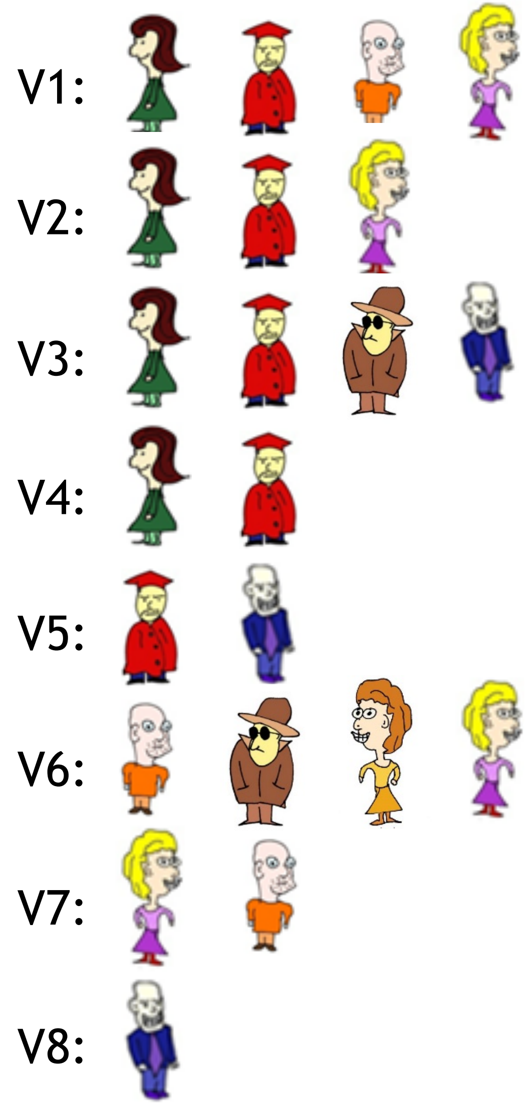

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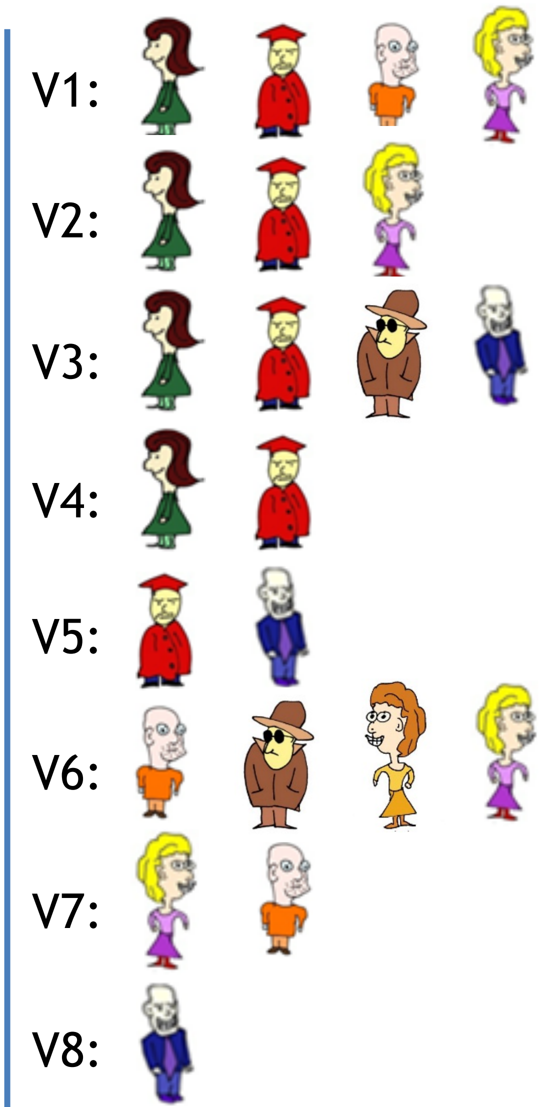

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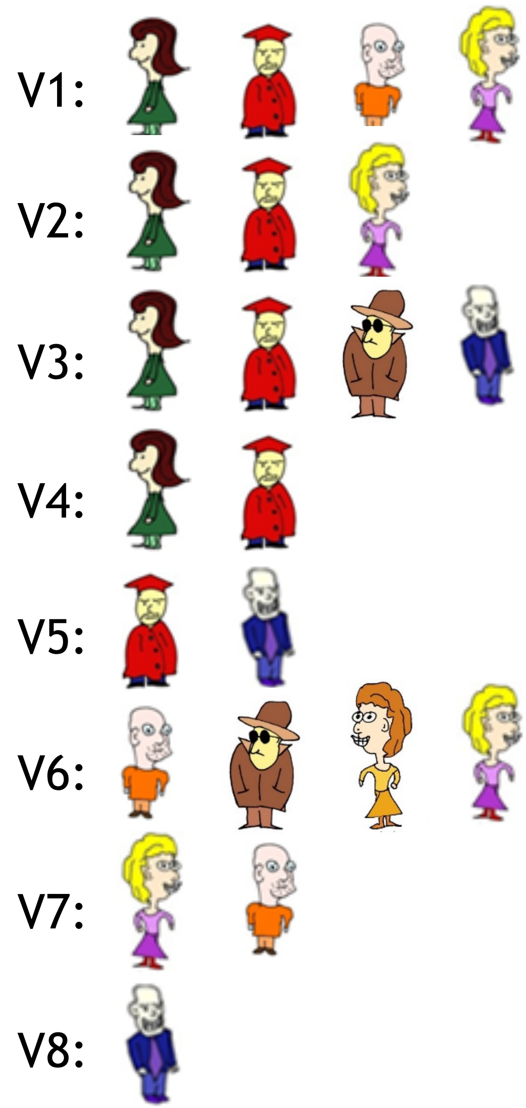

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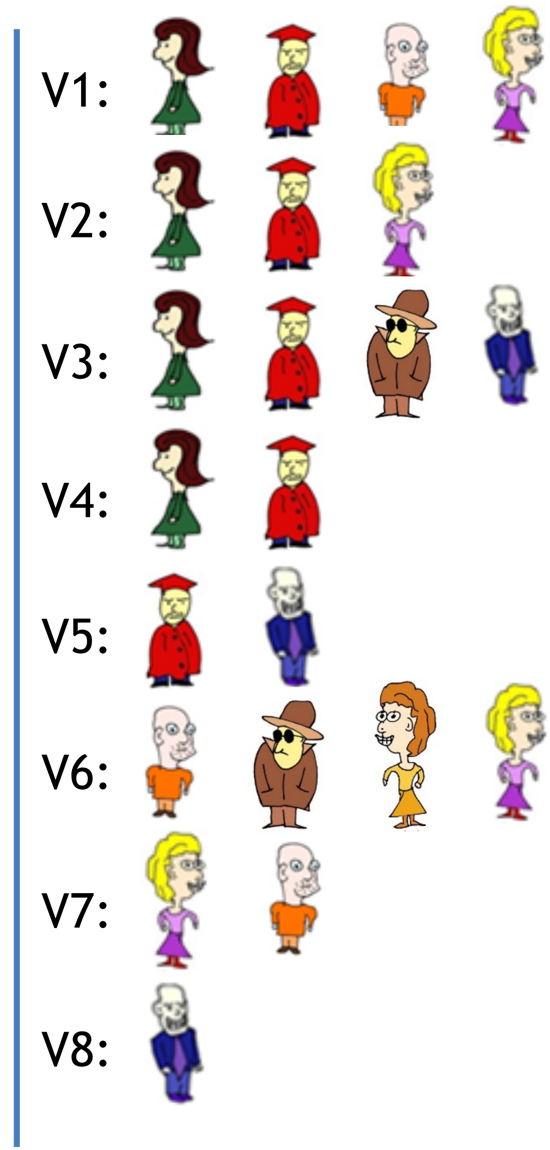

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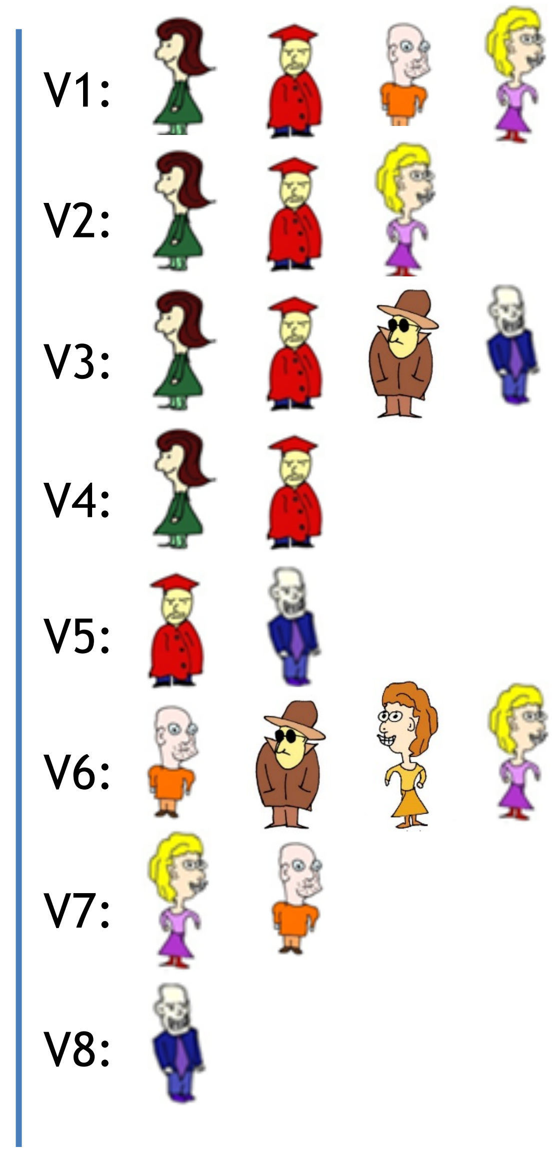

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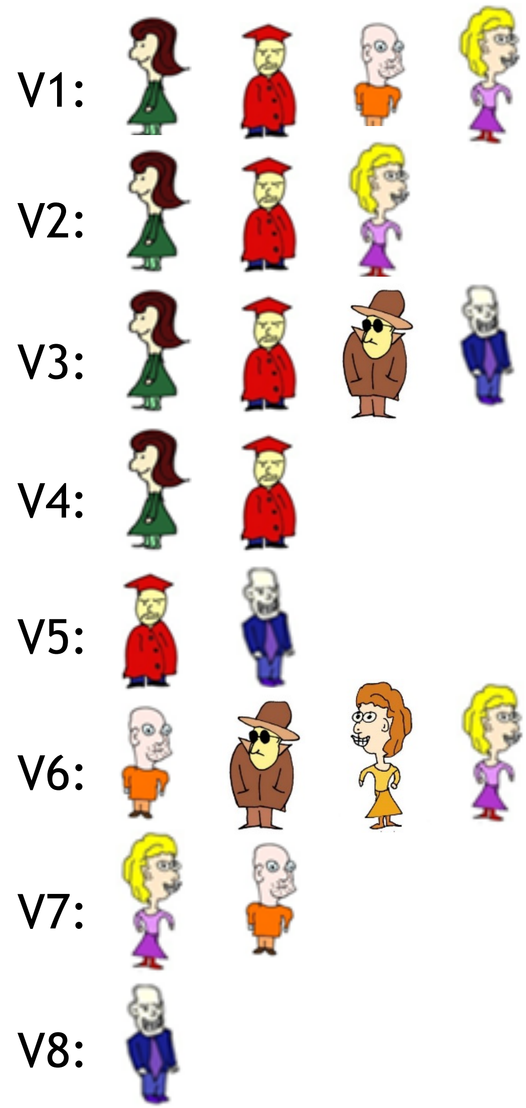

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Points per voter:

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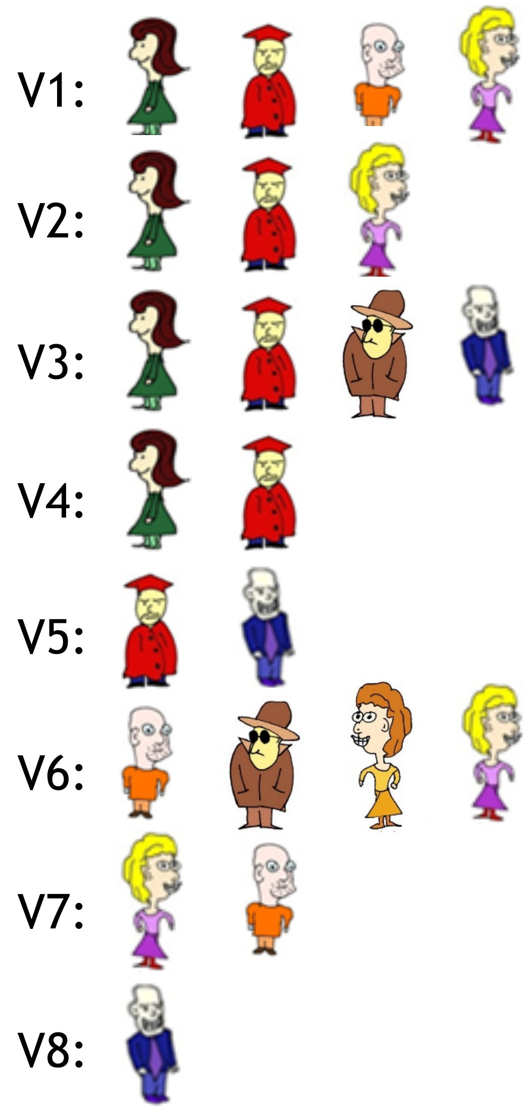

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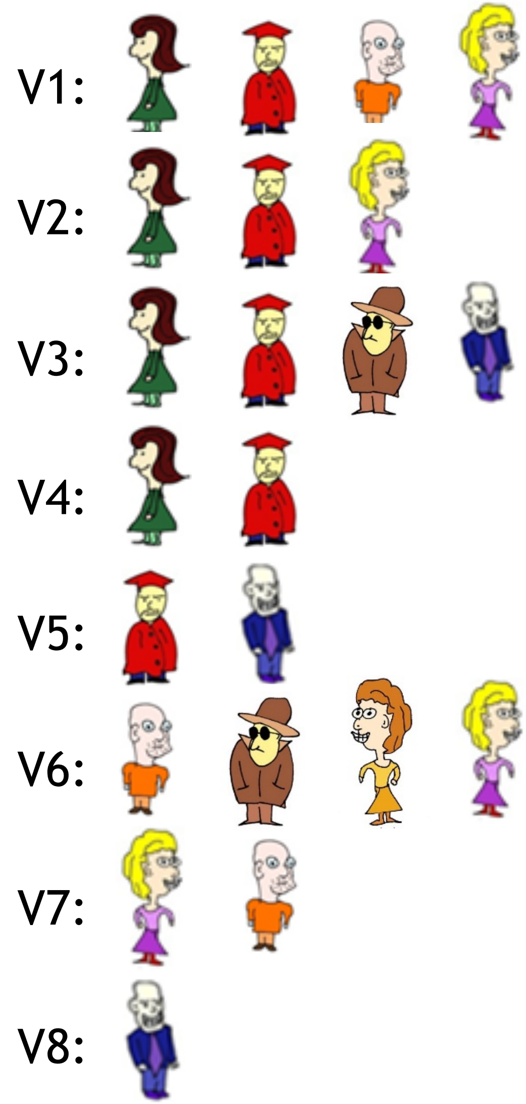

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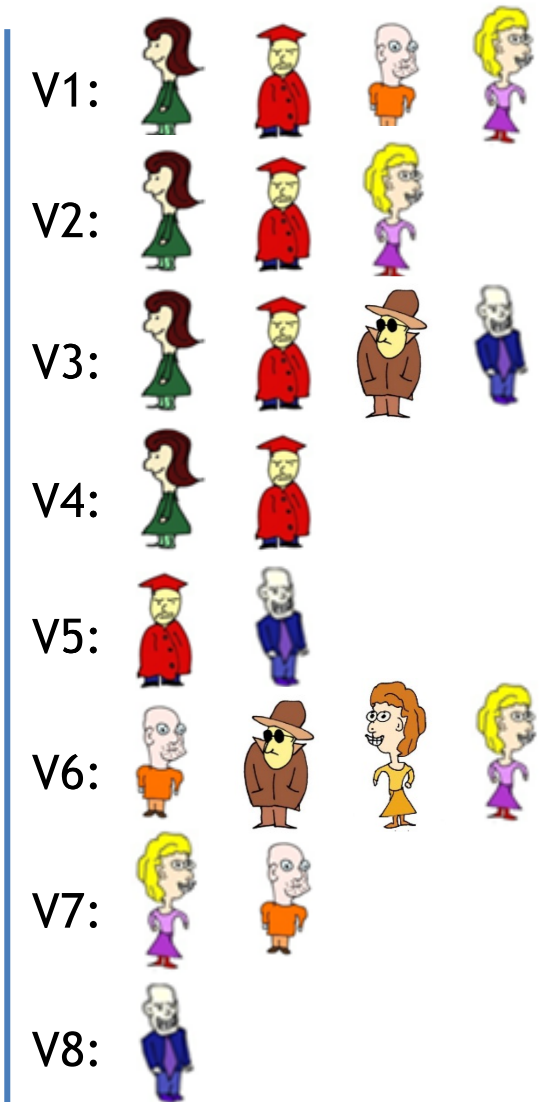

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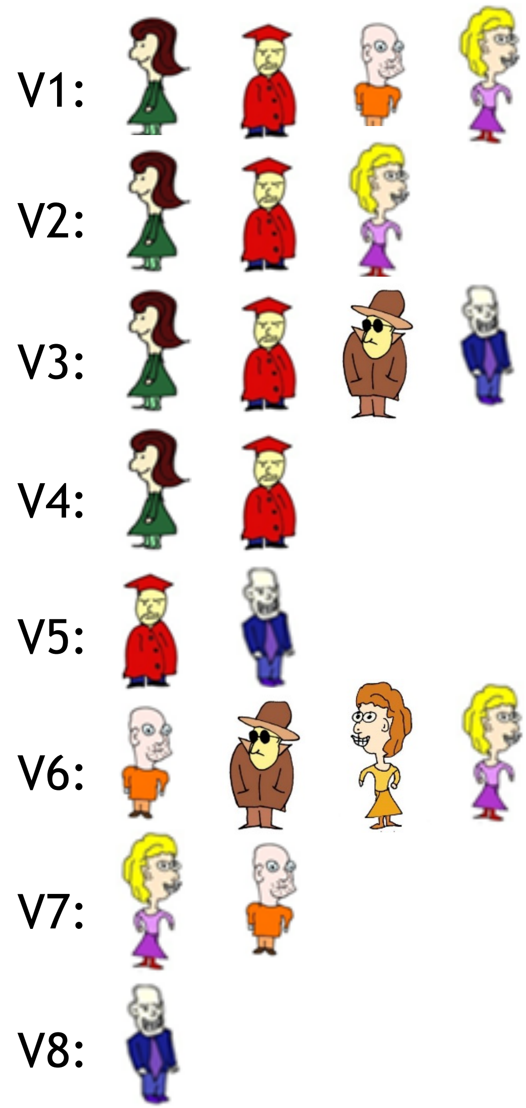
Sum of points = 8 + 5/6



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E.g.

Committee with the highest score wins the election.

Poin

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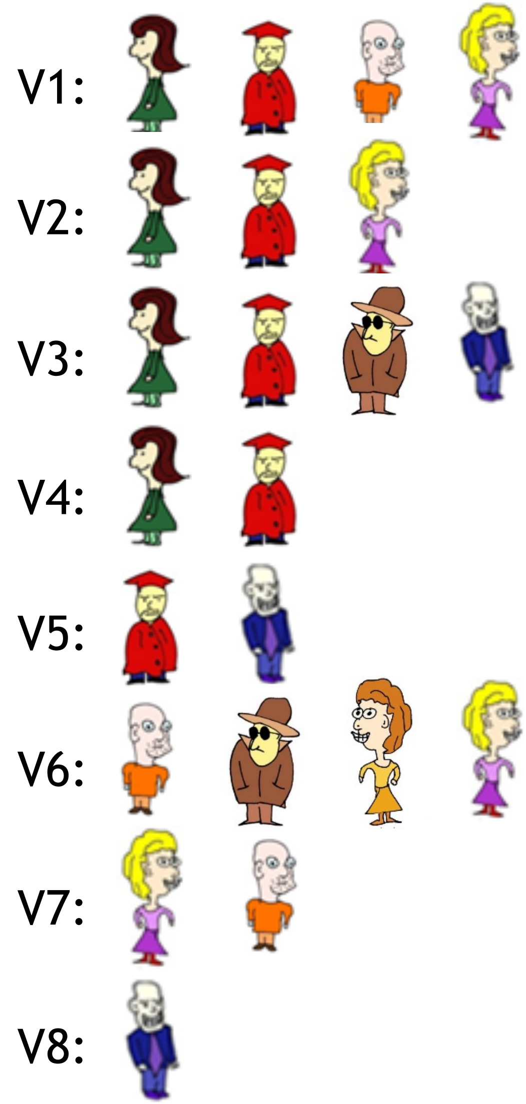

Proportional Approval Voting is welfarist

The welfare vector of a committee W is defined as:

$$\big(\,|A_1 \cap W|, |A_2 \cap W|, \ldots, |A_n \cap W|\,\big)$$

where:

A_i is the set of candidates approved by voter i

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A rule is welfarist if the decision which committee to elect can be made solely based on welfare vectors of the committees.

