Hypergraph functor and attachment

Sergejs Solovjovs\textsuperscript{1,2}

\textsuperscript{1}Department of Mathematics, University of Latvia
e-mail: sergejs.solovjovs@lu.lv

\textsuperscript{2}Institute of Mathematics and Computer Science, University of Latvia
e-mail: sergejs.solovjovs@lumii.lv

\textbf{AAA80 Workshop on General Algebra}
in connection with the
\textbf{Workshop on Non-Classical Algebraic Structures}

St. Banach Intern. Math. Center, Bedlewo, Poland
June 1 - 6, 2010
Outline

1. Introduction
2. Variety-based attachment
3. Variety-based hypergraph functor
4. Modified variety-based hypergraph functor
5. Conclusion
1965: L. A. Zadeh introduces fuzzy set as a map \( X \xrightarrow{\alpha} [0, 1] \) from a set \( X \) to the unit interval \([0, 1]\).

1967: J. A. Goguen replaces the unit interval with a lattice \( L \).


The main idea is to fuzzify the standard membership relation \( \in \).
**Fuzzy challenge**

1965: L. A. Zadeh introduces *fuzzy set* as a map $X \xrightarrow{\alpha} [0, 1]$ from a set $X$ to the unit interval $[0, 1]$.

1967: J. A. Goguen replaces the unit interval with a lattice $L$.

**Fuzzy mathematics requires new setting for well-known notions.**


The main idea is to fuzzify the standard *membership relation* “$\in$”. 

1965: L. A. Zadeh introduces fuzzy set as a map $X \xrightarrow{\alpha} [0, 1]$ from a set $X$ to the unit interval $[0, 1]$.

1967: J. A. Goguen replaces the unit interval with a lattice $L$.

Fuzzy mathematics requires new setting for well-known notions.


The main idea is to fuzzify the standard membership relation “$\in$”.
Fuzzy challenge

1965: L. A. Zadeh introduces fuzzy set as a map $X \xrightarrow{\alpha} [0, 1]$ from a set $X$ to the unit interval $[0, 1]$.

1967: J. A. Goguen replaces the unit interval with a lattice $L$.

Fuzzy mathematics requires new setting for well-known notions.


The main idea is to fuzzify the standard membership relation “$\in$”.
Quasi-coincidence relation

Definition 1 (P.-M. Pu and Y.-M. Liu)

Given a set $X$, a fuzzy point $x_a$ (a map from $X$ to the unit interval $\mathbb{I} = [0, 1]$, which takes value $a \neq 0$ at $x$ and 0 elsewhere) is said to be quasi-coincident with a fuzzy set $\alpha$ (a map $X \xrightarrow{\alpha} \mathbb{I}$) provided that $1 - \alpha(x) < a$.

1984: Y.-M. Liu shows that the quasi-coincidence relation is the unique membership relation, which satisfies the four principles of a “reasonable” membership relation.

1997: Y.-M. Liu and M.-K. Luo employ a completely distributive lattice $L$ equipped with an order-reversing involution $(\neg)'$ to generalize the definition to $a \not\in (\alpha(x))'$. 
Quasi-coincidence relation

Definition 1 (P.-M. Pu and Y.-M. Liu)

Given a set $X$, a fuzzy point $x_a$ (a map from $X$ to the unit interval $I = [0, 1]$, which takes value $a \neq 0$ at $x$ and 0 elsewhere) is said to be quasi-coincident with a fuzzy set $\alpha$ (a map $X \rightarrow I$) provided that $1 - \alpha(x) < a$.

1984: Y.-M. Liu shows that the quasi-coincidence relation is the unique membership relation, which satisfies the four principles of a “reasonable” membership relation.

1997: Y.-M. Liu and M.-K. Luo employ a completely distributive lattice $L$ equipped with an order-reversing involution $(-)'$ to generalize the definition to $a \not\leq (\alpha(x))'$.
Quasi-coincidence relation

**Definition 1 (P.-M. Pu and Y.-M. Liu)**

Given a set $X$, a fuzzy point $x_a$ (a map from $X$ to the unit interval $I = [0, 1]$, which takes value $a \neq 0$ at $x$ and 0 elsewhere) is said to be quasi-coincident with a fuzzy set $\alpha$ (a map $X \xrightarrow{\alpha} I$) provided that $1 - \alpha(x) < a$.

**1984:** Y.-M. Liu shows that the quasi-coincidence relation is the unique membership relation, which satisfies the four principles of a “reasonable” membership relation.

**1997:** Y.-M. Liu and M.-K. Luo employ a completely distributive lattice $L$ equipped with an order-reversing involution $(\cdot)'$ to generalize the definition to $a \not\leqslant (\alpha(x))'$.
2008: C. Guido removes the requirement on the existence of an involution and introduces a lattice-valued analogue of the desired relation under the name of attachment.

**Definition 2 (C. Guido)**

Let $L$ be a complete lattice.

- An attachment $A$ on $L$ is a family $(F_a)_{a \in L \setminus \{\bot\}}$ of completely prime filters of $L$, with an additional stipulation $F_\bot = \emptyset$.
- An $L$-point $x_a$ is said to be attached to an $L$-set $\alpha$ (denoted $x_a A \alpha$) provided that $\alpha(x) \in F_a$. 
2008: C. Guido removes the requirement on the existence of an involution and introduces a lattice-valued analogue of the desired relation under the name of attachment.

