Modes, modals, and barycentric algebras

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A mode $(A, \Omega)$ is an idempotent and entropic algebra: Each singleton subset $\{a\}$ of $A$ is a subalgebra, and each operation $\omega : A^{\omega \tau} \to A$ (with arity $\omega \tau$) is a homomorphism. A modal $(A, +, \Omega)$ is a semilattice $(A, +)$ and a mode $(A, \Omega)$ such that each mode operation $\omega$ distributes over the semilattice operation:

$$a_1 \ldots (a_i + a'_i) \ldots a_{\omega \tau} \omega = a_1 \ldots a_i \ldots a_{\omega \tau} \omega + a_1 \ldots a'_i \ldots a_{\omega \tau} \omega$$

for $1 \leq i \leq \omega \tau$. For a mode $A$, the set $AS$ of nonempty subalgebras of $A$ forms a modal $(AS, +, \Omega)$ under the join and complex products of subalgebras. In particular, modes enjoy the self-reproducing property that the subalgebra set $AS$ of a mode $A$ again forms a mode, with $A$ as a subalgebra.

The real line carries operations of complementation $p' = 1 - p$, dual multiplication $p \circ q = (p'q')'$, and implication $p \to q = \text{if } (p = 0) \text{ then } 1 \text{ else } q/p$.

A barycentric algebra $(A, I)$ is a set $A$ with a binary operation $xyp$ for each element $p$ of the open real unit interval $I$ satisfying idempotence $xxp = x$, skew-commutativity $xyp = yxp'$, and skew-associativity $xypzq = xyz(p \circ q \to q) p \circ q$. Examples are given by semilattices with $xyp = x \cup y$ and convex sets with $xyp = xp' + yp$. Barycentric algebras are modes.

Both theoretical aspects and applications of these structures will be discussed in the presentation.