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- Voters earn money with the constant speed (\$1 per time unit).

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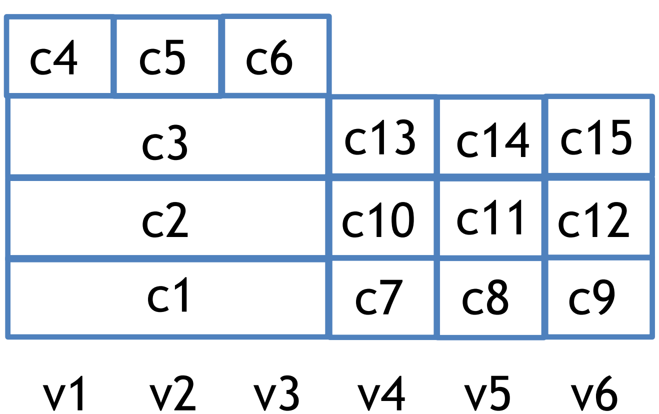

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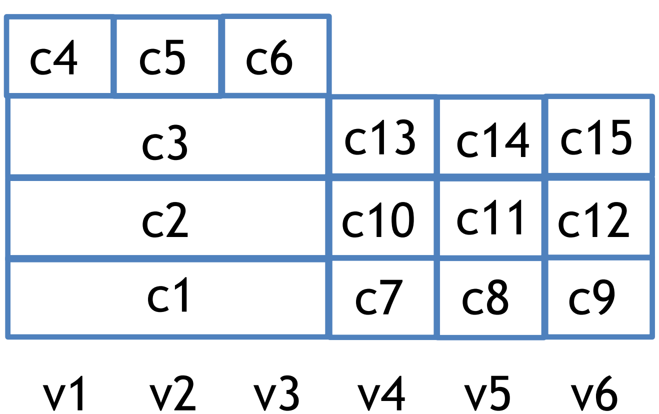
k = 12



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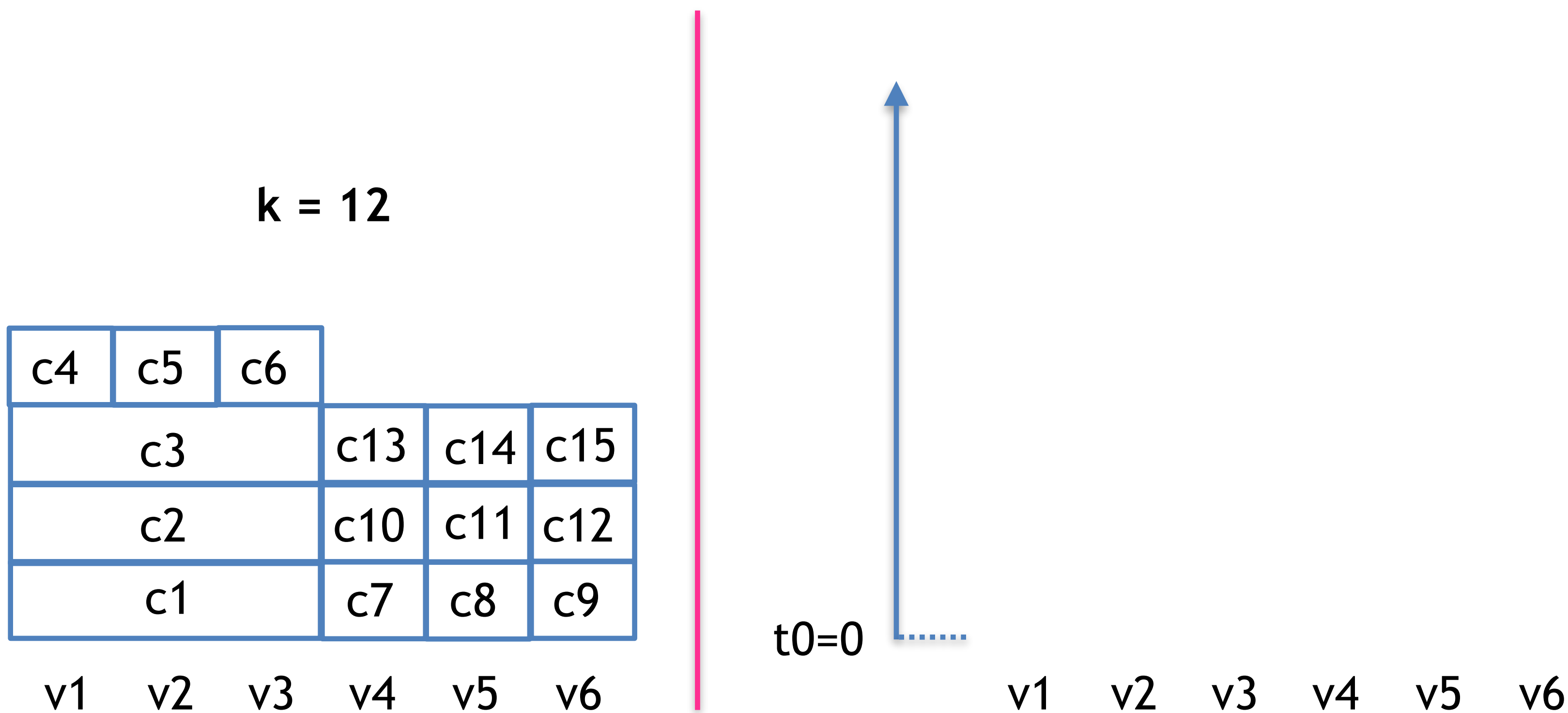

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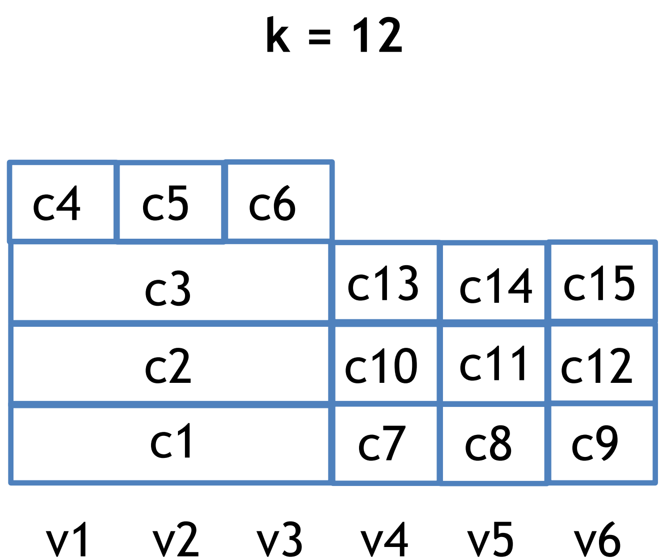

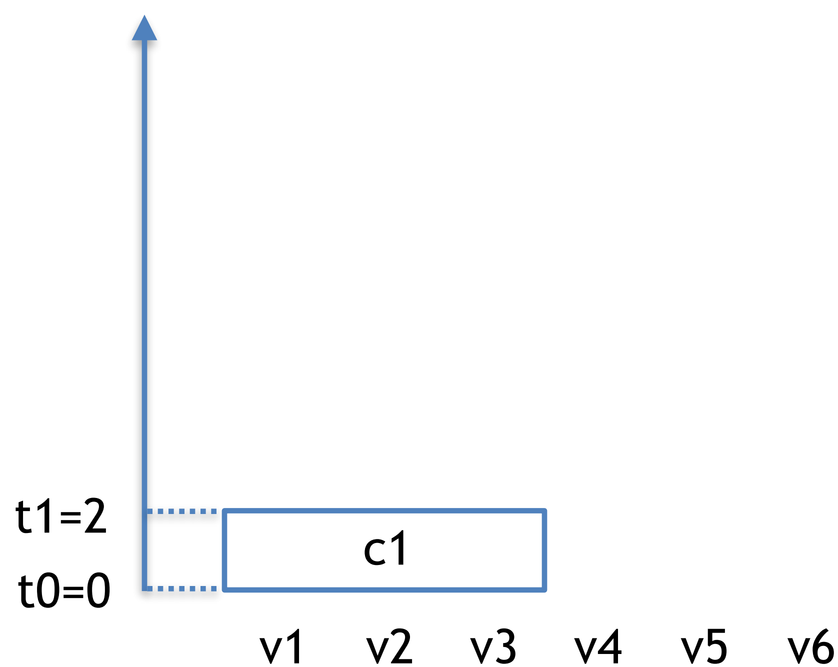

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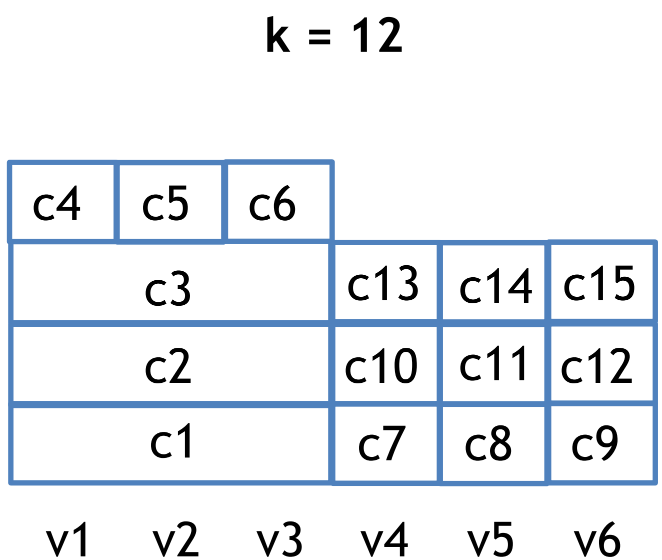

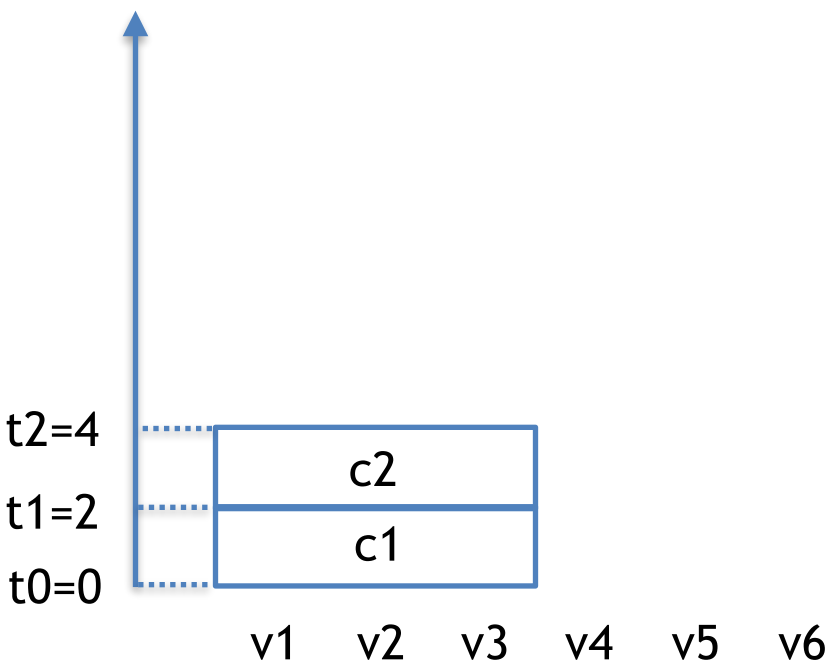

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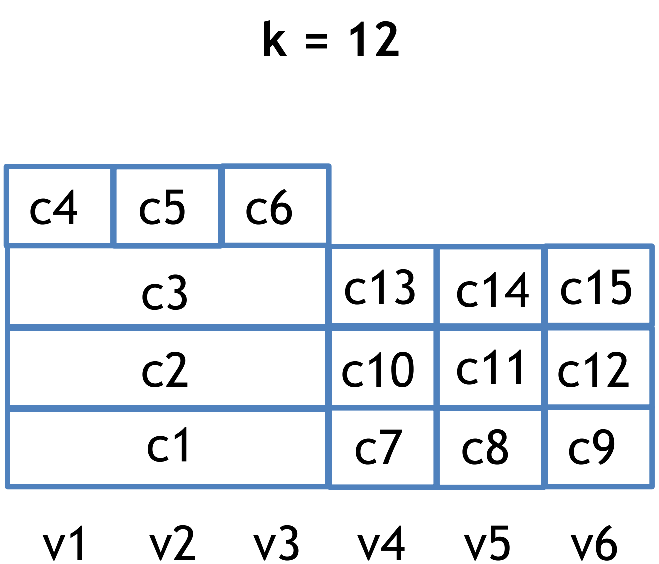

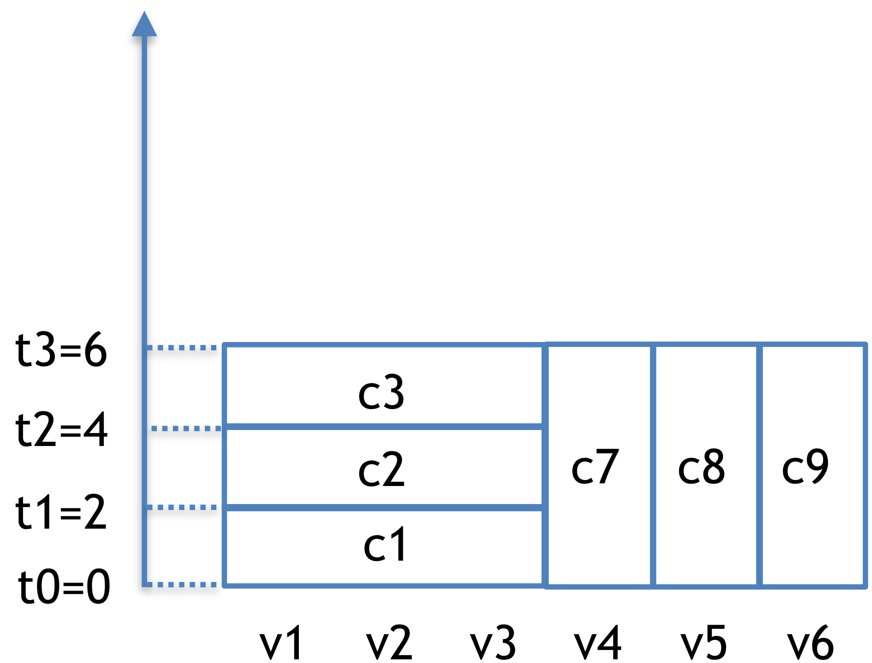

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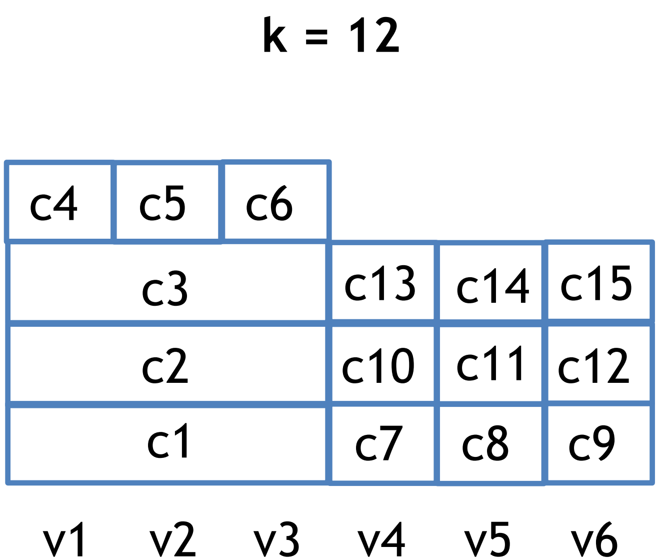

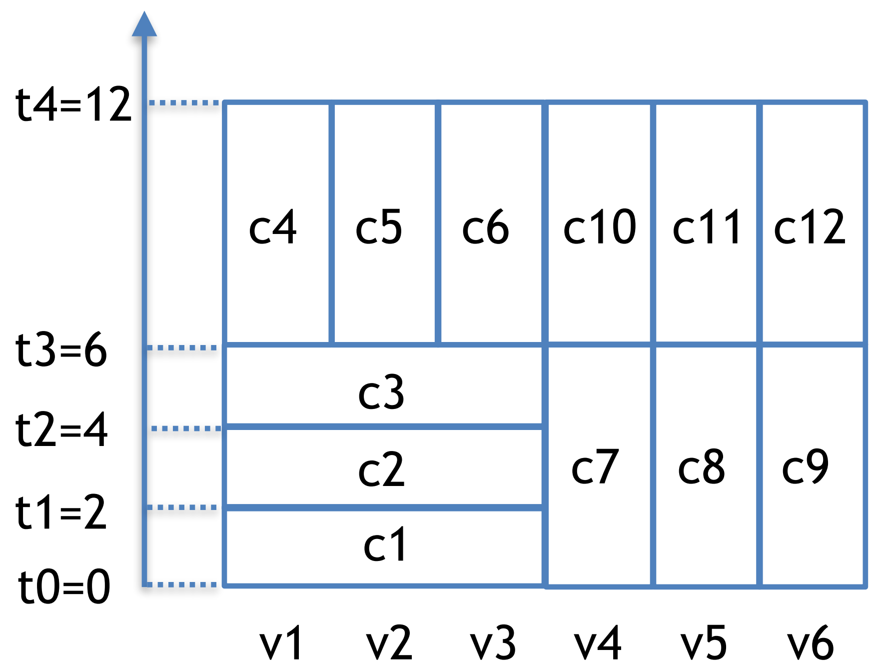

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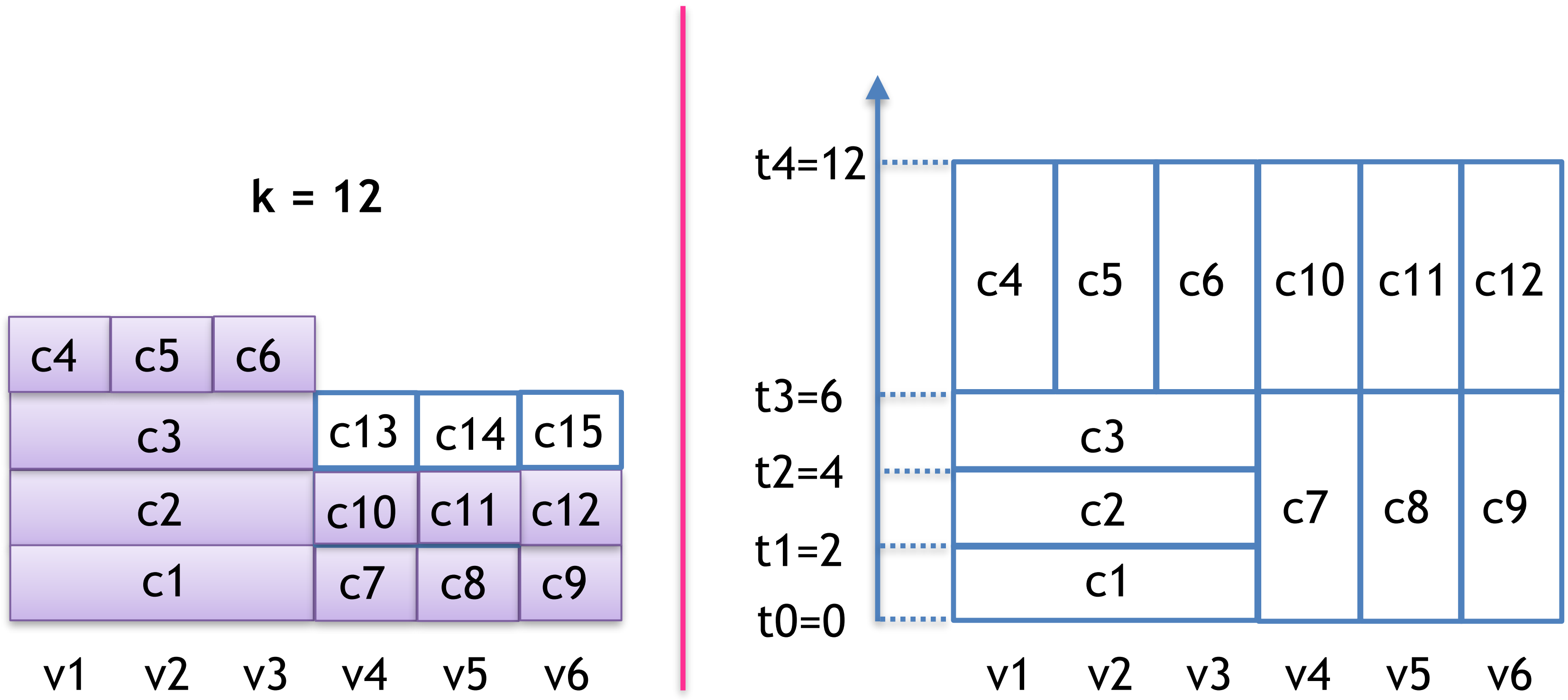

PAV versus Phragmén's Rule

Which of the two rules is better?