**Definition 2 (C. Guido)**

Let $L$ be a complete lattice.

- An attachment $\mathcal{A}$ on $L$ is a family $(F_a)_{a \in L \setminus \{\bot\}}$ of completely prime filters of $L$, with an additional stipulation $F_{\bot} = \emptyset$.
- An $L$-point $x_a$ is said to be attached to an $L$-set $\alpha$ (denoted $x_a \in_A \alpha$) provided that $\alpha(x) \in F_a$. 

Functor induced by attachment

Attachment of C. Guido

1. extends quasi-coincidence relation of P.-M. Pu and Y.-M. Liu (take $L = \mathbb{I}$ and let $F_a = \{b \mid 1 - a < b\}$ for every $a \in L$);

2. gives a functor $L\text{-Top} \xrightarrow{(-)^*} \text{Top}$, which takes an $L$-topological space $(X, \tau)$ to the space $(S_X, \tau^*)$, where $S_X$ is the set of all $L$-points of $X$ and $\tau^* = \{\alpha^* \mid \alpha \in \tau\}$ with $\alpha^* = \{x_a \mid x_a \in A \alpha\}$.

The functor $L\text{-Top} \xrightarrow{(-)^*} \text{Top}$ provides a common framework for some definitions of hypergraph functor of the fuzzy community.
Attachment on complete lattices

Functor induced by attachment

Attachment of C. Guido

1. extends quasi-coincidence relation of P.-M. Pu and Y.-M. Liu (take $L = \mathbb{I}$ and let $F_a = \{ b \mid 1 - a < b \}$ for every $a \in L$);

2. gives a functor $L\text{-}\mathbf{Top} \xrightarrow{(-)^*} \mathbf{Top}$, which takes an $L$-topological space $(X, \tau)$ to the space $(S_X, \tau^*)$, where $S_X$ is the set of all $L$-points of $X$ and $\tau^* = \{ \alpha^* \mid \alpha \in \tau \}$ with $\alpha^* = \{ x_a \mid x_a \in A \alpha \}$.

The functor $L\text{-}\mathbf{Top} \xrightarrow{(-)^*} \mathbf{Top}$ provides a common framework for some definitions of {hypergraph functor} of the fuzzy community.
Motivating idea

**Fact.** Given a frame $A$, there exists a one-to-one correspondence between completely prime filters of $A$ and points of $A$, which are frame homomorphisms from $A$ to $2 = \{\bot, \top\}$.

**Consequence.** An attachment $F$ can be represented (omitting the condition $F_\bot = \emptyset$) as a map $A \xrightarrow{F} \text{Frm}(A, 2)$.

**Advantage.** Attachment can be defined in any variety of algebras.
Motivating idea

**Fact.** Given a frame $A$, there exists a one-to-one correspondence between completely prime filters of $A$ and points of $A$, which are frame homomorphisms from $A$ to $2 = \{\bot, \top\}$.

**Consequence.** An attachment $F$ can be represented (omitting the condition $F_\bot = \emptyset$) as a map $A \xrightarrow{F} \text{Frm}(A, 2)$.

**Advantage.** Attachment can be defined in any variety of algebras.
### Fact
Given a frame $A$, there exists a one-to-one correspondence between completely prime filters of $A$ and points of $A$, which are frame homomorphisms from $A$ to $2 = \{\bot, \top\}$.

### Consequence
An attachment $F$ can be represented (omitting the condition $F_\bot = \emptyset$) as a map $A \overset{F}{\to} \text{Frm}(A, 2)$.

### Advantage
Attachment can be defined in any variety of algebras.
**Definition 3**

Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a (possibly proper) class of cardinal numbers.

- An **$\Omega$-algebra** is a pair $(A, (\omega^A_\lambda)_{\lambda \in \Lambda})$ comprising a set $A$ and a family of maps $A^{n_\lambda} \xrightarrow{\omega^A_\lambda} A$ ($n_\lambda$-ary primitive operations on $A$).

- An **$\Omega$-homomorphism** $(A, (\omega^A_\lambda)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega^B_\lambda)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that $f \circ \omega^A_\lambda = \omega^B_\lambda \circ f^{n_\lambda}$ for every $\lambda \in \Lambda$.

- **$\text{Alg}(\Omega)$** is the construct of $\Omega$-algebras and $\Omega$-homomorphisms, with the underlying functor denoted by $| - |$. 
Varieties of algebras

Definition 4

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of $\Omega$-homomorphisms with injective (resp. surjective) underlying maps.

- A **variety of $\Omega$-algebras** is a full subcategory of $\text{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images).
- The objects (resp. morphisms) of a variety are called **algebras** (resp. **homomorphisms**).
- The categorical dual of a given variety $\mathcal{A}$ is denoted by $\text{LoA}$, whose objects (resp. morphisms) are called **localic algebras** (resp. **homomorphisms**).
Category of attachments

Definition 5

Let $\mathbf{A}$ be a variety of algebras and let $\mathbf{A} \xrightarrow{(-)^*} \mathbf{Set}^{op}$ be a functor, which takes an algebra $A$ to its underlying set $|A|$.

- An (A-)attachment is a triple $F = (\Omega F, \Sigma F, \models)$, where $\Omega F$ and $\Sigma F$ are algebras, and $\Omega F \xrightarrow{\models} \mathbf{A}(\Omega F, \Sigma F)$ is a map.

- An attachment morphism $F_1 \xrightarrow{f} F_2$ is a pair of homomorphisms $(\Omega F_1, \Sigma F_1) \xrightarrow{(\Omega f, \Sigma f)} (\Omega F_2, \Sigma F_2)$ such that for every $a_1 \in \Omega F_1$, $a_2 \in \Omega F_2$, $(\models F_2(a_2))(\Omega f(a_1)) = (\Sigma f \circ \models F_1((\Omega f)^{op}(a_2)))(a_1)$.

- $\textbf{AttA}$ is the category of attachments and their morphisms. $\mathcal{|-|}$ is the underlying functor to the ground category $\mathbf{A} \times \mathbf{A}$. 
Powerset operators

Every set map $X \xrightarrow{f} Y$ extends to the operators:

- $\mathcal{P}(X) \xrightarrow{f^\rightarrow} \mathcal{P}(Y)$, $f^\rightarrow(S) = \{f(x) \mid x \in S\}$;
- $\mathcal{P}(Y) \xleftarrow{f^\leftarrow} \mathcal{P}(X)$, $f^\leftarrow(T) = \{x \mid f(x) \in T\}$.

The latter operator admits a more general setting.

**Proposition 6**

*Given a variety $\mathbf{A}$, every subcategory $\mathbf{C}$ of $\mathbf{LoA}$ induces a functor $\mathbf{Set} \times \mathbf{C} \xleftarrow{(-)^\leftarrow} \mathbf{LoA}$, $((X_1, A_1) \xrightarrow{(f, \varphi)} (X_2, A_2))^{\leftarrow} = A_1^{X_1} \xrightarrow{(f, \varphi)^{\leftarrow\text{op}}} A_2^{X_2}$ with $(f, \varphi)^{\leftarrow}(\alpha) = \varphi^{\text{op}} \circ \alpha \circ f$.***
Every set map \( X \xrightarrow{f} Y \) extends to the operators:

- \( \mathcal{P}(X) \xrightarrow{f^\rightarrow} \mathcal{P}(Y) \), \( f^\rightarrow(S) = \{ f(x) \mid x \in S \} \);
- \( \mathcal{P}(Y) \xrightarrow{f^\leftarrow} \mathcal{P}(X) \), \( f^\leftarrow(T) = \{ x \mid f(x) \in T \} \).

The latter operator admits a more general setting.

**Proposition 6**

Given a variety \( \mathbf{A} \), every subcategory \( \mathbf{C} \) of \( \mathbf{LoA} \) induces a functor \( \operatorname{Set} \times \mathbf{C} \xrightarrow{(-)^\leftarrow} \mathbf{LoA} \), ((\( X_1, A_1 \xrightarrow{(f, \varphi)} (X_2, A_2) \)))^\leftarrow = A_1^{X_1} ((f, \varphi)^\leftarrow)^{\mathbf{op}} \xrightarrow{A_2^{X_2}} \) with \( (f, \varphi)^\leftarrow(\alpha) = \varphi^{\mathbf{op}} \circ \alpha \circ f \).
Definition 7

Let \( C \) be a subcategory of \( \text{LoAttA} \).

- A \( C \)-topological space is a triple \((X, F, \tau)\), where \((X, F)\) is a \( \text{Set} \times C \)-object and \( \tau \) is a subalgebra of \( (\Omega F)^X \).

- A \( C \)-continuous map \((X_1, F_1, \tau_1) \xrightarrow{(f, g)} (X_2, F_2, \tau_2)\) is a \( \text{Set} \times C \)-morphism \((X_1, F_1) \xrightarrow{(f, g)} (X_2, F_2)\) with \(((f, (\Omega g)^{op})^{-1}) \rightarrow (\tau_2) \subseteq \tau_1\).

- \( C \text{-Top} \) is the category of \( C \)-spaces and \( C \)-continuous maps. \(|-|\) is the underlying functor to the ground category \( \text{Set} \times C \).

- Given a \( C \)-space \((X, F, \tau)\), \( F \) is called the basis of the space. For a \( C \)-attachment \( F \), \( F \text{-Top} \) is the subcategory of \( C \text{-Top} \) of \( C \)-spaces with basis \( F \) and \( C \)-maps \((f, g)\) with \( g = 1_F \).
Proposition 8

There exists a full embedding $A \xleftarrow{E_A} \text{AttA}$ given by the formula $E_A(A_1 \xrightarrow{\varphi} A_2) = (A_1, A_1, \models_1) \xrightarrow{(\varphi, \varphi)} (A_2, A_2, \models_2)$, with $\models_i(a) = 1_{A_i}$ for every $a \in A_i$.