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PAV versus Phragmén's Rule

Which of the two rules is better?

- Both Thiele and Phragmén argued that their rules are proportional by how they behave on party-list profiles.
- Historically PAV was preferred since it appeared simpler.
- Current research suggest that PAV is better in terms of proportionality.

Two Arguments in Favour of PAV

First Argument: Axioms for Cohesive Groups

How to define proportionality for more complex preferences?

V1:

V2:

V3:

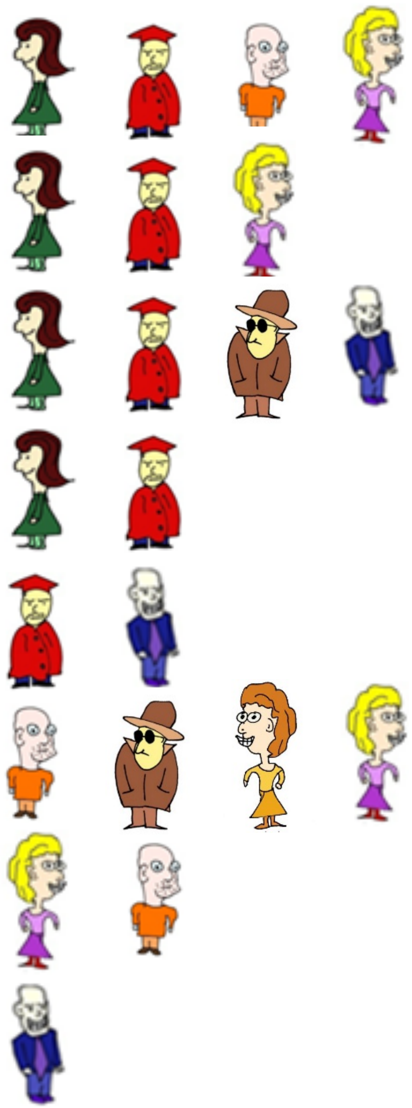
V4:

V5:

V6:

V7:

V8:



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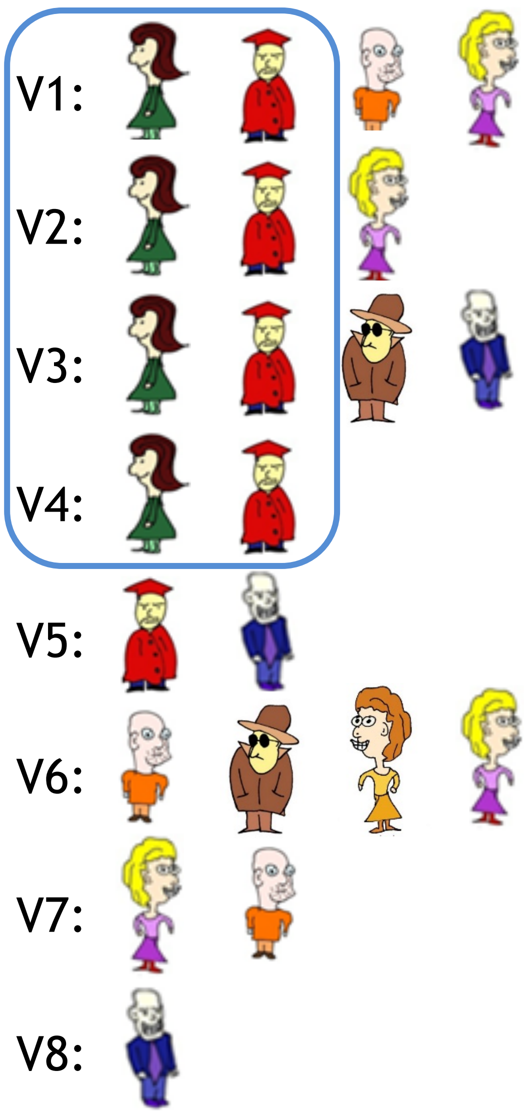

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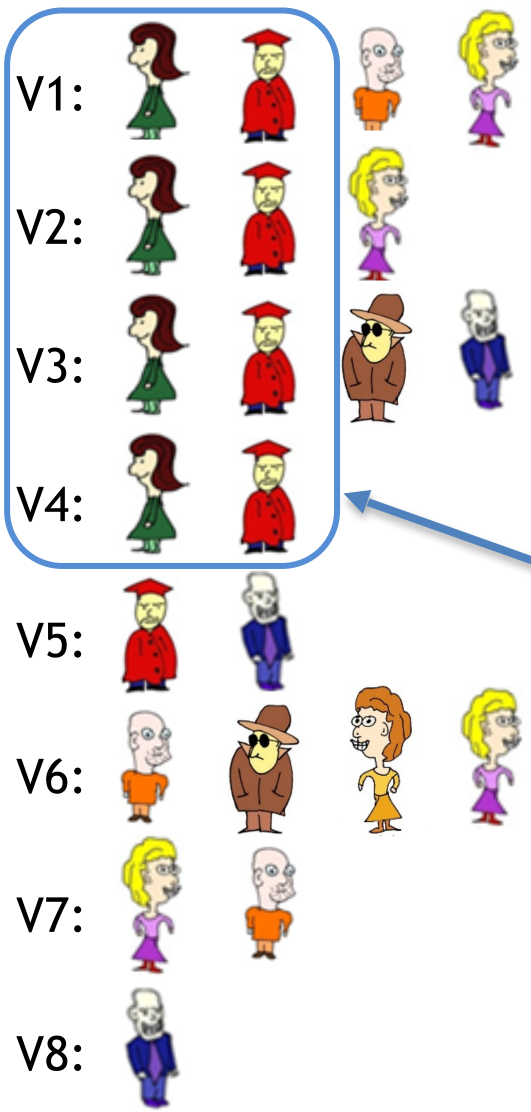

For k = 4 these voters should approve (on average) 2 candidates in the selected committee.

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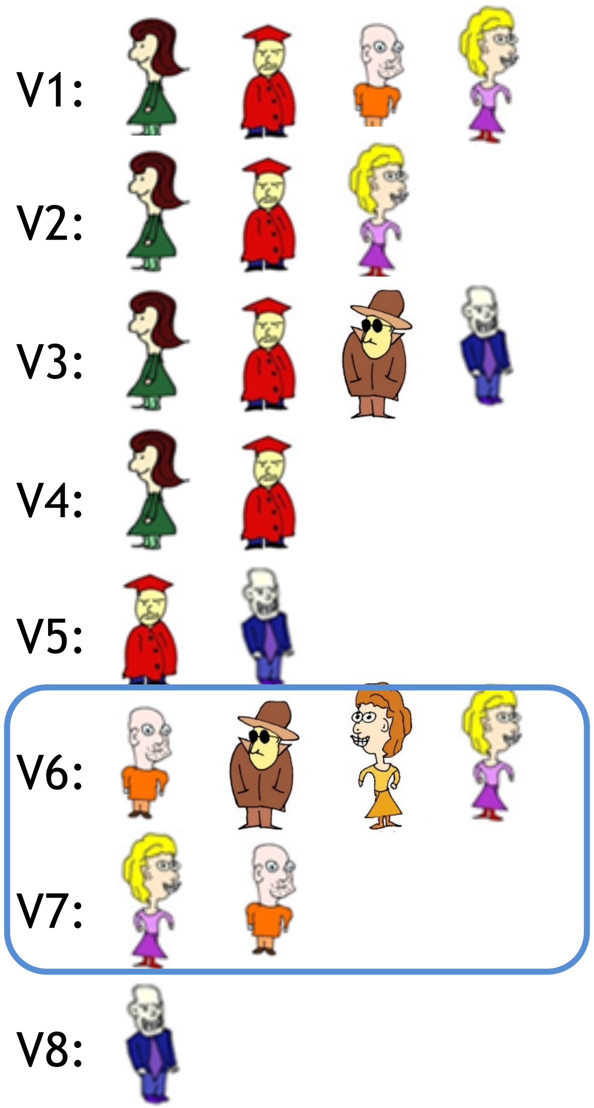

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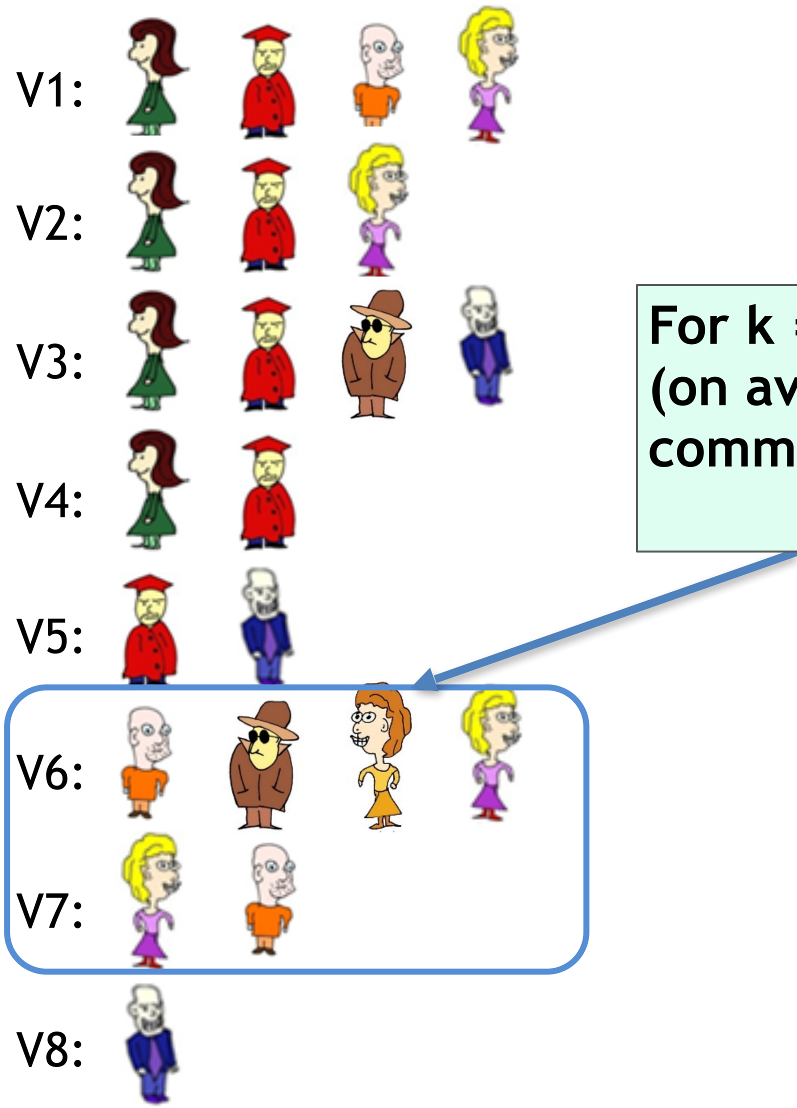
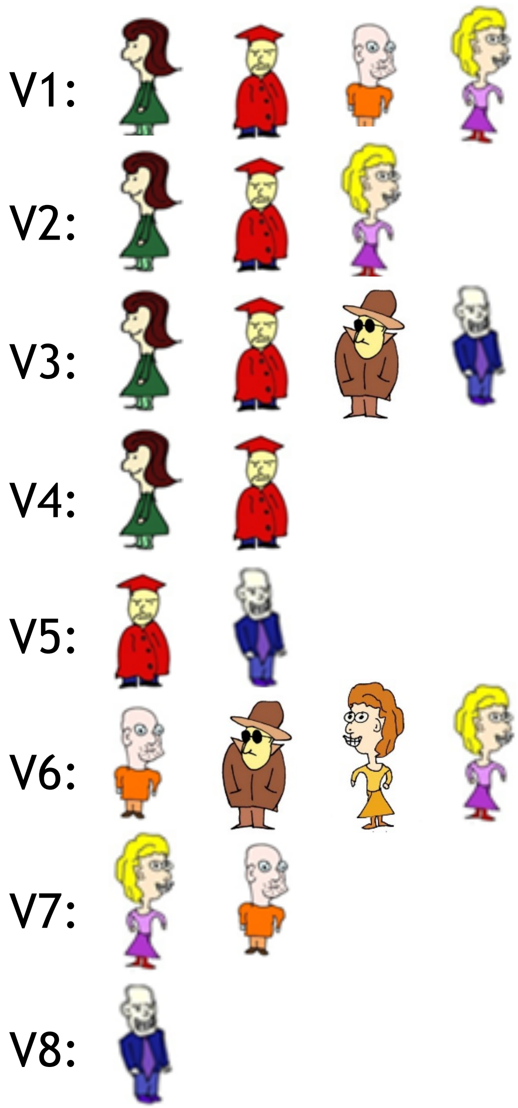


For k = 4 these voters should approve (on average) 1 candidate in the selected committee.

How to define proportionality for more complex preferences?



Definition: Each group with at least $\ell n/k$ voters who approve at least ℓ same candidates should have on average at least ℓ representatives in the elected committee.

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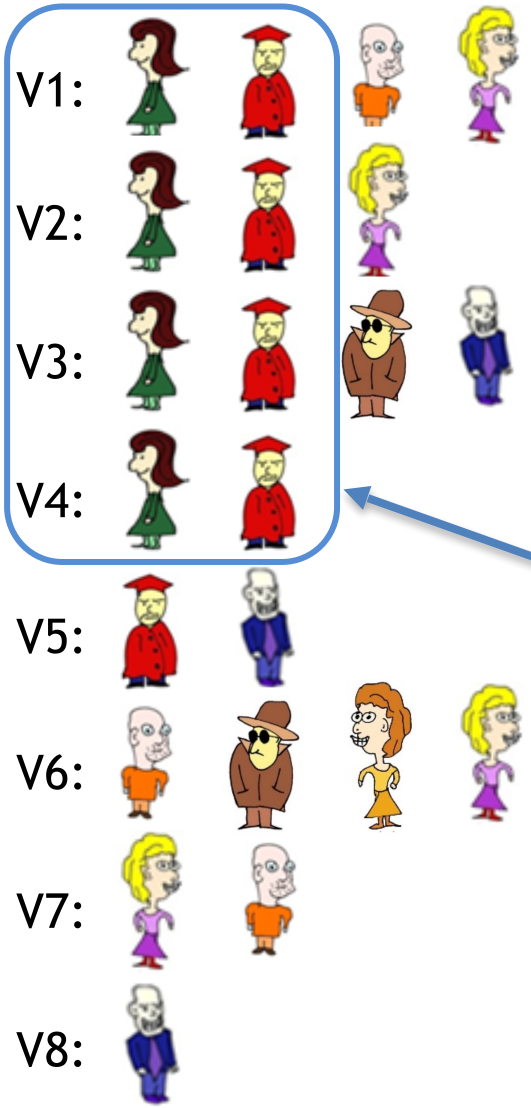

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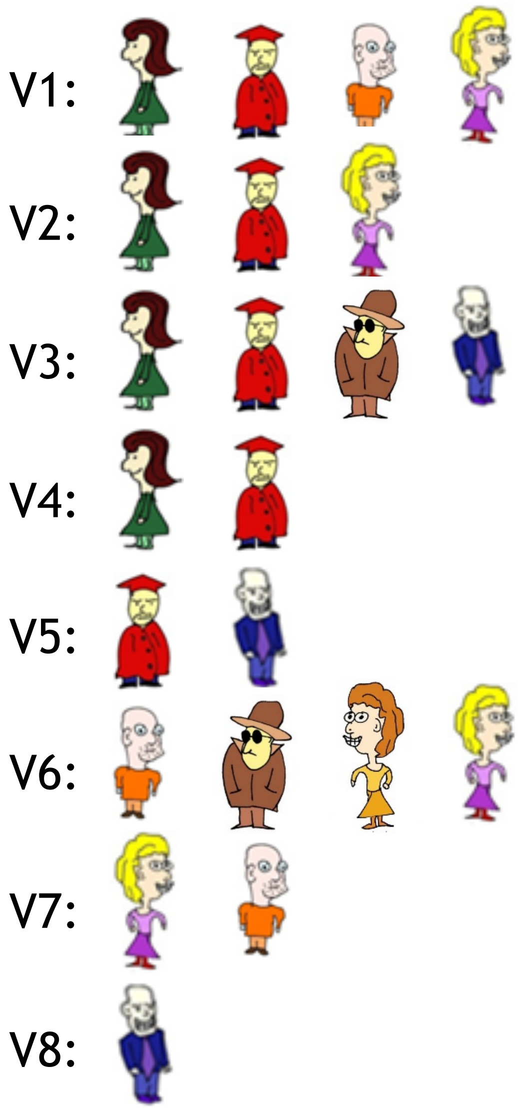

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Does there exist a system which satisfies this property?

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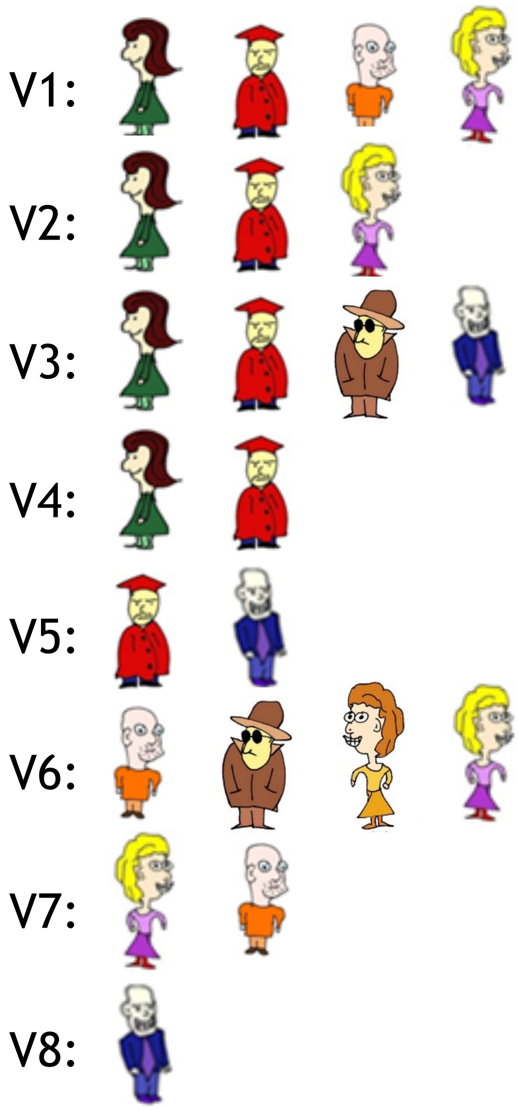

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v1: {a, d}	v7: {b, c}	
v2: {a}	v8: {c}	
v3: {a}	v9: {c}	**n = 12**
v4: {a, b}	v10: {c, d}	**k = 3**
v5: {b}	v11: {d}	
v6: {b}	v12: {d}	

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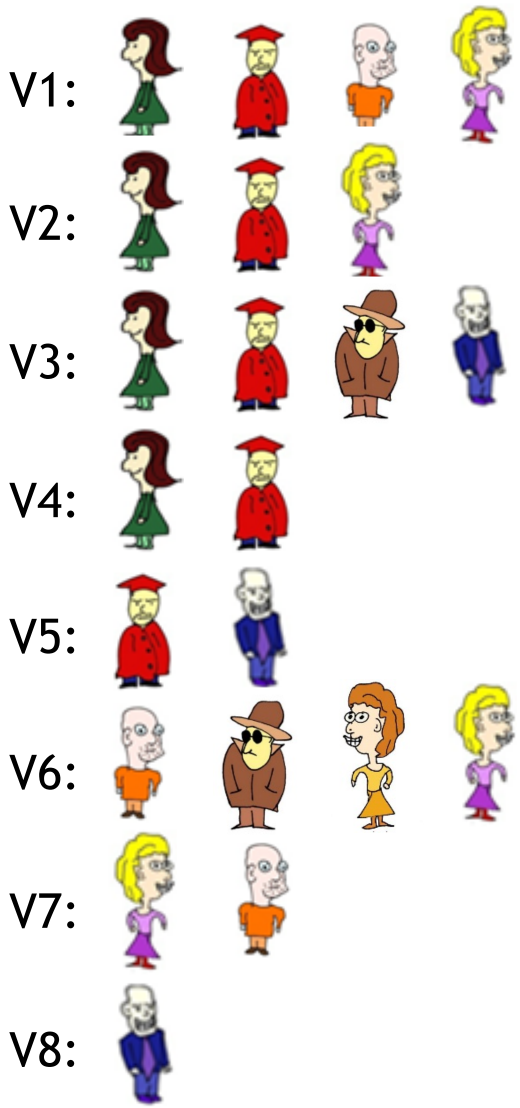

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But PAV satisfies a slightly weaker property!