Corollary 9

There exists the category $\text{LoA-Top}$ of variety-based topology, which makes the following diagram commute:

$$
\begin{array}{ccc}
\text{LoA-Top} & \xleftarrow{E} & \text{LoAttA-Top} \\
\downarrow \models & & \downarrow \models \\
\text{Set} \times \text{LoA} & \xleftarrow{1_{\text{Set}} \times \text{E}_A^\text{op}} & \text{Set} \times \text{LoAttA}.
\end{array}
$$
Topology induced by attachment

Variety-based topology

**Proposition 8**

There exists a full embedding $\mathbf{A} \xleftarrow{E_A} \mathbf{AttA}$ given by the formula

$E_A(A_1 \xrightarrow{\varphi} A_2) = (A_1, A_1, \models_1) \xrightarrow{\langle \varphi, \varphi \rangle} (A_2, A_2, \models_2)$, with $\models_i(a) = 1_{A_i}$ for every $a \in A_i$.

**Corollary 9**

There exists the category $\mathbf{LoA-Top}$ of variety-based topology, which makes the following diagram commute:

$$\begin{array}{ccc}
\mathbf{LoA-Top} & \xleftarrow{E} & \mathbf{LoAttA-Top} \\
\downarrow & & \downarrow \\
\mathbf{Set} \times \mathbf{LoA} & \xleftarrow{1_{\mathbf{Set}} \times E_A^{op}} & \mathbf{Set} \times \mathbf{LoAttA}.
\end{array}$$
Variety-based topological systems

Topological systems

Definition 10

Let $A$ be a variety and let $C$ be a subcategory of $\text{LoA}$.

- A $C$-topological system is a tuple $D = (\text{pt } D, \Sigma D, \Omega D, \models)$ with $(\text{pt } D, \Sigma D, \Omega D)$ in $\text{Set} \times C \times C$ and $\text{pt } D \times \Omega D \xrightarrow{=} \Sigma D$ a map such that $\Omega D \xrightarrow{=}(x, -) \Sigma D$ is a homomorphism for every $x \in \text{pt } D$.

- A $C$-continuous map $D_1 \xrightarrow{f} D_2$ is a $\text{Set} \times C \times C$-morphism $(\text{pt } D_1, \Sigma D_1, \Omega D_1) \xrightarrow{(\text{pt } f, (\Sigma f)^{\text{op}}, (\Omega f)^{\text{op}})} (\text{pt } D_2, \Sigma D_2, \Omega D_2)$ such that for $x \in \text{pt } D_1$, $b \in \Omega D_2$, $\models_1(x, \Omega f(b)) = \Sigma f \circ \models_2(\text{pt } f(x), b)$.

- $C$-$\text{TopSys}$ is the category of $C$-systems and $C$-continuous maps. $\models$ is the underlying functor to the ground category $\text{Set} \times C \times C$. 
Variety-based topological systems

Spatialization procedure for systems

Theorem 11

1. There exists a full embedding \( \text{LoA-Top} \xrightarrow{E_T} \text{LoA-TopSys} \).

2. There exists a functor \( \text{LoA-TopSys} \xrightarrow{\text{Spat}} \text{LoA-Top} \) given by
   \[
   \text{Spat}(D_1 \xrightarrow{f} D_2) = (\text{pt } D_1, \Sigma D_1, \tau_1) \xrightarrow{(\text{pt } f, (\Sigma f)^\text{op})} (\text{pt } D_2, \Sigma D_2, \tau_2)
   \]
   with \( \tau_i = \{|=;(-, b) | b \in \Omega D_i\} \).

3. \( \text{Spat} \) is a right-adjoint-left-inverse of \( E_T \).

4. The category \( \text{LoA-Top} \) is isomorphic to a full coreflective subcategory of the category \( \text{LoA-TopSys} \).
Theorem 12

There exists a functor $\text{LoAttA-Top} \xrightarrow{E_{AT}} \text{LoA-TopSys}$ given by

$$E_{AT}((X_1, F_1, \tau_1) \xrightarrow{(f,g)} (X_2, F_2, \tau_2)) = (X_1 \times |\Omega F_1|, \Sigma F_1, \tau_1, \models_1) \xrightarrow{(f \times (\Omega g)^{op},(\Sigma g)^{op},((f, (\Omega g)^{op})^{op}))^{op}} (X_2 \times |\Omega F_2|, \Sigma F_2, \tau_2, \models_2),$$

where $\models_i((x,a), \alpha) = (\models_i(a))(\alpha(x))$.

- The functors $\xrightarrow{E_{AT}}$ produce the embedding $\xrightarrow{E_T}$.
- In general, the functor $E_{AT}$ is not an embedding.
Theorem 12

There exists a functor $\text{LoAttA-Top} \xrightarrow{E_{AT}} \text{LoA-TopSys}$ given by

$$E_{AT}((X_1, F_1, \tau_1) \xrightarrow{(f,g)} (X_2, F_2, \tau_2)) = (X_1 \times |\Omega F_1|, \Sigma F_1, \tau_1, \models_1) \xrightarrow{(f \times (\Omega g)^{op}, (\Sigma g)^{op}, ((f, (\Omega g)^{op})\leftarrow)^{op})} (X_2 \times |\Omega F_2|, \Sigma F_2, \tau_2, \models_2),$$

where $\models_i((x, a), \alpha) = (\models_i(a))(\alpha(x))$.