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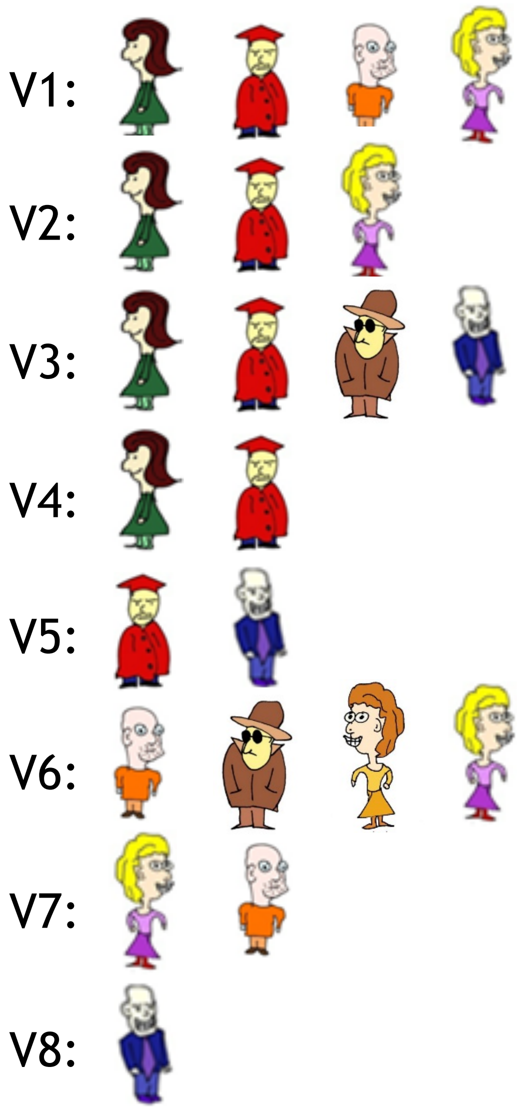

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But PAV satisfies a slightly weaker property!

Phragmén's Rule would satisfy it only if we replaced $\ell - 1$ with $(\ell - 1)/2$.

Two Arguments in Favour of PAV

Second Argument: Axiomatic Extensions of Apportionment Methods

Proportionality for party-list systems

Each voter can cast her vote on a single party:
(assume we have ***n*** voters and ***k*** parliamentary seats)

Lower-quota: The party that gets x votes should get $\left\lfloor \frac{x}{n} \cdot k \right\rfloor$ seats.

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The D'Hondt method of apportionment satisfies lower-quota.

D’Hondt method: An example

Party 1: **6 votes,** *Party 2*: **7 votes,** *Party 3*: **39 votes,** *Party 4*: **48 votes**

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Party 2 gets 0 seats

Party 3 gets 4 seats

Party 4 gets 6 seats

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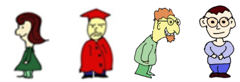
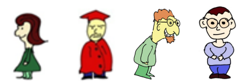
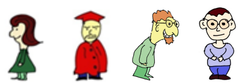
The D’Hondt method satisfies lower-quota.

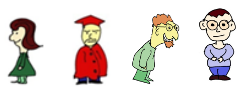
Let’s look at this instance

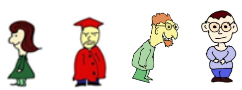
We have 9 voters, 9 candidates, and our goal is to select a committee of **size k = 4**.

V1: 

V2: 

V3: 

V4: 

V5: 

V6: 

V7: 

V8: 

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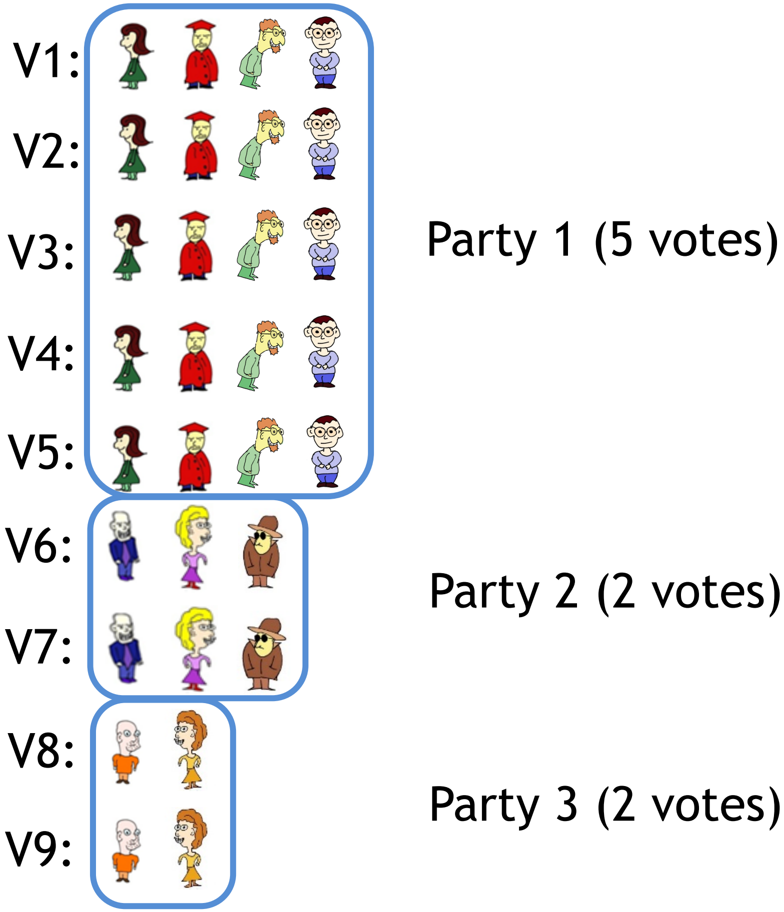

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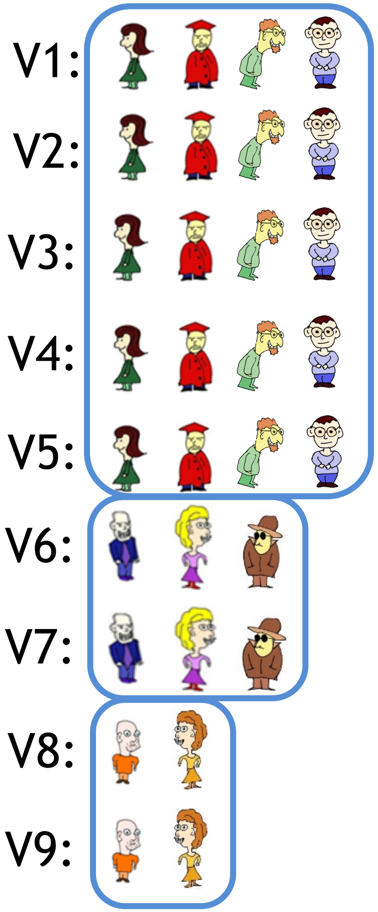


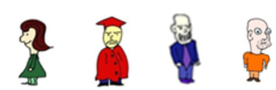
Party 1 (5 votes)

Party 2 (2 votes)

Party 3 (2 votes)

- Party 1 gets 2 seats.
- Party 2 gets 1 seat.
- Party 3 gets 1 seat.

For example



Some basic axiomatic properties: **Symmetry**

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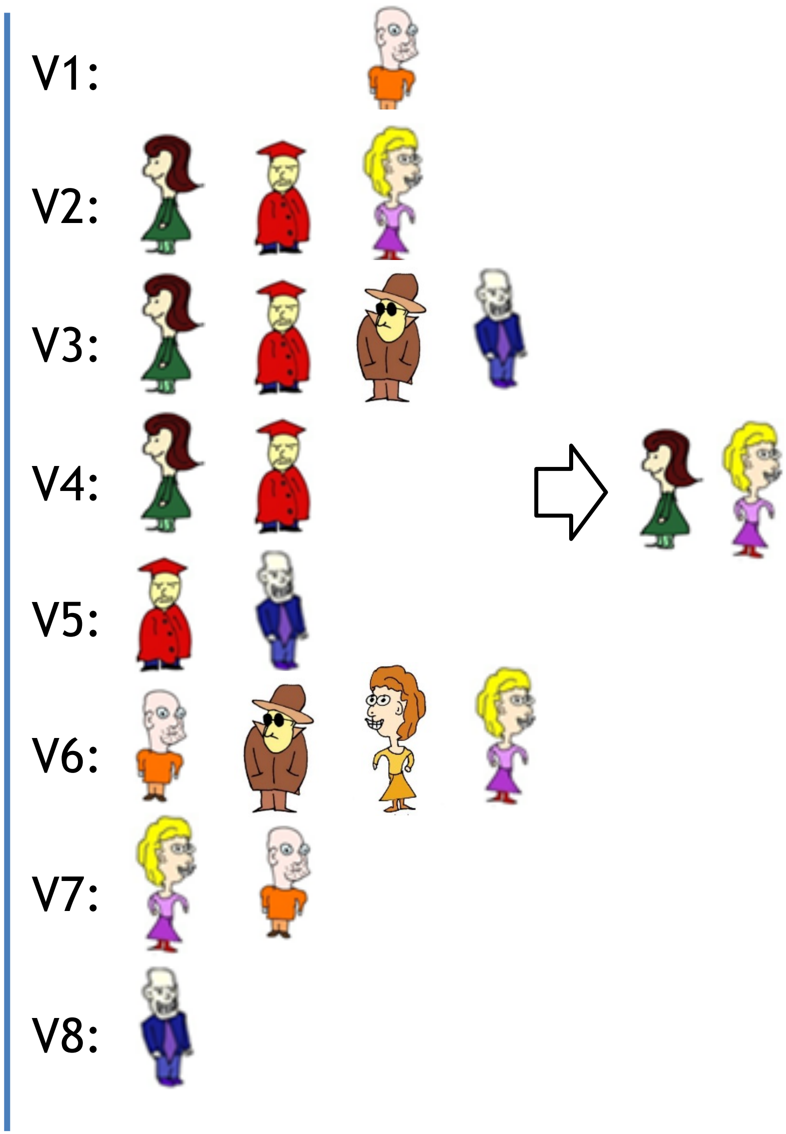

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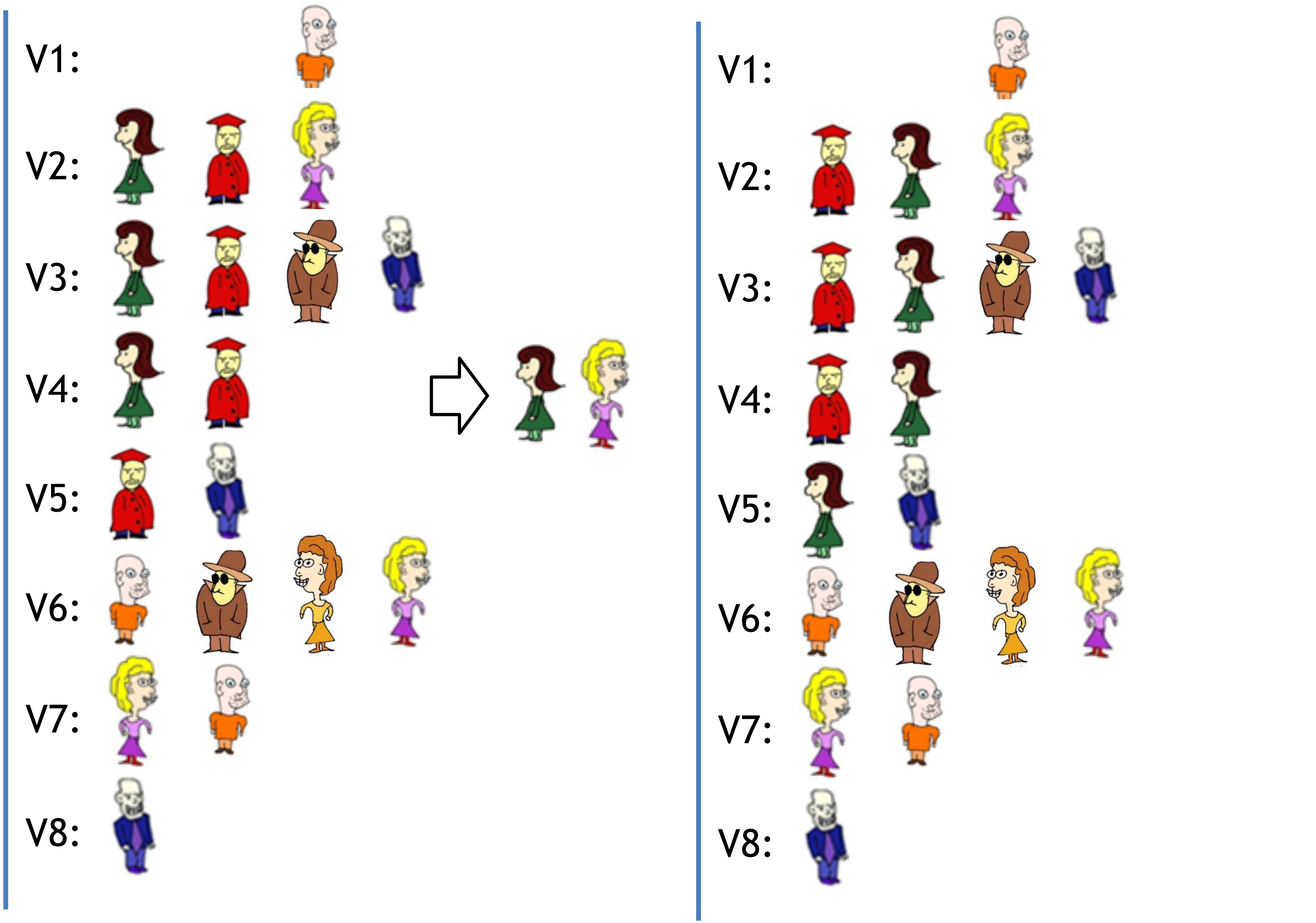

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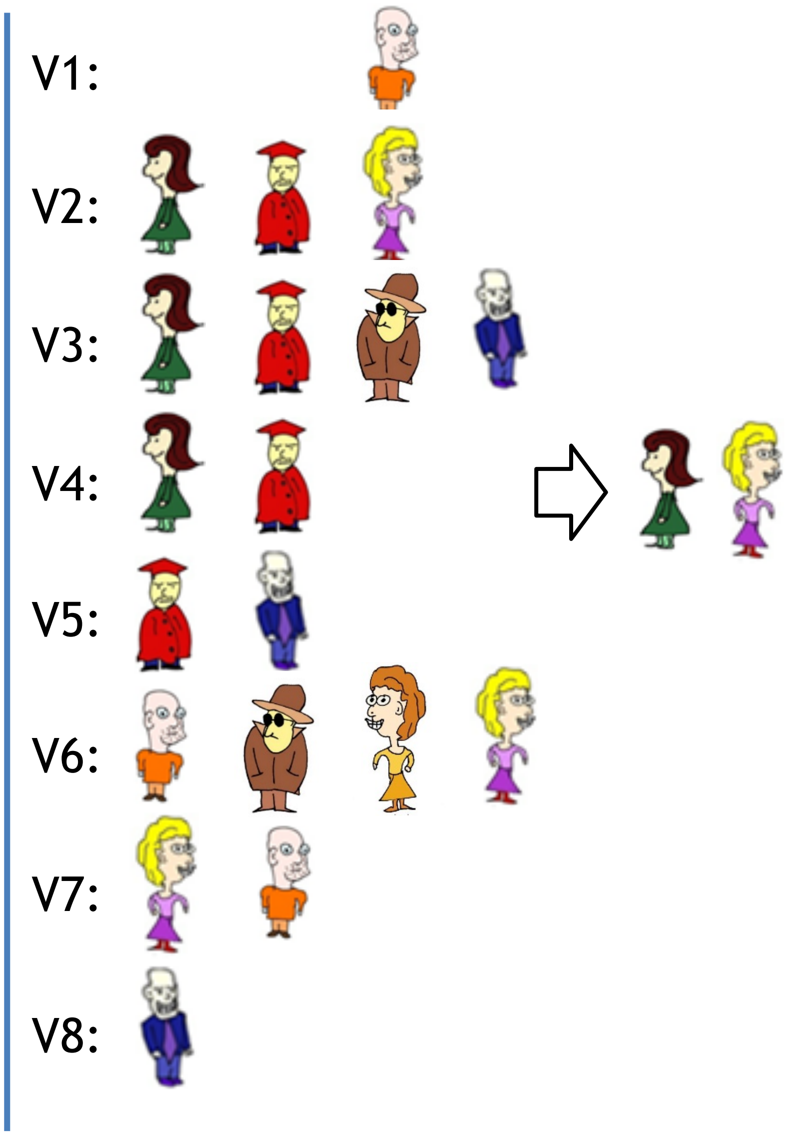

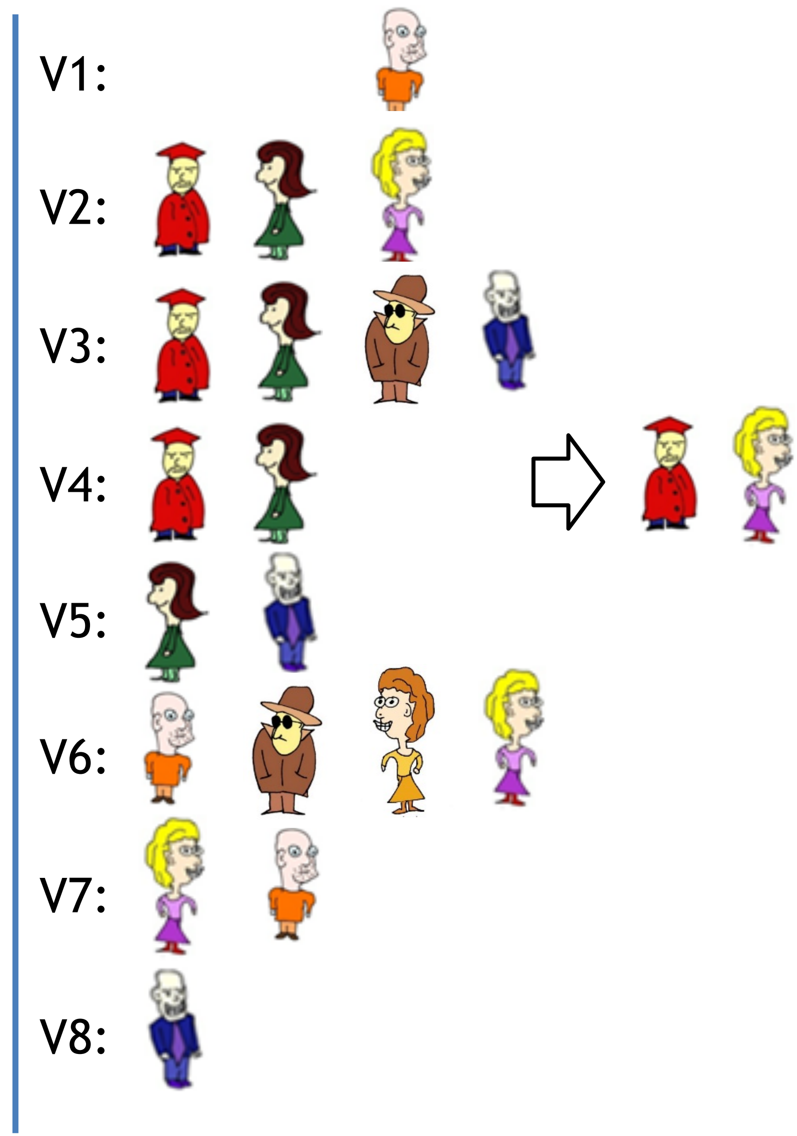

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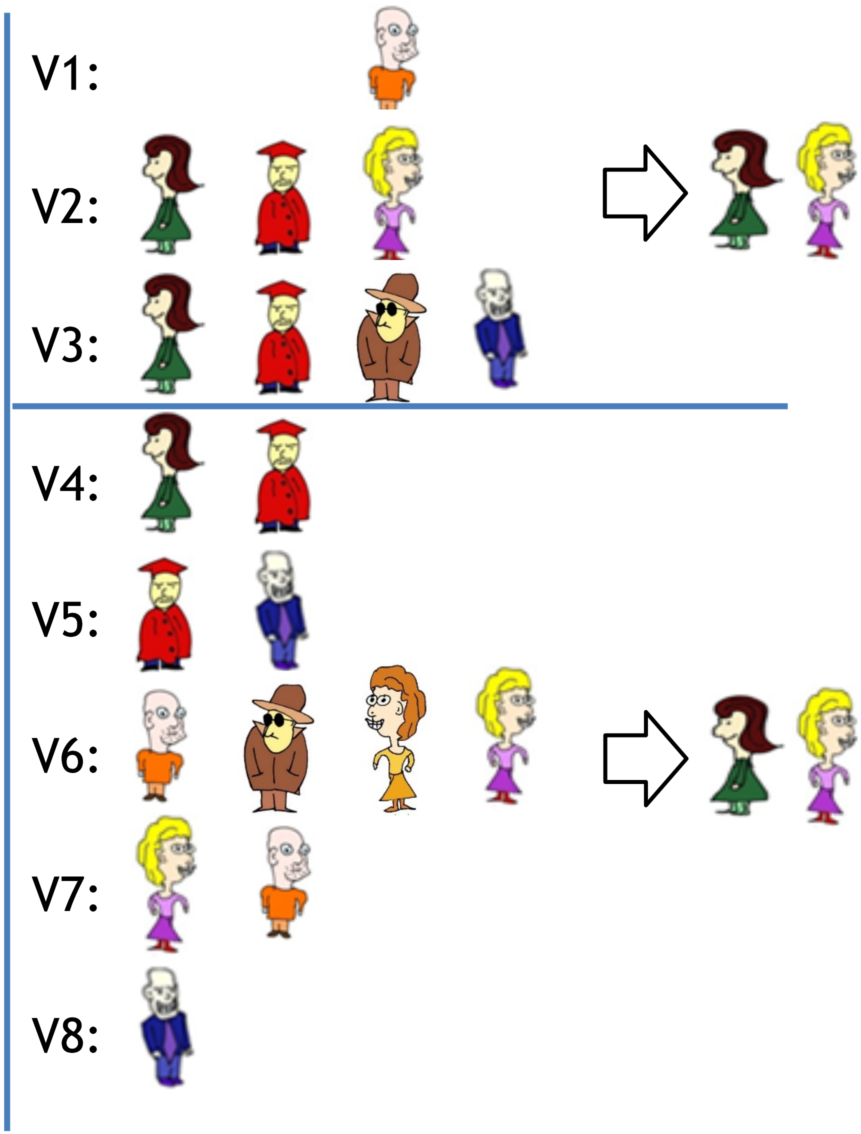

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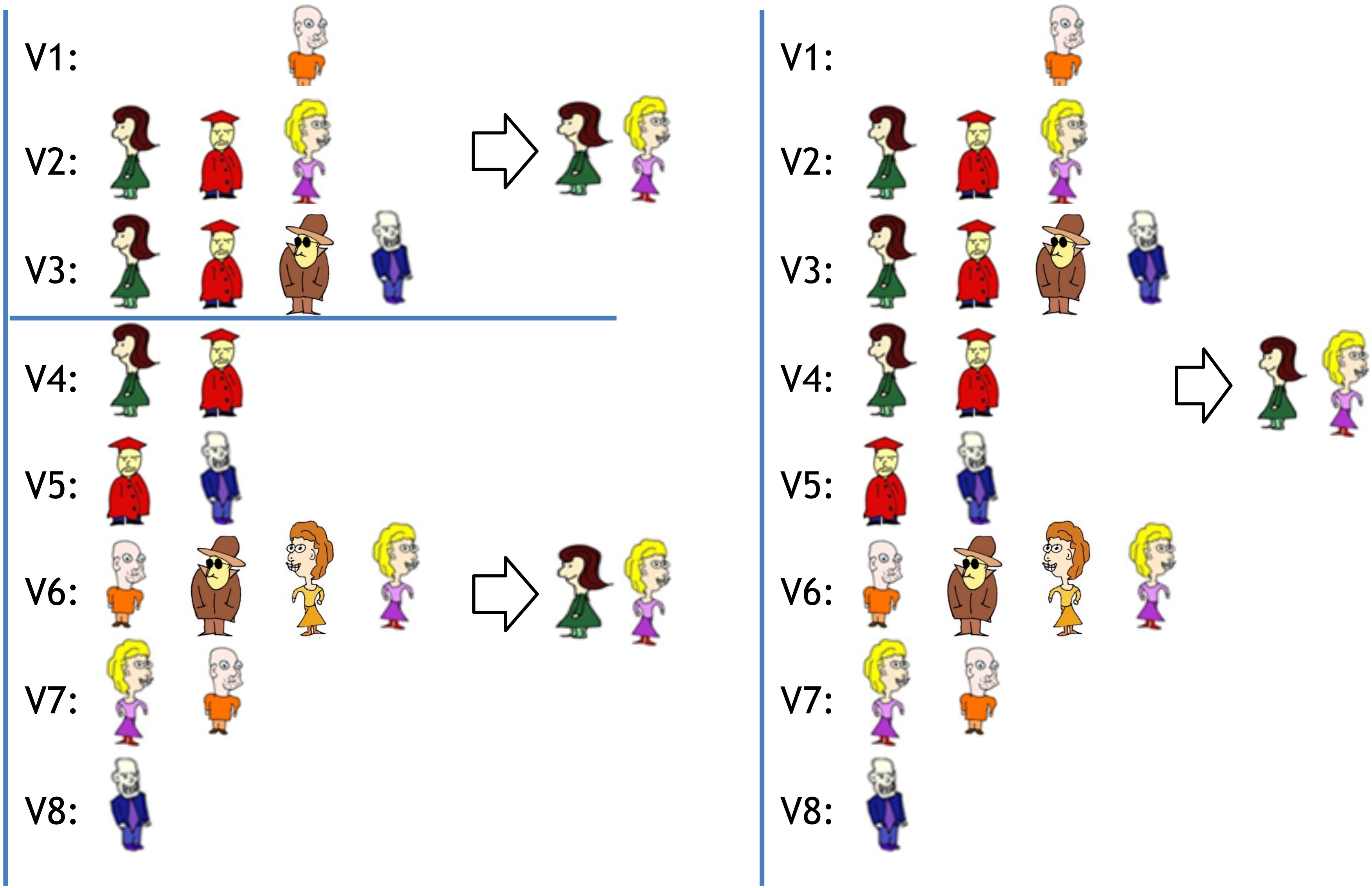

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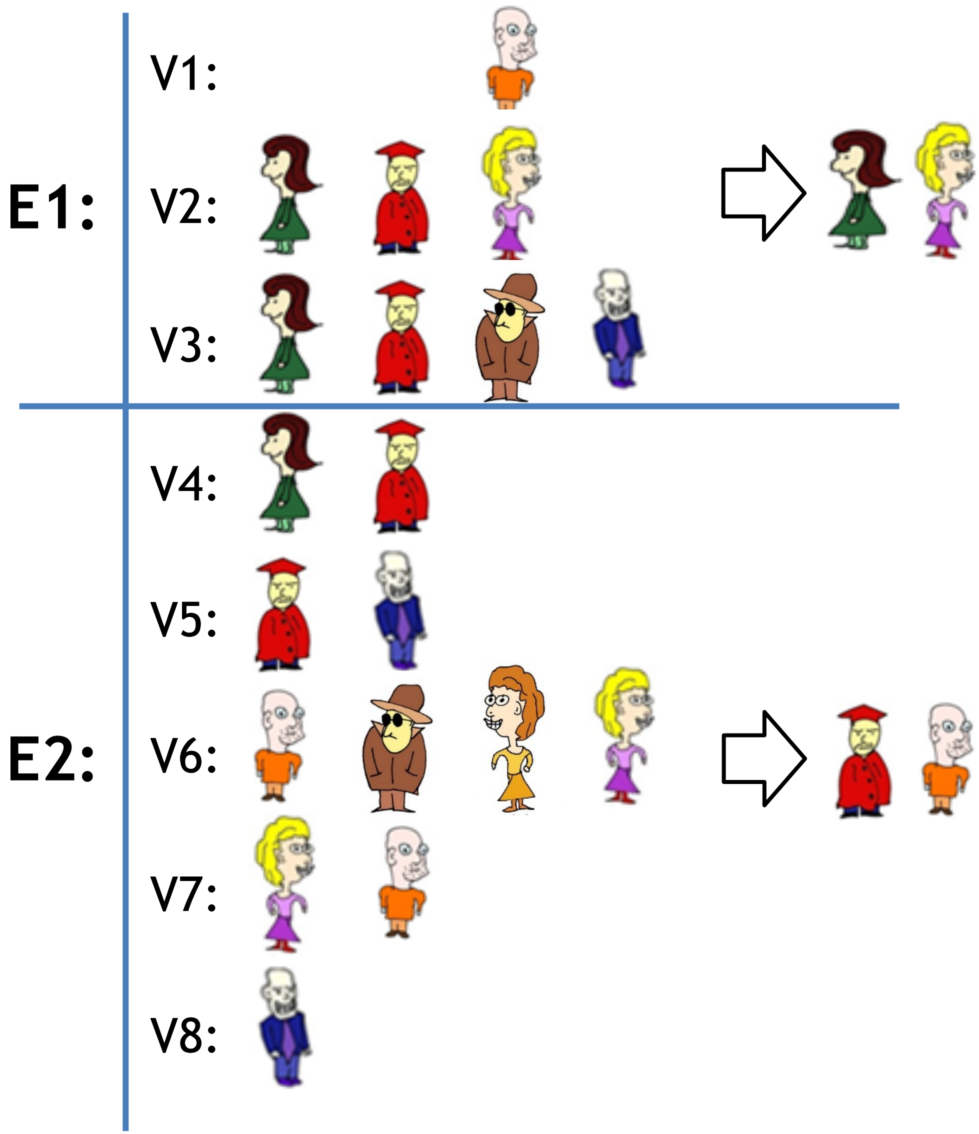

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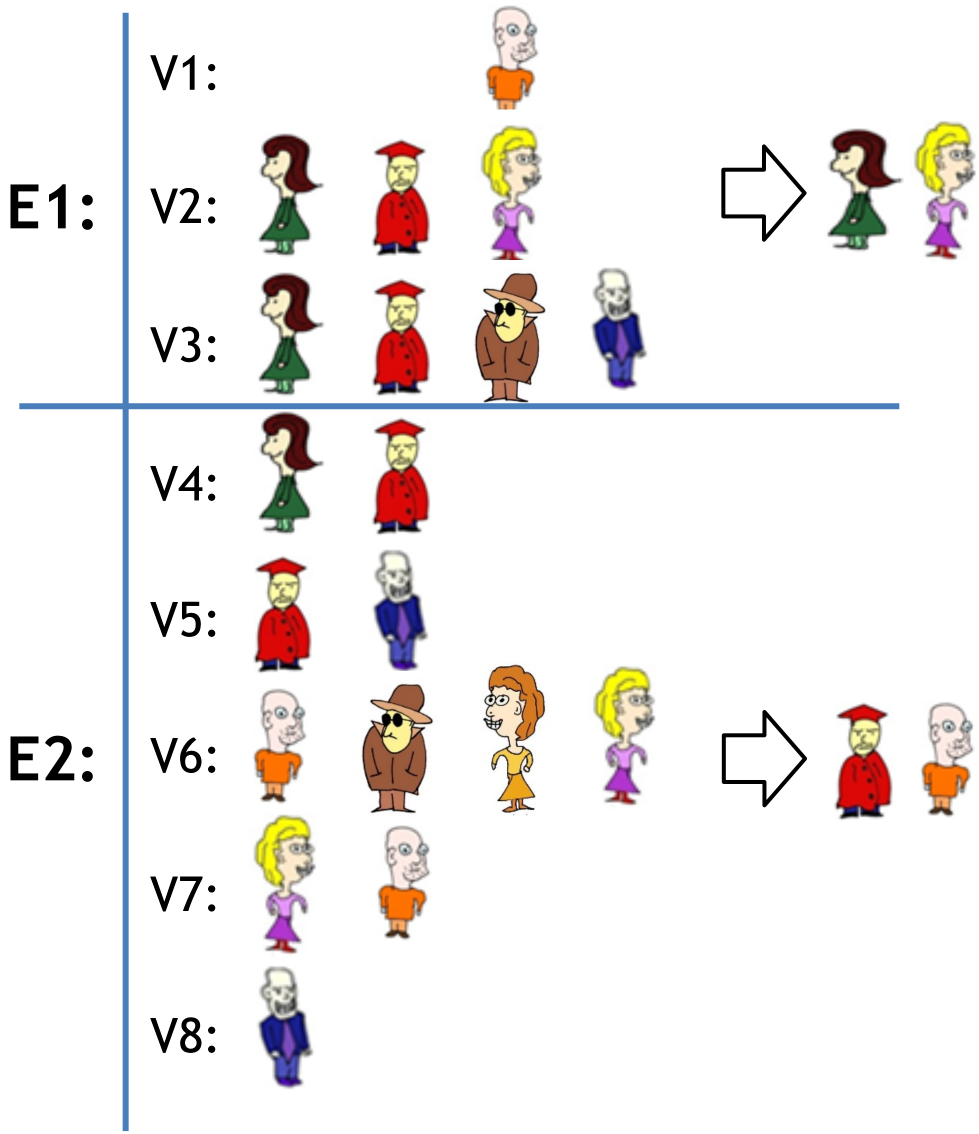
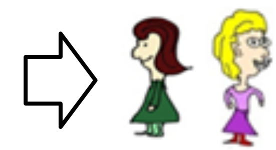


Then, there exists (possibly very large) value n such that:

n·E1 + E2:



Axiomatic Characterisations

Theorem: Proportional Approval Voting is the only ABC ranking rule that satisfies symmetry, consistency, continuity and D'Hondt proportionality.

[LS17] M. Lackner, P. Skowron, Consistent Approval-Based Multi-Winner Rules, Arxiv 2017.

Axiomatic Characterisations

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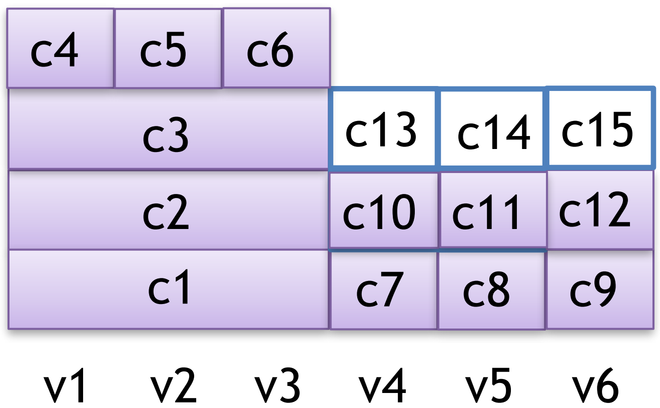
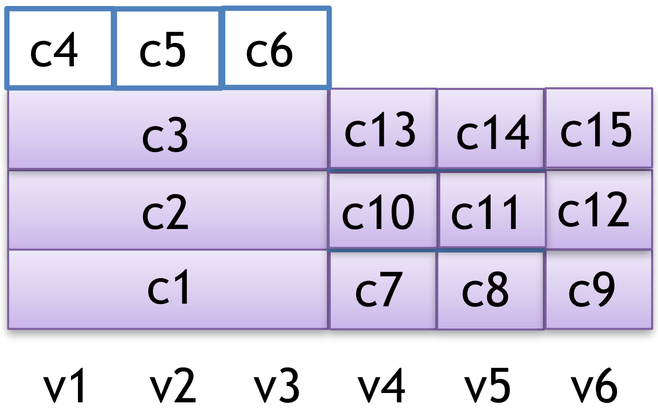
PAV versus Phragmén's Rule

PAV versus Phragmén's Rule

k = 12



Phragmén's Rule



Thiele's Rule (PAV)

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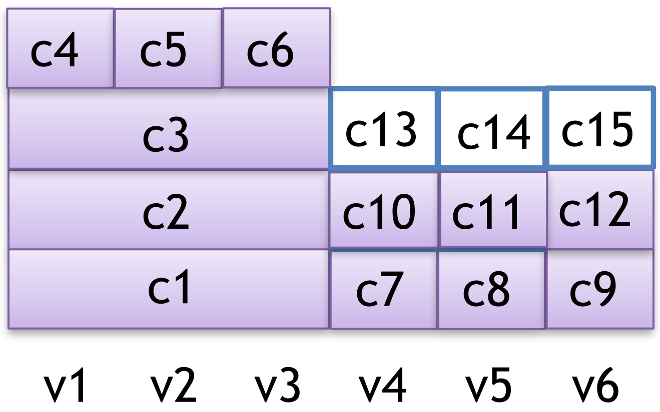

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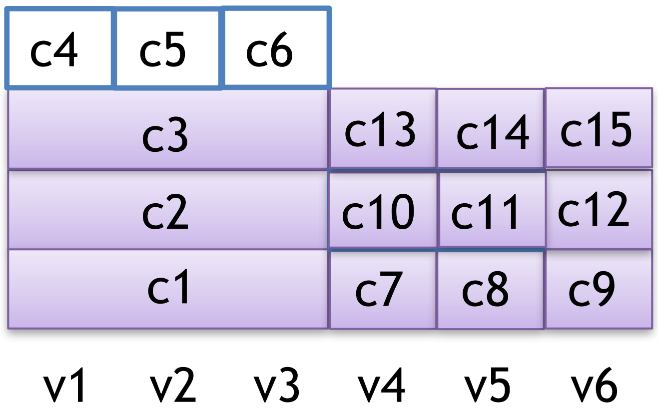

Thiele’s Rule (PAV)

Proportionality with respect to power

Proportionality with respect to welfare

PAV versus Phragmén's Rule

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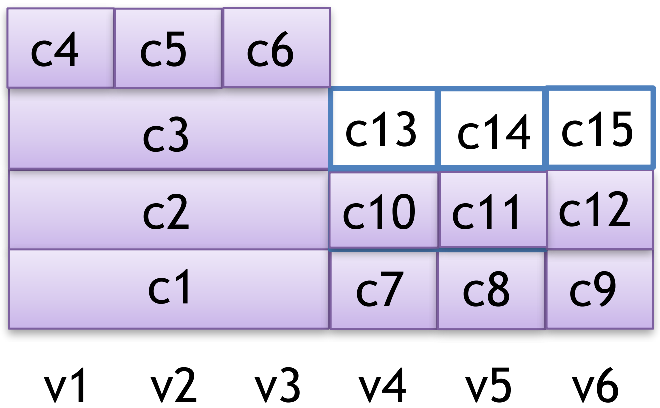

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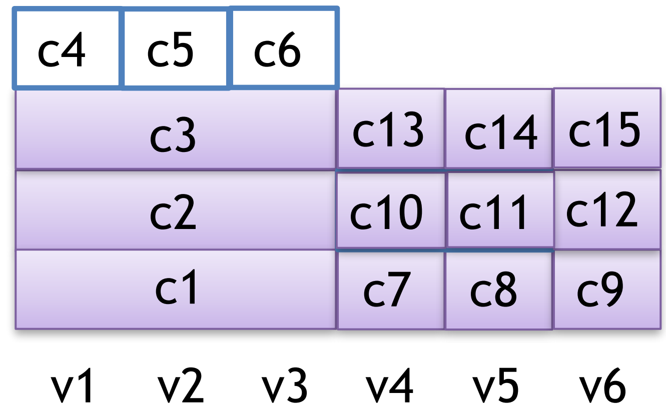

Thiele's Rule (PAV)

Proportionality with respect to power

- priceability,
- laminar proportionality

Proportionality with respect to welfare

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- EJR

PAV versus Phragmén's Rule

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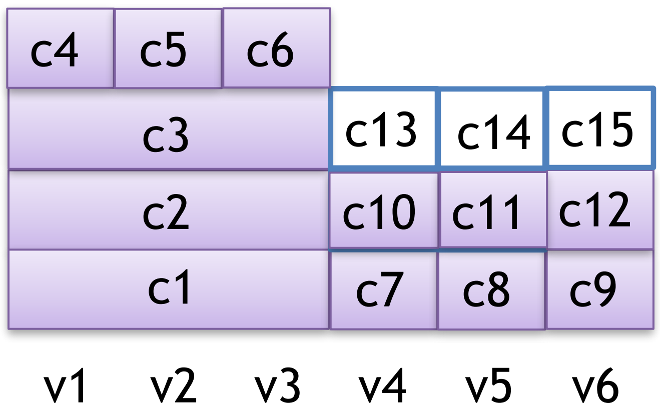

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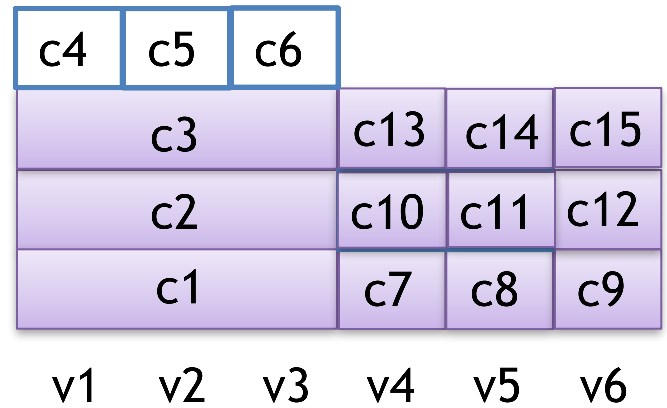