- The functors $\text{LoA-Top} \xleftarrow{E_T} \text{LoAttA-Top} \xrightarrow{E_{AT}} \text{LoA-TopSys}$ produce the embedding $\text{LoA-Top} \xleftarrow{E_T} \text{LoA-TopSys}$.
- In general, the functor $E_{AT}$ is not an embedding.
**Definition 13 (Variable-basis approach)**

The variable-basis hypergraph functor \( \text{LoAttA-Top} \xrightarrow{H} \text{LoA-Top} \) is the composition \( \text{LoAttA-Top} \xrightarrow{E_{AT}} \text{LoA-TopSys} \xrightarrow{\text{Spat}} \text{LoA-Top} \), explicitly given by the formula

\[
H((X_1, F_1, \tau_1) \xrightarrow{(f, g)} (X_2, F_2, \tau_2)) = (X_1 \times |\Omega F_1|, \Sigma F_1, \hat{\tau}_1) \xrightarrow{(f \times (\Omega g)^{op}, (\Sigma g)^{op})} (X_2 \times |\Omega F_2|, \Sigma F_2, \hat{\tau}_2),
\]

where \( \hat{\tau}_i = \{ \hat{u} = (\vdash_i(-))(u(-)) | u \in \tau_i \} \).

**Definition 14 (Fixed-basis approach)**

Given a \( \text{LoAttA} \)-object \( F \), the fixed-basis hypergraph functor \( F-\text{Top} \xrightarrow{H^F} \Sigma F-\text{Top} \) is the restriction of \( H \), explicitly given by

\[
H^F((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1 \times |\Omega F|, \hat{\tau}_1) \xrightarrow{f \times 1_{|\Omega F|}} (X_2 \times |\Omega F|, \hat{\tau}_2).
\]
Definition 13 (Variable-basis approach)

The variable-basis hypergraph functor \( \text{LoAttA-Top} \xrightarrow{H} \text{LoA-Top} \) is the composition \( \text{LoAttA-Top} \xrightarrow{E_{AT}} \text{LoA-TopSys} \xrightarrow{\text{Spat}} \text{LoA-Top} \), explicitly given by the formula

\[
H((X_1, F_1, \tau_1) \xrightarrow{(f, g)} (X_2, F_2, \tau_2)) = (X_1 \times |\Omega F_1|, \Sigma F_1, \hat{\tau}_1) \xrightarrow{(f \times (\Omega g)^{op}, (\Sigma g)^{op})} (X_2 \times |\Omega F_2|, \Sigma F_2, \hat{\tau}_2),
\]

where \( \hat{\tau}_i = \{ \hat{u} = (|\text{L}_i|(-))(u(-)) | u \in \tau_i \} \).

Definition 14 (Fixed-basis approach)

Given a \( \text{LoAttA} \)-object \( F \), the fixed-basis hypergraph functor \( \text{F-Top} \xrightarrow{H^F} \Sigma \text{F-Top} \) is the restriction of \( H \), explicitly given by

\[
H^F((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1 \times |\Omega F|, \hat{\tau}_1) \xrightarrow{f \times 1 |\Omega F|} (X_2 \times |\Omega F|, \hat{\tau}_2).
\]
Example 15

Given the unit interval $\mathbb{I} = [0, 1]$, define $\mathbb{I} \xrightarrow{\lhd} \text{Frm}(\mathbb{I}, 2)$ by

$$(\lhd(a))(b) = \begin{cases} \top, & a < b \\ \bot, & \text{otherwise} \end{cases}$$

for every $a \in \mathbb{I}\setminus\{1\}$ and take any of the maps for $\lhd(1)$. The attachment provides the functor $\mathbb{I}\text{-}\text{Top} \xrightarrow{\mathcal{H}_F} \text{Top}$, which is a modified version of the hypergraph functor of R. Lowen.

For an $F$-space $(X, \tau)$, the obtained space $(X \times \mathbb{I}, \hat{\tau})$ has the topology of the sets $\hat{u} = \{(x, a) \mid a < u(x)\} \cup \{(x, 1) \mid b < u(x)\}$ for every $u \in \tau$, provided that $\lhd(1) = \lhd(b)$. 
Other hypergraph functors

Example 16
Given a complete chain $L$, use the construction from Example 15, to obtain a functor which is a modified version of the hypergraph functor of S. E. Rodabaugh.

Example 17
- Let $L$ be a frame such that there is a bijection $L \overset{h}{\rightarrow} \text{Frm}(L, 2)$. Example 15 gives a modification of the hypergraph functors of W. Kotzé and T. Kubiak as well as of U. Höhle.
- Given an $F$-space $(X, \tau)$, the obtained space can be written as $(X \times \text{Frm}(L, 2), \hat{\tau})$ with $\hat{\tau} = \{\hat{u} \mid u \in \tau\}$, $\hat{\tau} = \{(x, p) \mid p(u(x)) = T\}$. 
Examples of hypergraph functors

Other hypergraph functors

Example 16

Given a complete chain $L$, use the construction from Example 15, to obtain a functor which is a modified version of the hypergraph functor of S. E. Rodabaugh.

Example 17

- Let $L$ be a frame such that there is a bijection $L \rightarrow \text{Frm}(L, 2)$. Example 15 gives a modification of the hypergraph functors of W. Kotzé and T. Kubiak as well as of U. Höhle.
- Given an $F$-space $(X, \tau)$, the obtained space can be written as $(X \times \text{Frm}(L, 2), \hat{\tau})$ with $\hat{\tau} = \{ \hat{u} \mid u \in \tau \}$, $\hat{u} = \{ (x, p) \mid p(u(x)) = \top \}$. 
Example 18

Given a frame $L$ and a $\text{Frm}$-attachment $F = (L, 2, \models)$, the functor $L\text{-Top} \xrightarrow{H^F} \text{Top}$ is a modification of the functor $(-)^*$ of C. Guido.

The difference

C. Guido constructs his functor using $L$-points $x_a$ with $a \neq \bot$.

Our approach is based on the product set $X \times L$, which includes the set $S_X$ of fuzzy points and allows $a = \bot$. 
Example 18

Given a frame $L$ and a $\text{Frm}$-attachment $F = (L, 2, \vdash)$, the functor $L\text{-Top} \xrightarrow{H^F} \text{Top}$ is a modification of the functor $(-)^* \text{ of C. Guido}.$

The difference

C. Guido constructs his functor using $L$-points $x_a$ with $a \neq \bot$.

Our approach is based on the product set $X \times L$, which includes the set $S_X$ of fuzzy points and allows $a = \bot$. 
Spatial attachments

**Definition 19**

Let $F$ be an $\mathbf{A}$-attachment.

- $F$ is called Ω-spatial provided that for every $a, b \in \Omega F$ such that $a \neq b$, $\models(a) \neq \models(b)$.

- $F$ is called Σ-spatial provided that for every $a, b \in \Omega F$ such that $a \neq b$, there exists $c \in \Omega F$ with $(\models(c))(a) \neq (\models(c))(b)$.

- $F$ is called spatial provided that it is both Ω- and Σ-spatial.

Let $F = (L, 2, \models)$ be a Σ-spatial $\mathbf{Frm}$-attachment. Then $L$ is a spatial frame in the sense of P. T. Johnstone and $F$ is a spatial attachment in the sense of C. Guido.
Spatial attachments

Definition 19

Let $F$ be an $A$-attachment.

- $F$ is called $\Omega$-spatial provided that for every $a, b \in \Omega F$ such that $a \neq b$, $\vdash (a) \neq \vdash (b)$.
- $F$ is called $\Sigma$-spatial provided that for every $a, b \in \Omega F$ such that $a \neq b$, there exists $c \in \Omega F$ with $(\vdash (c))(a) \neq (\vdash (c))(b)$.
- $F$ is called spatial provided that it is both $\Omega$- and $\Sigma$-spatial.

Let $F = (L, 2, \vdash)$ be a $\Sigma$-spatial $\text{Frm}$-attachment. Then $L$ is a spatial frame in the sense of P. T. Johnstone and $F$ is a spatial attachment in the sense of C. Guido.
Examples of spatial attachments

Example 20

Define \( \forall \xrightarrow{\lhd} \text{Frm}(\forall, 2) \) by \((\forall \lhd (a))(b) = \top \) for \( \frac{a}{2} < b \); otherwise, \((\forall \lhd (a))(b) = \bot \). \( F = (\forall, 2, \lhd) \) is \( \Omega \)-spatial and not \( \Sigma \)-spatial.

Example 21

Define \( \forall \xrightarrow{\lhd} \text{Frm}(\forall, 2) \) by

- For \( a \in [0, \frac{1}{2}] \): \((\forall \lhd (a))(b) = \top \) for \( 2a < b \); otherwise, \((\forall \lhd (a))(b) = \bot \);
- For \( a \in (\frac{1}{2}, 1] \): \((\forall \lhd (a))(b) = \top \) for \( a < b \); otherwise, \((\forall \lhd (a))(b) = \bot \).

\( F = (\forall, 2, \lhd) \) is \( \Sigma \)-spatial and not \( \Omega \)-spatial.
Examples of spatial attachments

Example 20

Define $\mathcal{I} \xrightarrow{\vdash} \text{Frm}(\mathcal{I}, 2)$ by $(\vdash(a))(b) = \top$ for $\frac{a}{2} < b$; otherwise, $(\vdash(a))(b) = \bot$. $F = (\mathcal{I}, 2, \vdash)$ is $\Omega$-spatial and not $\Sigma$-spatial.

Example 21

Define $\mathcal{I} \xrightarrow{\vdash} \text{Frm}(\mathcal{I}, 2)$ by

- $a \in [0, \frac{1}{2}]$: $(\vdash(a))(b) = \top$ for $2a < b$; otherwise, $(\vdash(a))(b) = \bot$;
- $a \in (\frac{1}{2}, 1]$: $(\vdash(a))(b) = \top$ for $a < b$; otherwise, $(\vdash(a))(b) = \bot$.

$F = (\mathcal{I}, 2, \vdash)$ is $\Sigma$-spatial and not $\Omega$-spatial.
Properties of hypergraph functor

Theorem 22

Let $F$ be an $A$-attachment. If $F$ is $\Sigma$-spatial and $\Omega F$ is non-empty, then the fixed-basis hypergraph functor $F\text{-}\text{Top} \xrightarrow{H^F} \Sigma F\text{-}\text{Top}$ is an embedding.