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Two New Notions of Proportionality

Fair distribution of power

(failed by PAV)

Laminar Proportionality: Examples

It describes how the rule should behave on certain well-behaved profiles

Laminar Proportionality: Examples

k = 8

c4	c8	c12
c3	c7	c11
c2	c6	c10
c1	c5	c9
v1 v2 v3	v4 v5 v6	v7 v8

Party list profiles

Laminar Proportionality: Examples

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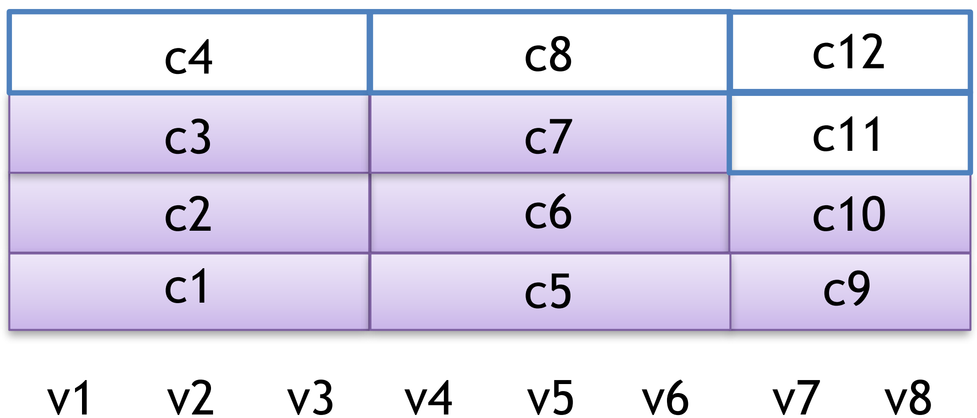

Party list profiles

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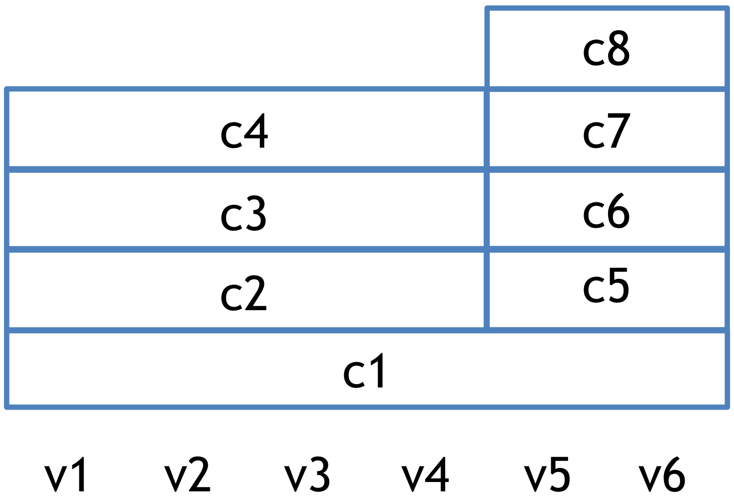

Party lists with a common leader

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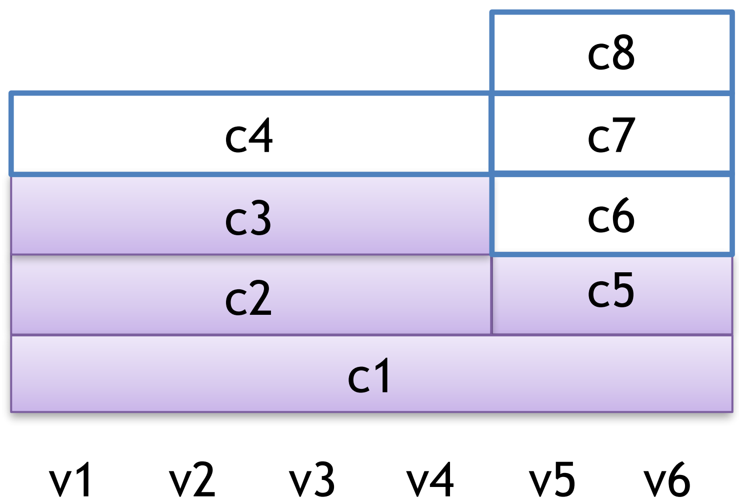

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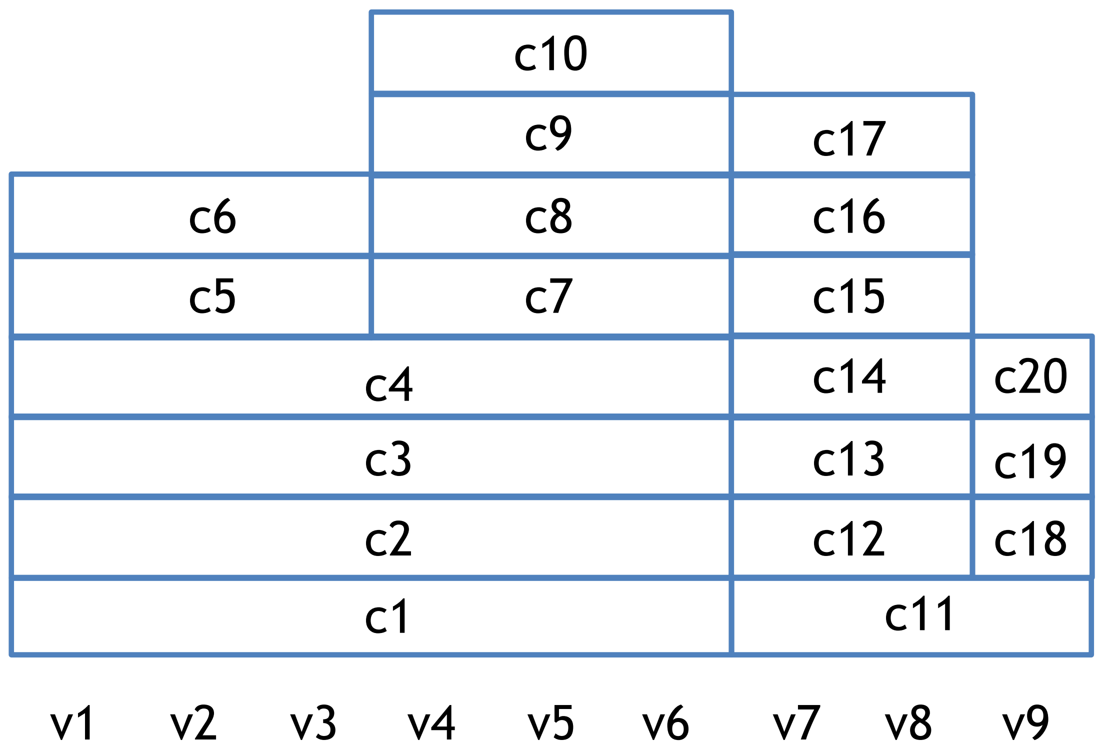

Subdivided parties

Laminar Proportionality: Examples

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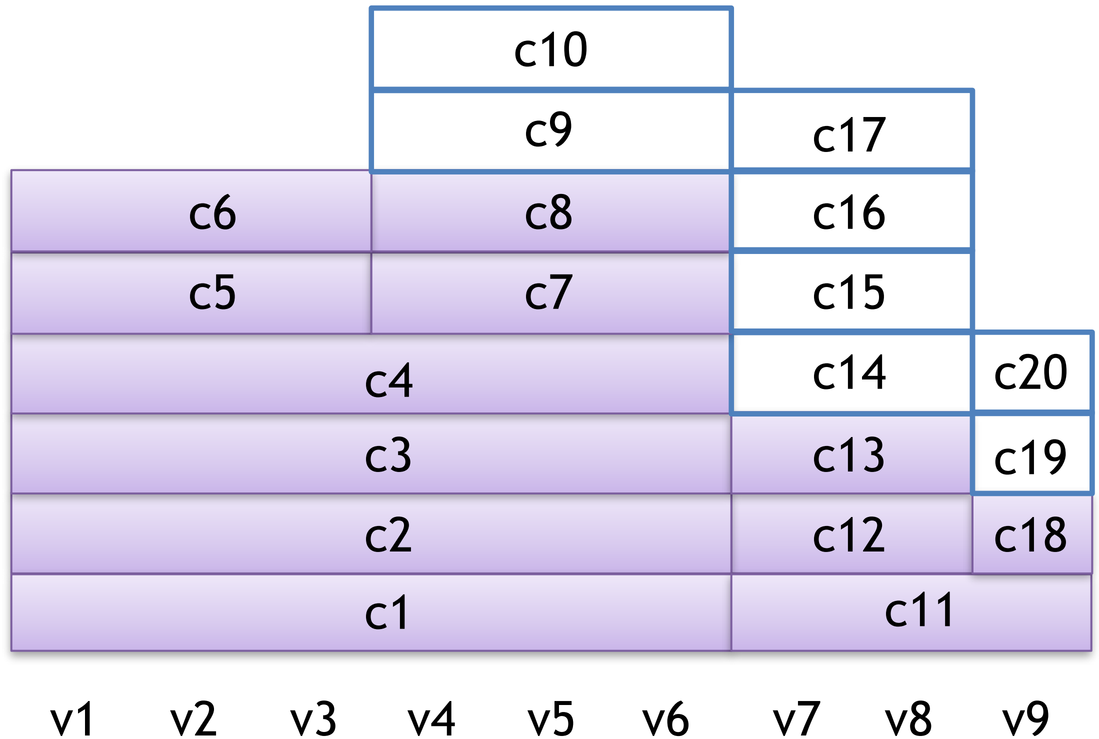

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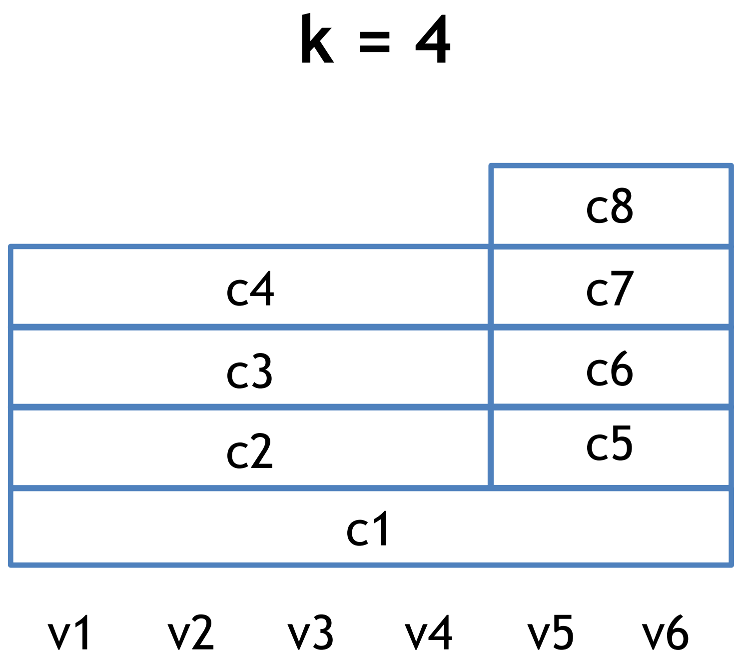

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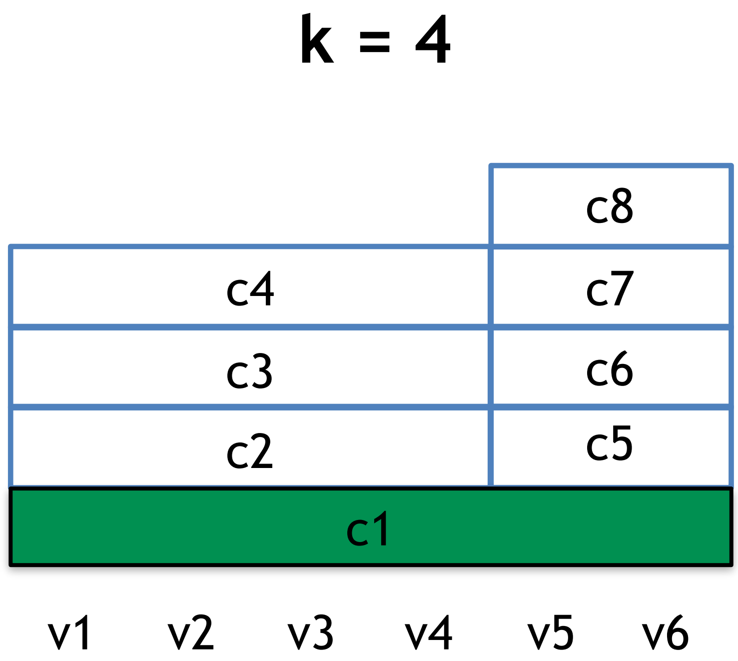

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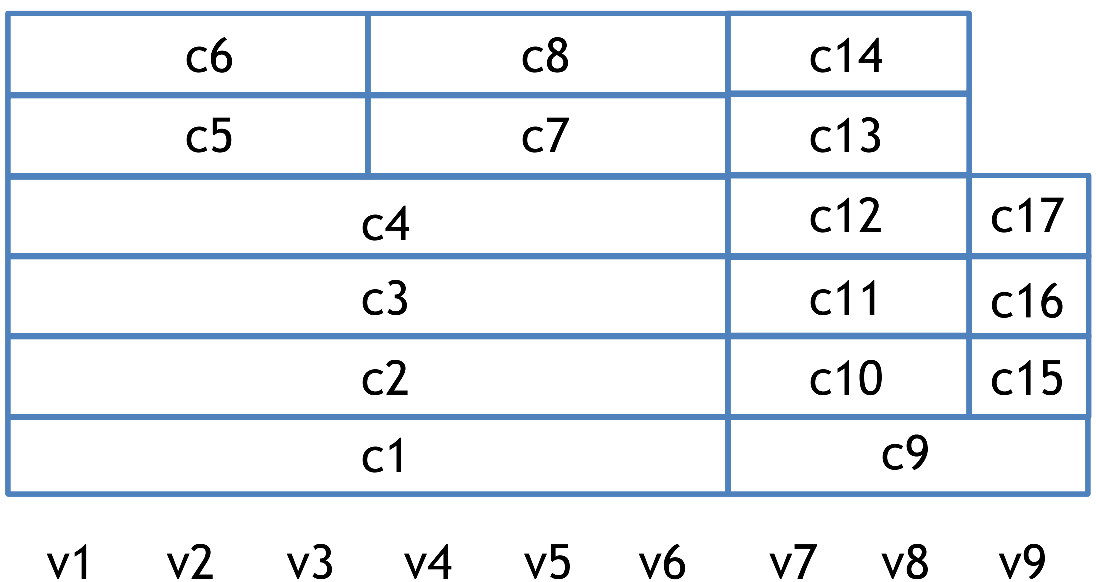

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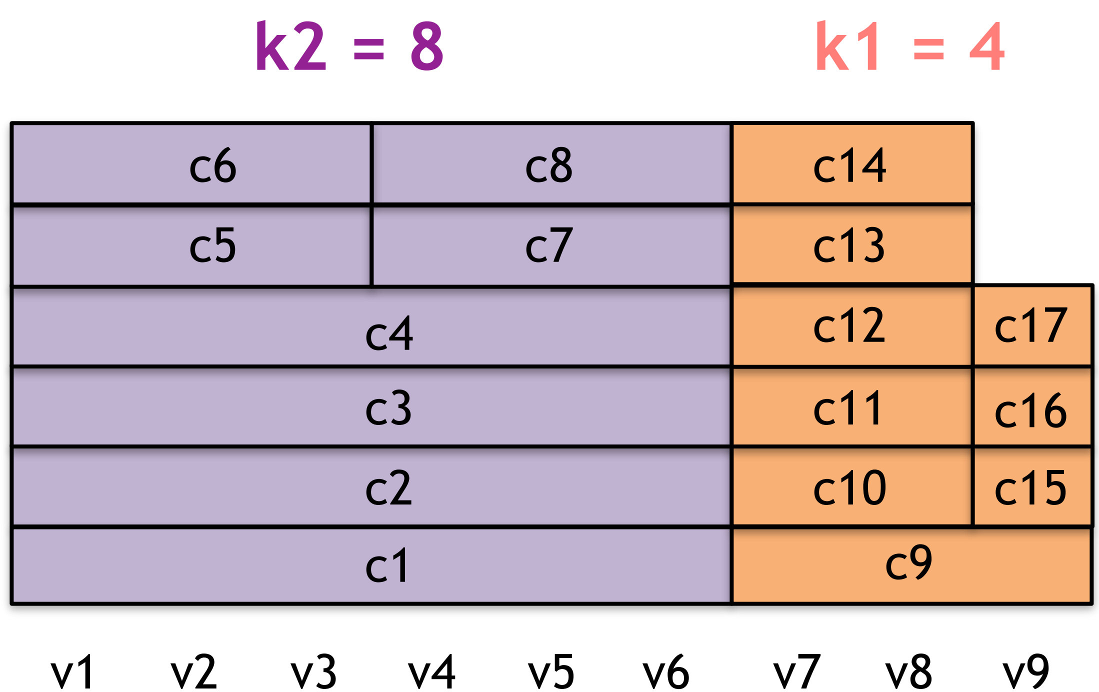

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We say that a rule is laminar proportional if it behaves well on laminar profiles.

Welfarist Rules

The welfare vector of a committee W is defined as:

$$\left(|A_1 \cap W|, |A_2 \cap W|, \ldots, |A_n \cap W| \right)$$

where:

A_i is the set of candidates approved by voter i

($|A_i \cap W|$ is the number of representatives of i)

A rule is welfarist if the decision which committee to elect can be made solely based on welfare vectors of the committees.

No welfarist rule can be laminar proportional

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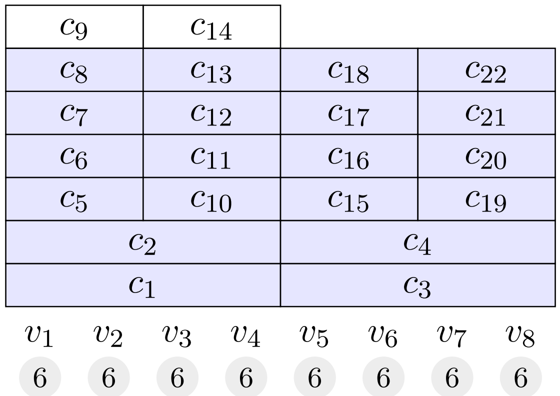

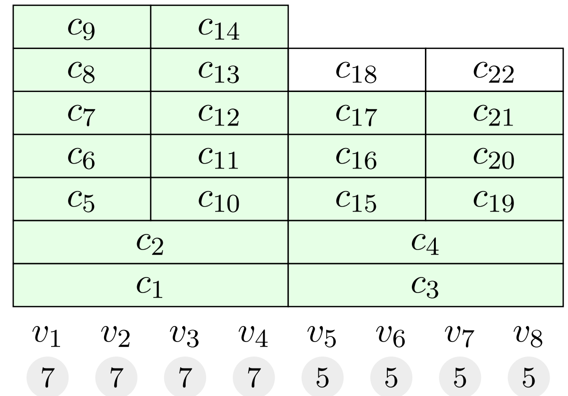

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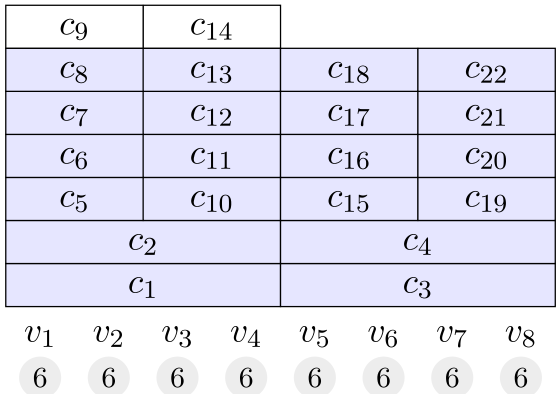

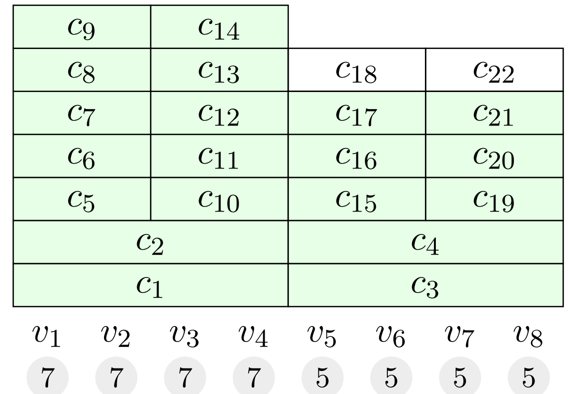

Welfare (6, 6, 6, 6, 6, 6 ,6 ,6) is preferred over welfare (7, 7, 7, 7, 5, 5, 5, 5)