Theorem 23

For any spatial $A$-attachment $F$, the fixed-basis hypergraph functor $F\text{-}\text{Top} \xrightarrow{H^F} \Sigma F\text{-}\text{Top}$ has a right adjoint.
Properties of hypergraph functor

Theorem 22

Let \( F \) be an \( A \)-attachment. If \( F \) is \( \Sigma \)-spatial and \( \Omega F \) is non-empty, then the fixed-basis hypergraph functor \( F \text{-Top} \xrightarrow{H^F} \Sigma F \text{-Top} \) is an embedding.

Theorem 23

For any spatial \( A \)-attachment \( F \), the fixed-basis hypergraph functor \( F \text{-Top} \xrightarrow{H^F} \Sigma F \text{-Top} \) has a right adjoint.
Definition 24

Let $F$ be an $A$-attachment. The modified fixed-basis hypergraph functor $F$-$\text{Top} \xrightarrow{\mathcal{H}^F} \Sigma F$-$\text{Top}$ is defined by the following formula:

$$\mathcal{H}^F(((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2))) = (X_1 \times A(\Omega F, \Sigma F), \tilde{\tau}_1 \circledx 1_{A(\Omega F, \Sigma F)})$$

$$(X_2 \times A(\Omega F, \Sigma F), \tilde{\tau}_2),$$

where $\tilde{\tau}_i = \{ \tilde{\alpha} | \alpha \in \tau_i \}$, $\tilde{\alpha}(x, p) = p(\alpha(x))$.

Example 25

If $F = (L, 2, \Vdash)$ is a $\text{Frm}$-attachment, then the functor $\mathcal{H}^F$ gives the hypergraph functors of U. Höhle, and W. Kotzé and T. Kubiak.
## Modified hypergraph functor

### Definition 24

Let $F$ be an $A$-attachment. The **modified fixed-basis hypergraph functor** $F$-$\text{Top} \xrightarrow{\mathcal{H}^F} \Sigma F$-$\text{Top}$ is defined by the following formula:

$$\mathcal{H}^F((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1 \times A(\Omega F, \Sigma F), \tilde{\tau}_1) \xrightarrow{f \times 1_{A(\Omega F, \Sigma F)}} (X_2 \times A(\Omega F, \Sigma F), \tilde{\tau}_2),$$

where $\tilde{\tau}_i = \{ \tilde{\alpha} | \alpha \in \tau_i \}$, $\tilde{\alpha}(x, p) = p(\alpha(x))$.

### Example 25

If $F = (L, 2, \models)$ is a $\text{Frm}$-attachment, then the functor $\mathcal{H}^F$ gives the hypergraph functors of U. Höhle, and W. Kotzé and T. Kubiak.
Theorem 26

Suppose $F$ is an $A$-attachment. If $F$ is $\Sigma$-spatial and the hom-set $A(\Omega F, \Sigma F)$ is non-empty, then the fixed-basis hypergraph functor $F\text{-}\text{Top} \xrightarrow{\mathcal{H}^F} \Sigma F\text{-}\text{Top}$ is an embedding.

Theorem 27

For any spatial $A$-attachment $F$, the fixed-basis hypergraph functor $F\text{-}\text{Top} \xrightarrow{\mathcal{H}^F} \Sigma F\text{-}\text{Top}$ has a right adjoint.
Properties of modified hypergraph functor

Theorem 26

Suppose $F$ is an $A$-attachment. If $F$ is $\Sigma$-spatial and the hom-set $A(\Omega F, \Sigma F)$ is non-empty, then the fixed-basis hypergraph functor $F-\text{Top} \xrightarrow{\mathcal{H}^F} \Sigma F-\text{Top}$ is an embedding.

Theorem 27

For any spatial $A$-attachment $F$, the fixed-basis hypergraph functor $F-\text{Top} \xrightarrow{\mathcal{H}^F} \Sigma F-\text{Top}$ has a right adjoint.
Suppose $F$ is an $A$-attachment.

**Proposition 28**

There exists a natural transformation $H^F \xrightarrow{\nu} \mathcal{H}^F$, which is given by

$$H^F(X, \tau) \xrightarrow{\nu(X, \tau) = (1 \times \Pi)} \mathcal{H}^F(X, \tau).$$

**Corollary 29**

$\Omega F \xrightarrow{\Pi} A(\Omega F, \Sigma F)$ is bijective iff $H^F \xrightarrow{\nu} \mathcal{H}^F$ is a natural isomorphism.

**Theorem 30**

Let $F$ be spatial and let $\langle H^F, G^F, \eta^F, \epsilon^F \rangle, \langle \mathcal{H}^F, G^F, \theta^F, \epsilon^F \rangle$ be the adjunctions generated by the functors $H^F, \mathcal{H}^F$ respectively. There exists a transformation $\langle H^F, G^F, \eta^F, \epsilon^F \rangle \xrightarrow{(\lambda, \kappa)} \langle \mathcal{H}^F, G^F, \theta^F, \epsilon^F \rangle.$
Suppose $F$ is an $A$-attachment.

**Proposition 28**

There exists a natural transformation $H^F \xrightarrow{\eta} \mathcal{H}^F$, which is given by

$$H^F(X, \tau) \xrightarrow{\eta(X, \tau) = (1 \times \bot)} \mathcal{H}^F(X, \tau).$$

**Corollary 29**

$\Omega F \xrightarrow{\bot} A(\Omega F, \Sigma F)$ is bijective iff $H^F \xrightarrow{\eta} \mathcal{H}^F$ is a natural isomorphism.

**Theorem 30**

Let $F$ be spatial and let $\langle H^F, G^F, \eta^F, \epsilon^F \rangle$, $\langle \mathcal{H}^F, \mathcal{G}^F, \theta^F, \epsilon^F \rangle$ be the adjunctions generated by the functors $H^F$, $\mathcal{H}^F$ respectively. There exists a transformation $\langle H^F, G^F, \eta^F, \epsilon^F \rangle \xrightarrow{(\iota, \kappa)} \langle \mathcal{H}^F, \mathcal{G}^F, \theta^F, \epsilon^F \rangle$. 
Suppose $F$ is an $A$-attachment.

**Proposition 28**

There exists a natural transformation $H^F \xrightarrow{\iota} \mathcal{H}^F$, which is given by

$$H^F(X, \tau) \xrightarrow{\iota(X, \tau) \equiv (1 \times x \times \iota)} \mathcal{H}^F(X, \tau).$$

**Corollary 29**

$\Omega F \xrightarrow{\iota} A(\Omega F, \Sigma F)$ is bijective iff $H^F \xrightarrow{\iota} \mathcal{H}^F$ is a natural isomorphism.

**Theorem 30**

Let $F$ be spatial and let $\langle H^F, G^F, \eta^F, \epsilon^F \rangle, \langle \mathcal{H}^F, \mathcal{G}^F, \theta^F, \varepsilon^F \rangle$ be the adjunctions generated by the functors $H^F, \mathcal{H}^F$ respectively. There exists a transformation $\langle H^F, G^F, \eta^F, \epsilon^F \rangle \xrightarrow{(\iota, \kappa)} \langle \mathcal{H}^F, \mathcal{G}^F, \theta^F, \varepsilon^F \rangle$. 
Suppose $F$ is an $A$-attachment.

**Proposition 28**

There exists a natural transformation $H^F \overset{\iota}{\rightarrow} \mathcal{H}^F$, which is given by

$$H^F(X, \tau) \overset{\iota(X, \tau) = (1 \times \iota)}{\longrightarrow} \mathcal{H}^F(X, \tau).$$

**Corollary 29**

$\Omega F \overset{\iota}{\rightarrow} A(\Omega F, \Sigma F)$ is bijective iff $H^F \overset{\iota}{\rightarrow} \mathcal{H}^F$ is a natural isomorphism.

**Theorem 30**

Let $F$ be spatial and let $\langle H^F, G^F, \eta^F, \epsilon^F \rangle$, $\langle \mathcal{H}^F, \mathcal{G}^F, \theta^F, \varepsilon^F \rangle$ be the adjunctions generated by the functors $H^F$, $\mathcal{H}^F$ respectively. There exists a transformation $\langle H^F, G^F, \eta^F, \epsilon^F \rangle \overset{(\iota, \kappa)}{\rightarrow} \langle \mathcal{H}^F, \mathcal{G}^F, \theta^F, \varepsilon^F \rangle$. 
Contribution of the talk

- The talk introduced the notion of variety-based attachment.
- The concept generalizes the membership relation “∈” and its extensions of P.-M. Pu and Y.-M. Liu as well as of C. Guido.
- Variety-based attachment provides a common framework for all approaches to hypergraph functor of the fuzzy community.
Motivating example

Step 1. Given a finite chain $L$, define an attachment $\mathcal{A}$ on $L$ by $F_\bot = \emptyset$ and $F_a = \uparrow a = \{ b \in L \mid a \leq b \}$ for every $a \in L \setminus \{ \bot \}$.

Step 2. Given an $L$-point $x_a$ and an $L$-set $\alpha$: $x_a \mathcal{A} \alpha$ iff $\alpha(x) \in F_a$ iff $a \leq \alpha(x)$ iff $a \in \downarrow \alpha(x)$.

Step 3. If $\alpha(x) \neq \top$, then $\downarrow \alpha(x)$ is a completely prime ideal of $L$, which is the dual notion to that of completely prime filter.

Step 4. The family $G_\top = \emptyset$ and $G_a = \downarrow a$ for every $a \in L \setminus \{ \top \}$ is a possible substitute for $\mathcal{A}$.

Problem 31

What will be the concept of dual variety-based attachment?
Motivating example

Step 1. Given a finite chain $L$, define an attachment $\mathcal{A}$ on $L$ by $F_{\perp} = \emptyset$ and $F_a = \uparrow a = \{b \in L | a \leq b\}$ for every $a \in L \setminus \{\perp\}$.

Step 2. Given an $L$-point $x_a$ and an $L$-set $\alpha$: $a_x \mathcal{A} \alpha$ iff $\alpha(x) \in F_a$ iff $a \leq \alpha(x)$ iff $a \in \downarrow \alpha(x)$.

Step 3. If $\alpha(x) \neq \top$, then $\downarrow \alpha(x)$ is a completely prime ideal of $L$, which is the dual notion to that of completely prime filter.

Step 4. The family $G_{\top} = \emptyset$ and $G_a = \downarrow a$ for every $a \in L \setminus \{\top\}$ is a possible substitute for $\mathcal{A}$.

Problem 31

What will be the concept of dual variety-based attachment?


Thank you for your attention!