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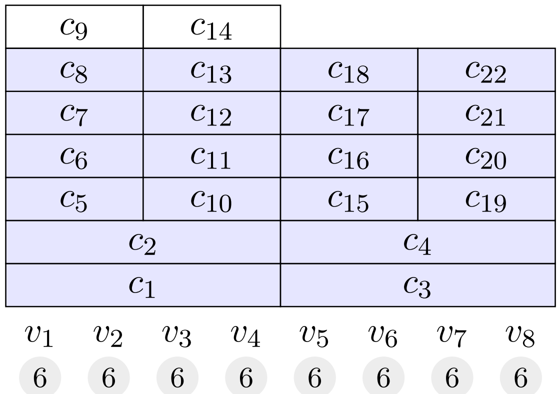

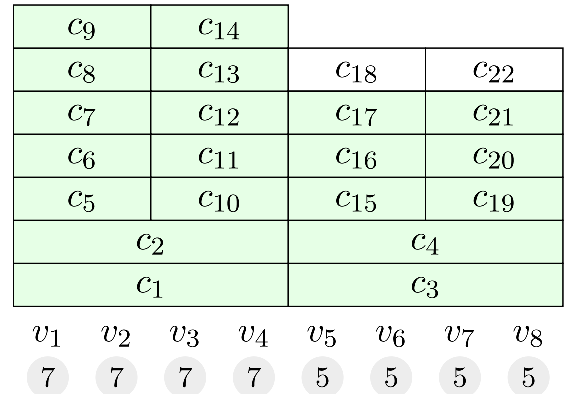

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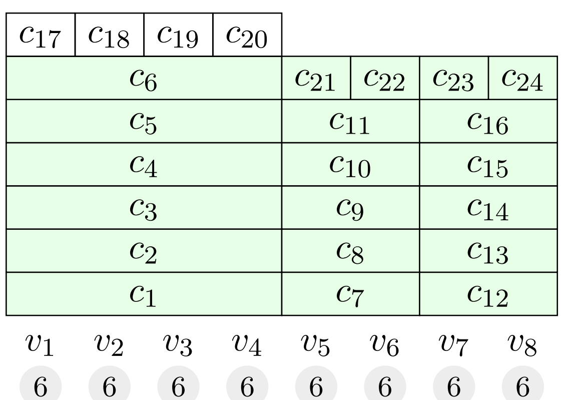

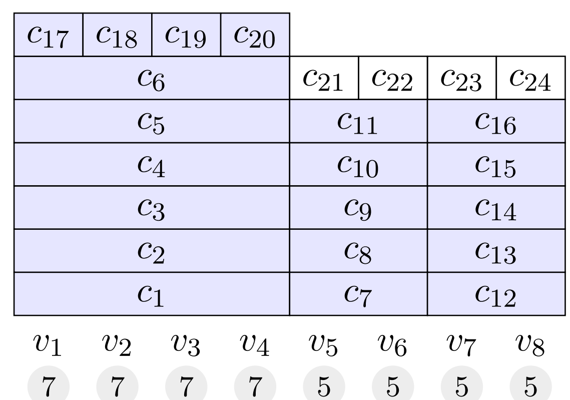

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Priceability

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A price system is a pair $ps = (p, \{p_i\}_{i \in [n]})$, where $p > 0$ is a price, and for each voter $i \in [n]$, there is a payment function $p_i : C \to [0, 1]$ such that:

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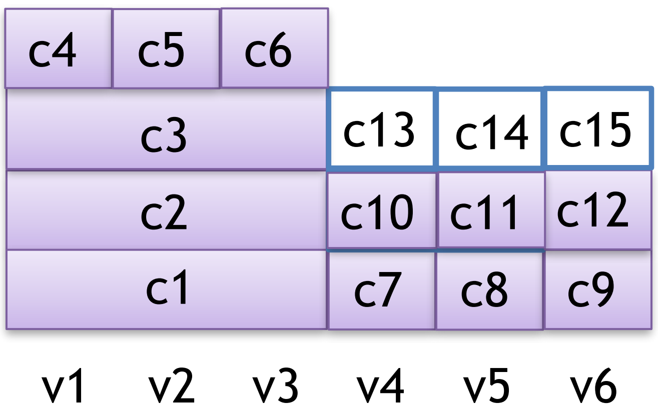
3. There exists no unelected candidate whose supporters, in total, have a remaining unspent budget of more than p

Priceability: Example

k = 12



Phragmén's Rule

The price is p = 0.5.

1. v1 pays 1/6 for c1, c2 and c3 and 1/2 for c4

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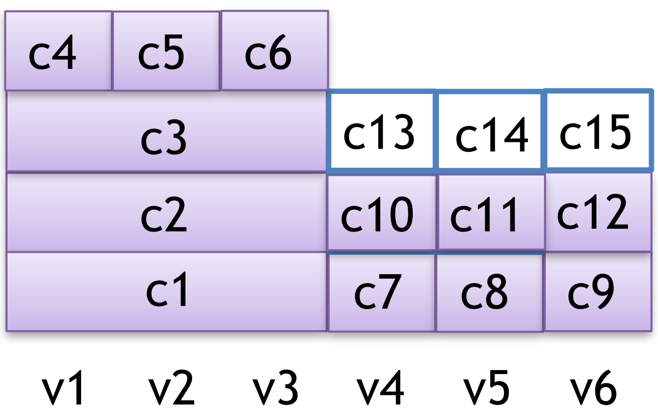

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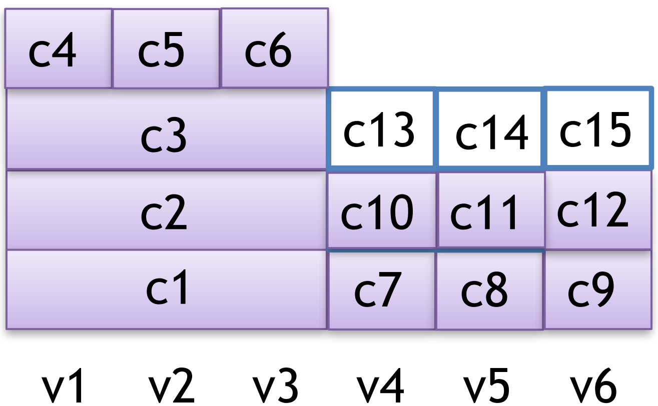

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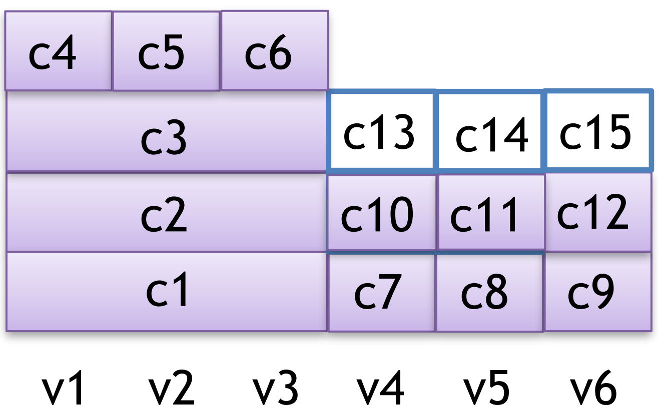

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5. v5 pays 1/2 for c8 and c11
6. V6 pays 1/2 for c9 and c12

No welfarist rule can be priceable

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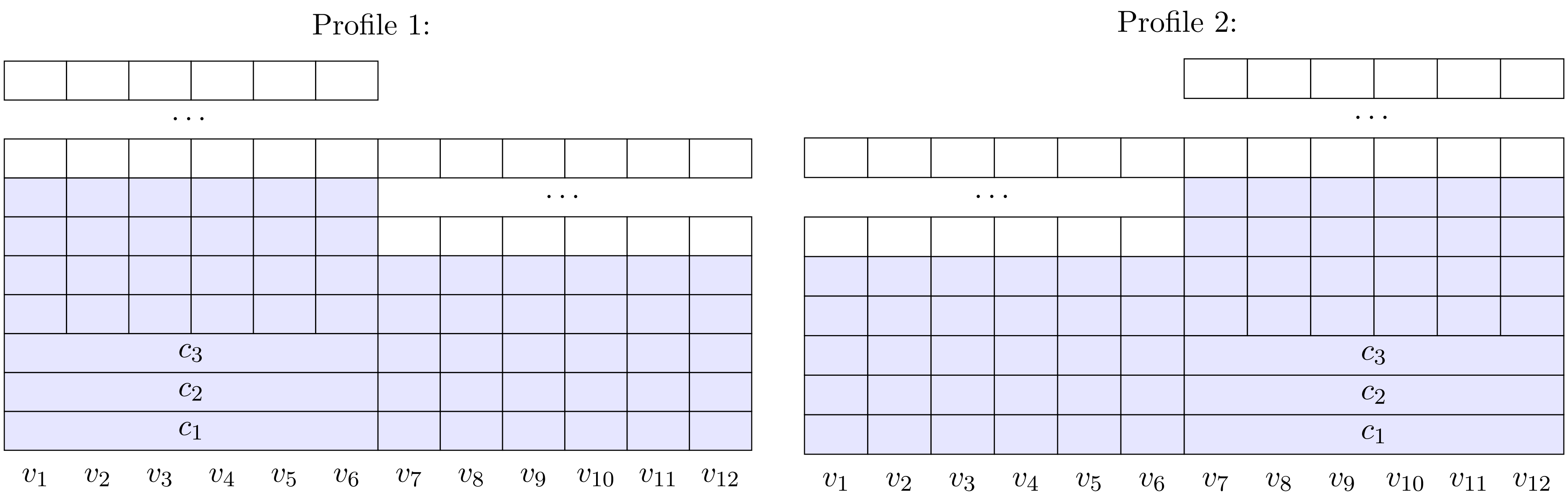

Core

Core: Definition

We say that a committee **W** is in the core if there exists no group of voters **S** and a subset of candidates **T** such that:

1. $\frac{|\mathbf{T}|}{k} \leq \frac{|\mathbf{S}|}{n}$,and

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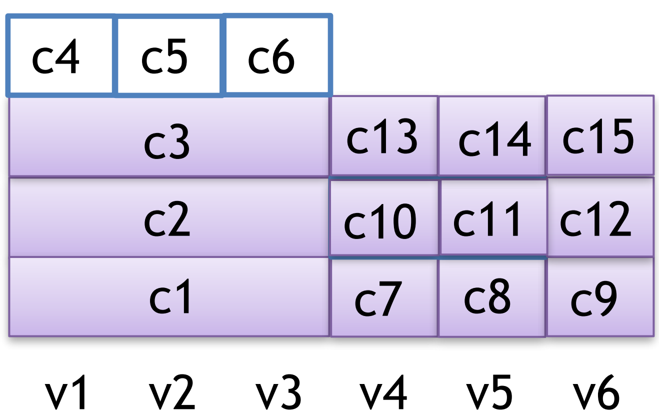

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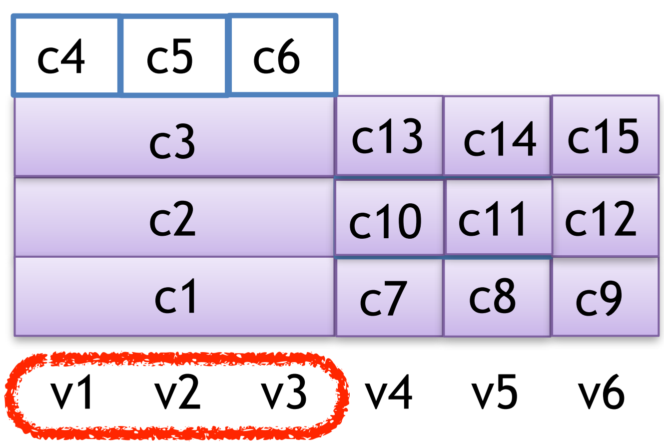

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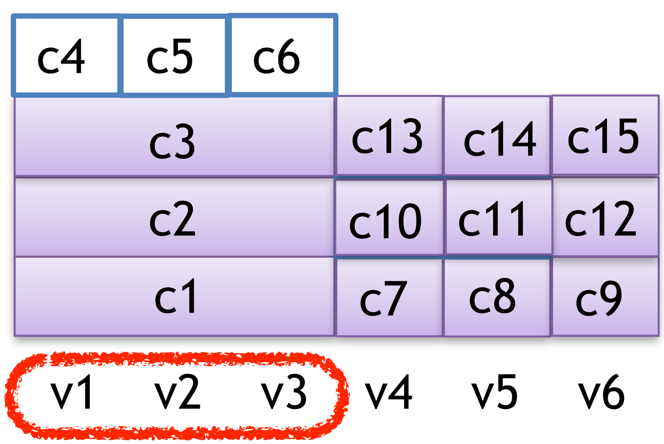

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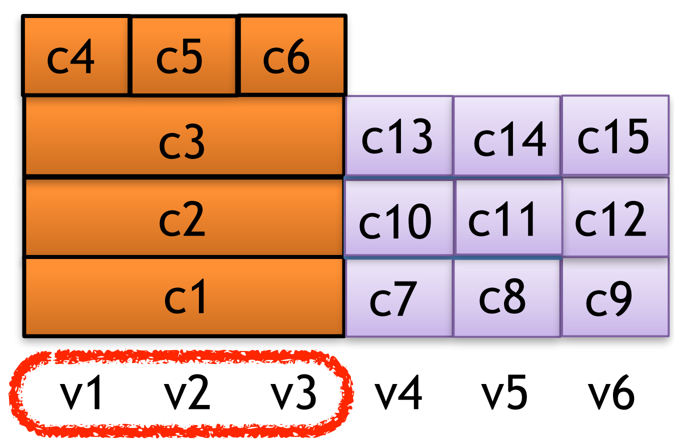

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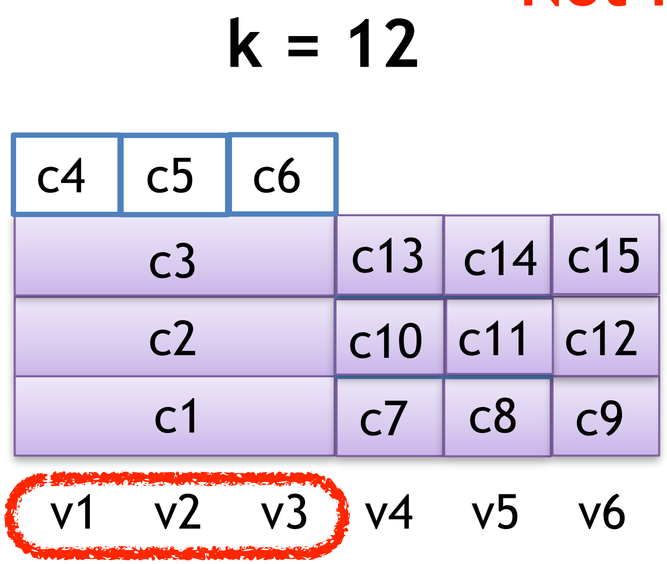
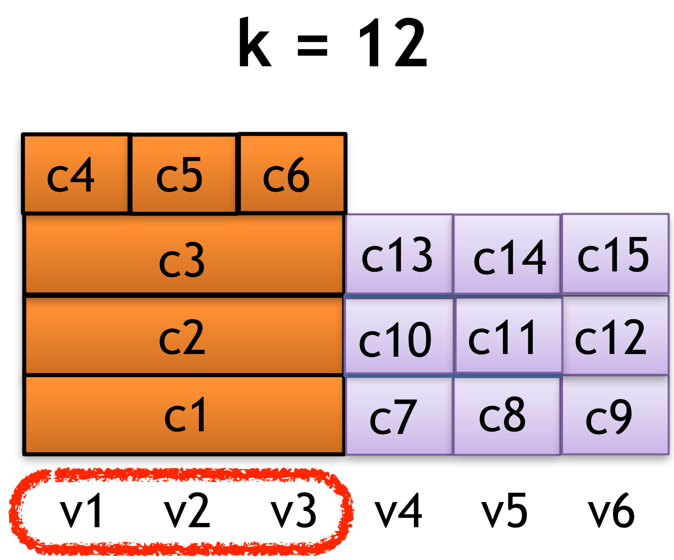
Not in the core!





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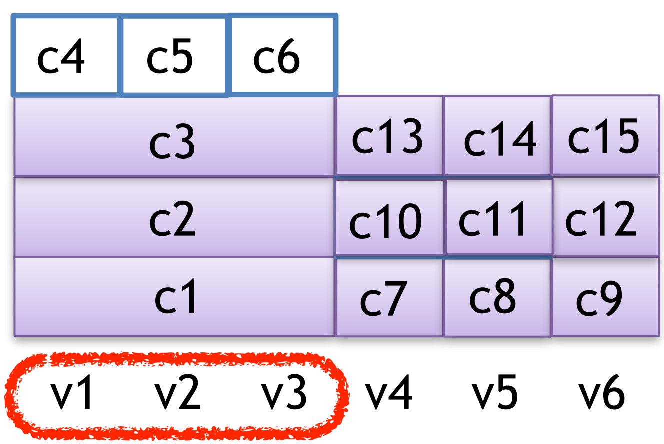

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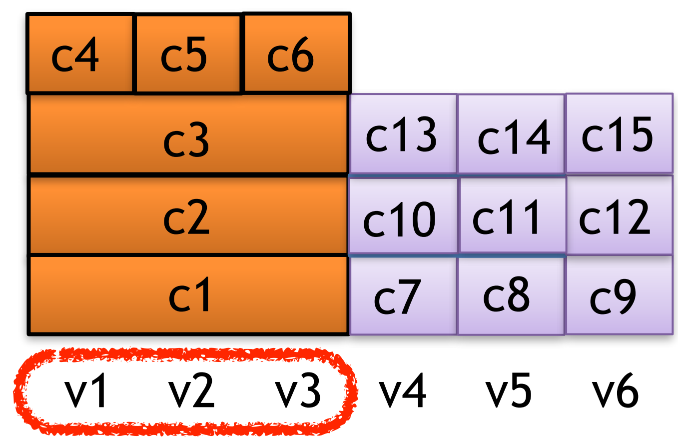

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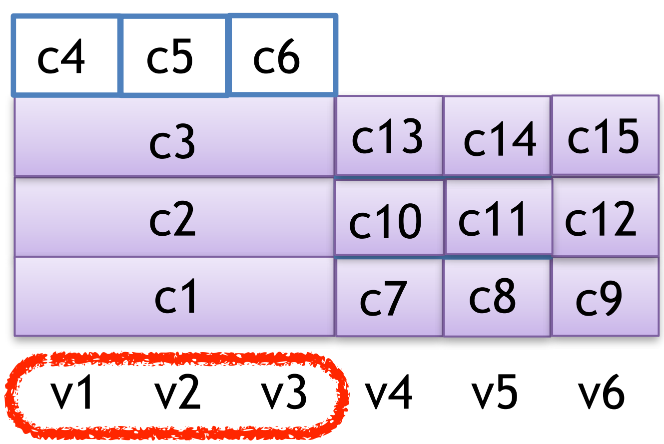

Theorem: PAV gives the best possible Approximation of the core subject to Satisfying the Pigou-Dalton principle!

No welfarist rule can satisfy the core

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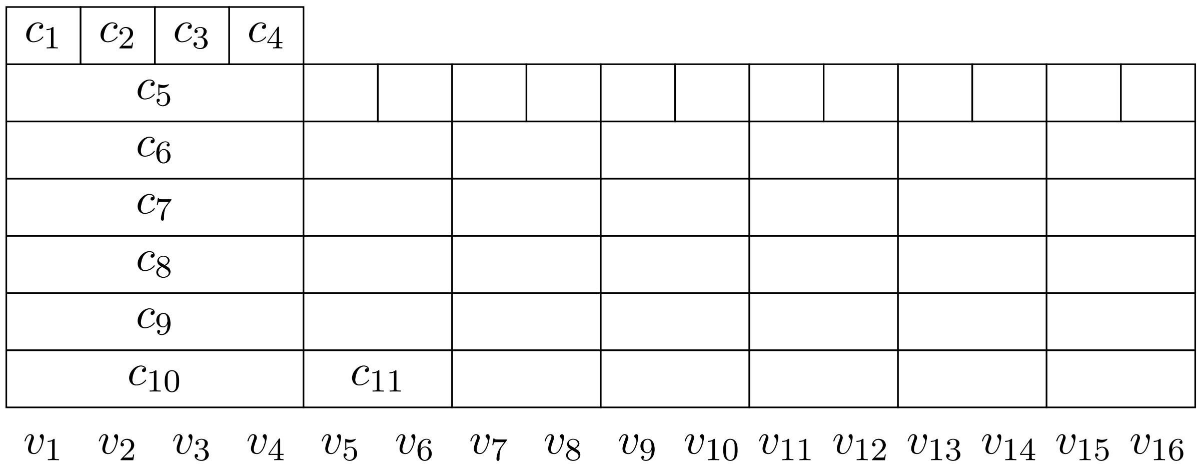
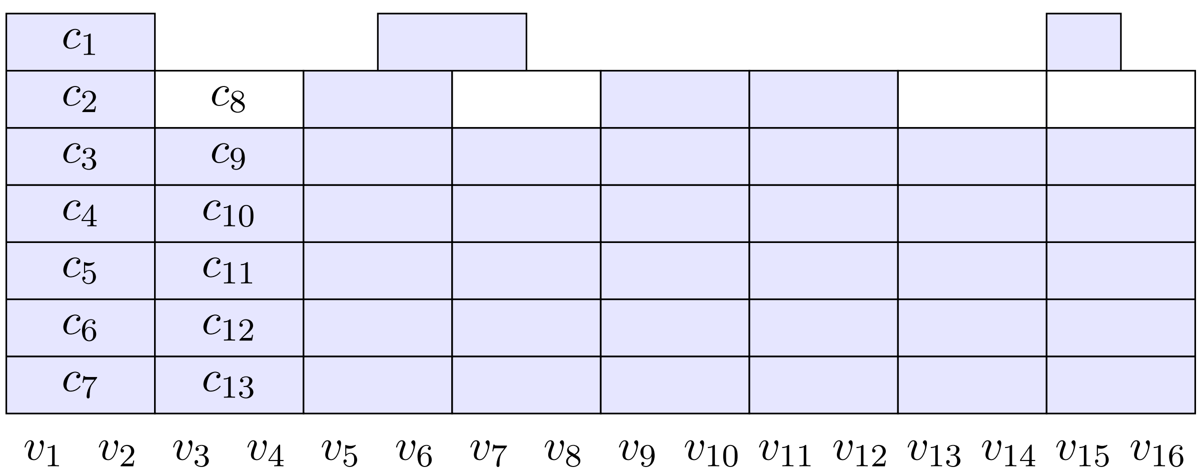
Profile 1:



Profile 2:



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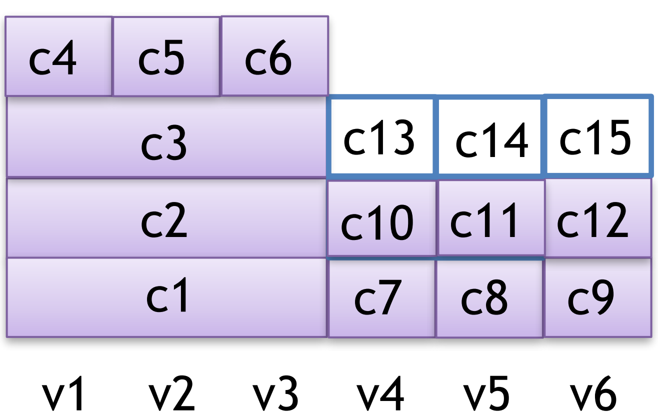
k = 12



Our new method: Rule X

- Each voter is initially given \$k.
- Buying a candidate c costs \$n in total. This cost is spread equally among the voters who approve c and who still have money.

  Formally, we find a minimal price p(c) such that if each voter who approves c pays p(c) or all the money she is left with, then c gets \$n.

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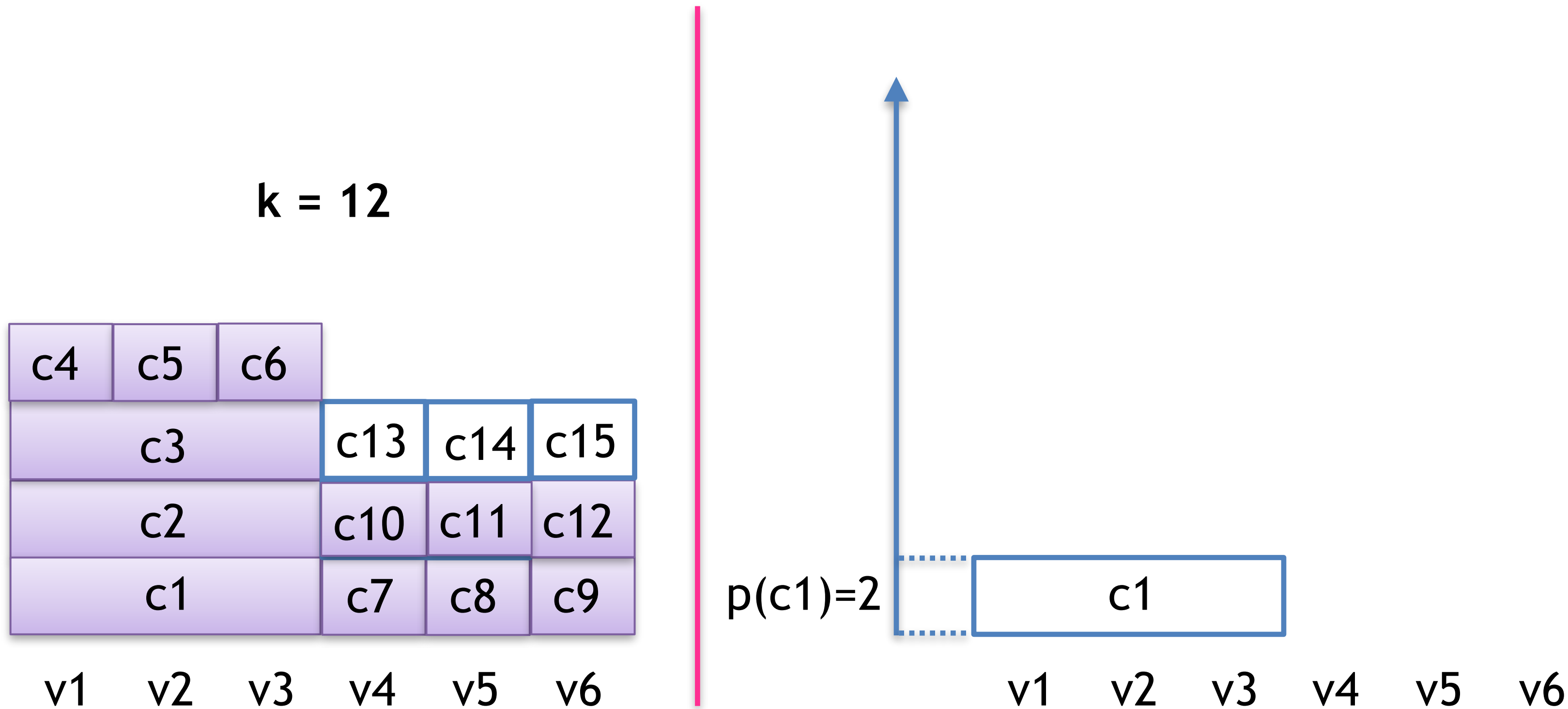

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- Each voter **is initially given $k**.
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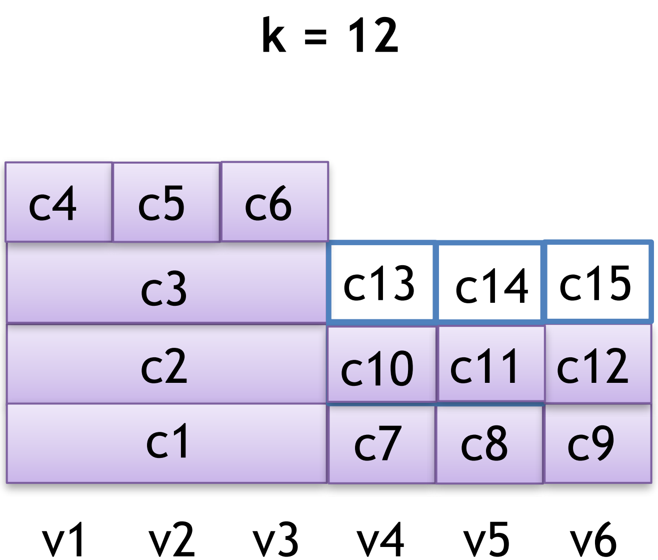

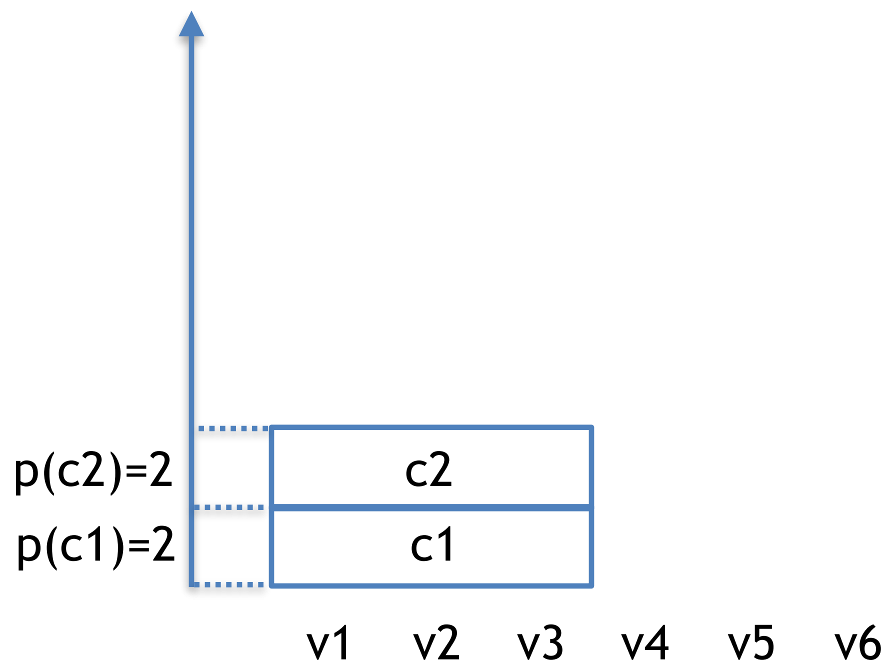

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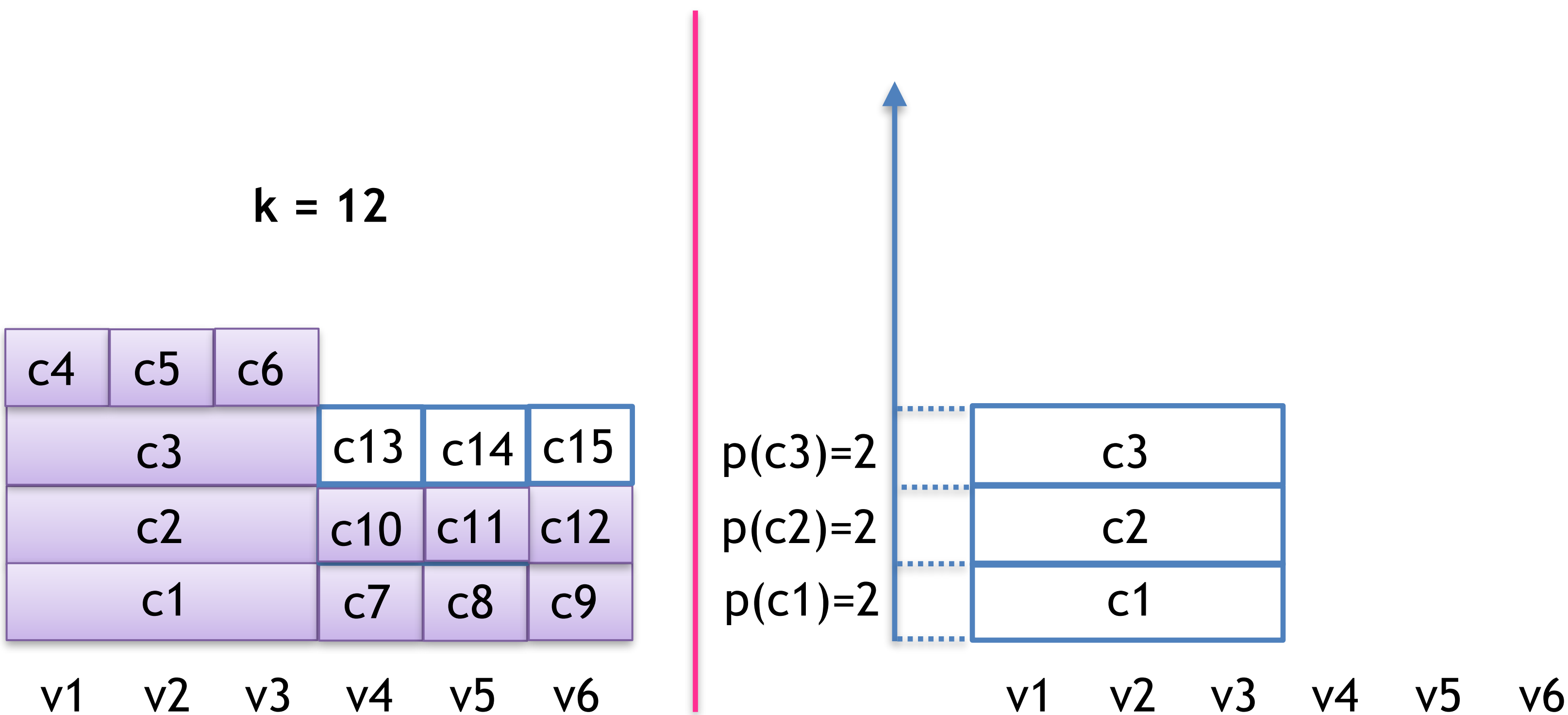

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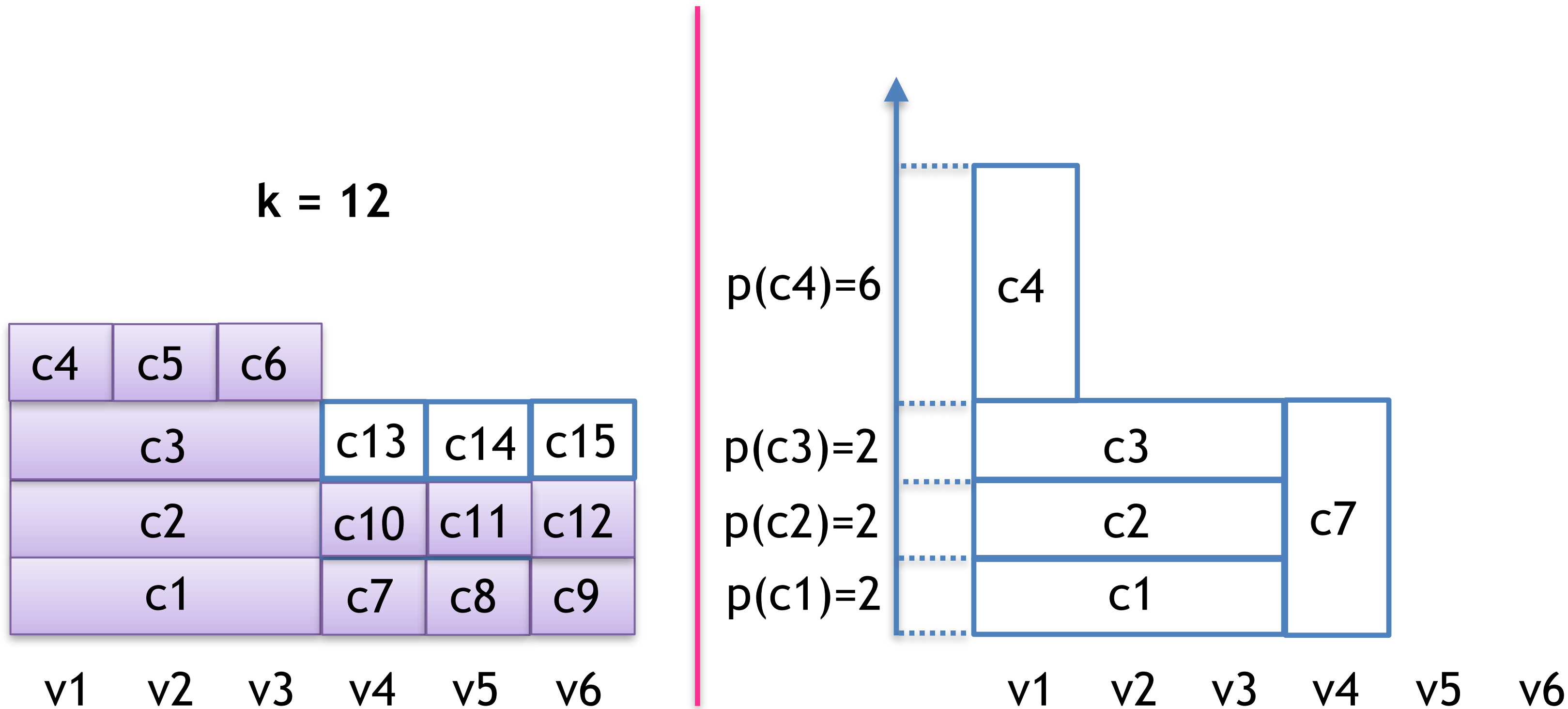

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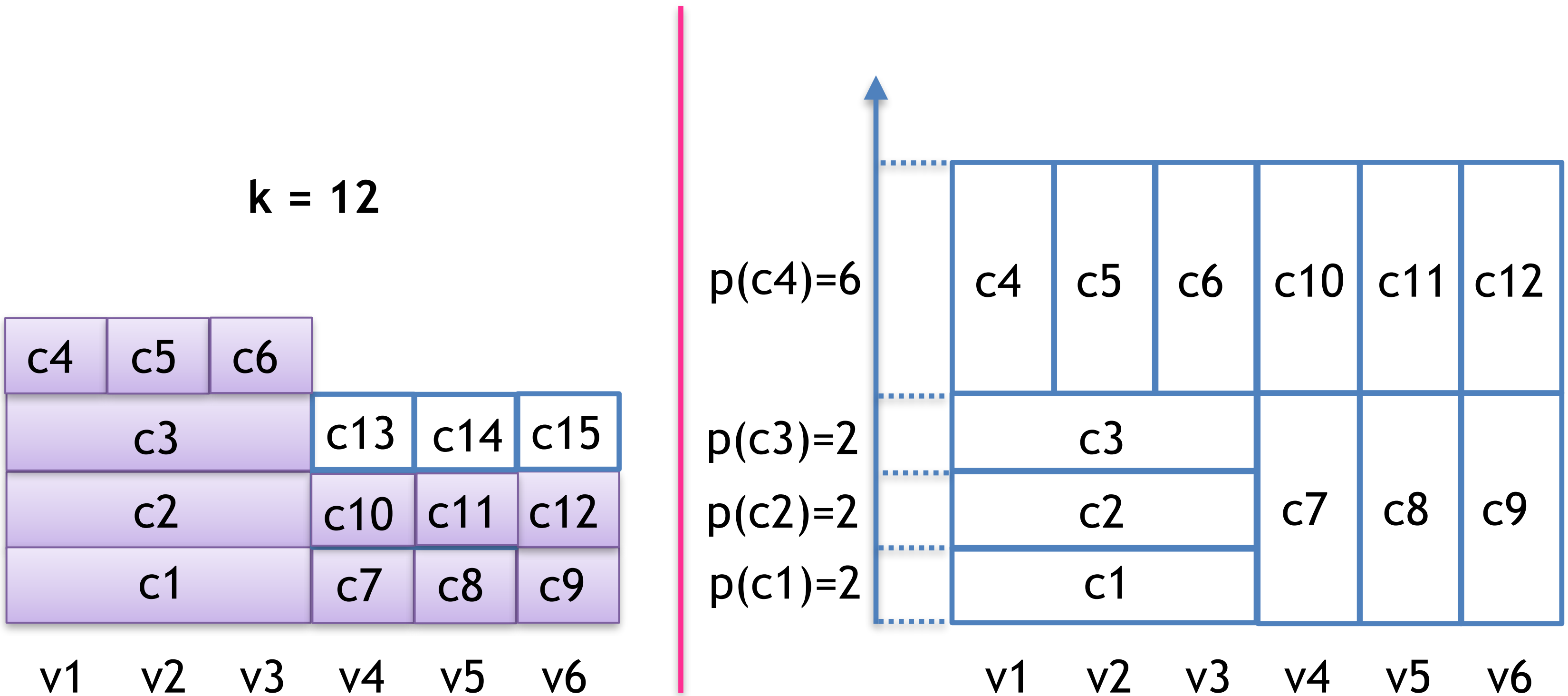

Comparison of committee rules

	Thiele's method (PAV)	Phragmén's method	Our method
laminar proportional		✓	✓
priceable		✓	✓
PJR	✓	✓	✓
EJR	✓		✓
core with constrained deviations			✓
core	2-approx.	?	$O(\log k)$-approx.
welfarist	✓		
Pareto-optimal	✓		
Pigou–Dalton	✓		
computation	NP-complete	polynomial time	polynomial time

Table 1: The rules we consider and properties that they satisfy.

Thiele versus Phragmén

Borda versus Condorcet

Open questions:

- Does there always exist a Pareto-optimal priceable committee?
- What is the best possible core-approximation among welfarist rules?