Anti-inverse Subsemigroups of the Full Transformation Semigroup

Ilinka Dimitrova

e-mail: ilinka_dimitrova@yahoo.com
South-West University, Blagoevgrad, Bulgaria

Jörg Koppitz

e-mail: koppitz@rz.uni-potsdam.de
Potsdam University, Germany

The notion of anti-inverse elements in a semigroup was introduced by J.C. Sharp in 1977 ([6]). In [2], Bogdanović, Milić, and Pavlović studied the structure and considered some properties of those semigroups. In 1982, the anti-inverse semigroups were studied by Blagojevich ([1]). An element \(a\) of a semigroup \(S\) is called anti-inverse if there exists an element \(b \in S\) such that \(aba = b\) and \(bab = a\) (in this case \(a\) and \(b\) are called mutual anti-inverse). The semigroup \(S\) is called anti-inverse if each element of \(S\) is anti-inverse.

For particular classes of semigroups, the anti-regular elements are studied (see [2], [6]). Clearly, each band is anti-inverse because of the idempotent low. Moreover, each element of a band has a unique anti-inverse element. An abelian group is anti-inverse if and only if each element is inverse to itself. But not each group is anti-inverse. An abelian semigroup \(S\) is anti-inverse if and only if \(x = x^3\) for each element \(x \in S\).

We want to initiate the study of anti-inverse transformation semigroups. Here, we characterize the anti-inverse elements of the full transformation semigroup \(T_n\). A \(J\)-class in \(T_n\) consists of all transformations with the same rank. Moreover, we determine all anti-inverse semigroups in the \(J\)-classes. Let note that the semigroup of all order-preserving (order-preserving or order-reversing) transformations as well as the semigroup of all orientation-preserving (orientation-preserving or orientation-reversing) transformations are of particular interest (see for example [3], [4], [5]). In order to illustrate our result, we will describe the anti-inverse semigroups within the \(J\)-classes of these transformation semigroups. The last part is devoted semigroups with elements of order 2. We give a description of its maximal anti-inverse subsemigroups containing particular transformations with rank \(\leq 3\).

We will try to keep the standard notation. Let \(X_n = \{1 < \cdots < n\}\) be a finite chain with \(n\) - elements. The full transformation semigroup \(T_n\) is the set of all mappings, written on the right, of \(X_n\) into itself with the composition of mappings as multiplication. For every transformation \(\alpha \in T_n\), by \(\ker \alpha\) and \(\text{im} \alpha\) we denote the kernel and the image of \(\alpha\), respectively.

Each \(J\)-class of \(T_n\) has the form

\[ J_k := \{\alpha \in T_n : |\text{im} \alpha| = k\} \text{ for } 1 \leq k \leq n. \]

For \(1 \leq k \leq n\) let us denote by \(\Lambda_k\) the collection of all subsets of \(X_n\) of cardinality \(k\). Let \(A \in \Lambda_k\) and let \(\pi\) be an equivalence relation on \(X_n\) with
Moreover, let 1 \sim \inversesemigroup iff there is a semigroup \( S \) such that \( \alpha \beta \alpha = \beta \) and \( \beta \alpha \beta = \alpha \). These equations imply \( \text{im} \alpha = \text{im} \beta \), \( \text{ker} \alpha = \text{ker} \beta \), and \( \text{im} \alpha \# \text{ker} \beta \). Therefore, we have that \( \alpha, \beta \in H_\alpha \) and \( H_\alpha \) contains an idempotent.

**Definition 1.** For \( \varepsilon \in E(T_n) \) we put
\[
H^*_\varepsilon := \{ \alpha \in H_\varepsilon : \text{there is a} \ \beta \in H_\varepsilon \text{ with } \alpha^4 = \beta^4 = \alpha^2 \beta^2 = \alpha \beta \alpha \beta^3 = \varepsilon \}.
\]

**Proposition 2.** Let \( S \) be a subsemigroup of \( T_n \). Then \( S \) is an anti-inverse semigroup if and only if \( S \subseteq \bigcup \{ H^*_\varepsilon : \varepsilon \in E(T_n) \} \).

Proposition 2 shows that the anti-inverse subsemigroups of \( T_n \) are Clifford semigroups (completely regular semigroups).

Let \( 1 \leq k \leq n \in N \) and \( S \subseteq J_k \). Then \( S \) is a semigroup in \( J_k \) iff there are congruence relations \( \pi_1, \ldots, \pi_r \) on \( X_n \) with \( |X_n/\pi_i| = k \) \( (i = 1, \ldots, r) \) and sets \( A_1, \ldots, A_s \in \Lambda_k \) such that \( A_j \# \pi_i \) for \( 1 \leq i \leq r \) and \( 1 \leq j \leq s \), and \( S = \{ \alpha \in J_k : \text{ker} \alpha \in \{ \pi_1, \ldots, \pi_r \} \text{ and } \text{im} \alpha \in \{ A_1, \ldots, A_s \} \} \).

**Theorem 3.** Let \( 1 \leq k \leq n \in N \) and \( S \subseteq J_k \subseteq T_n \). Then \( S \) is an anti-inverse semigroup iff there is a semigroup \( \tilde{S} \) in \( E(T_n) \cap J_k \) and a semigroup \( K \) in \( H^*_\varepsilon \) for some \( \varepsilon \in \tilde{S} \) such that \( S = \bigcup \{ \varepsilon \alpha \varepsilon : \alpha \in K \} : \tilde{\varepsilon} \in \tilde{S} \} \).

A transformation \( \alpha \in T_n \) is called order-preserving (respectively, order-reversing) if for all \( x, y \in X_n \), \( x \leq y \) implies \( x \alpha \leq y \alpha \) (respectively, \( x \alpha \geq y \alpha \)).

The set \( O_n := \{ \alpha \in T_n : \alpha \text{ is order-preserving} \} \) forms a subsemigroup of \( T_n \), which is called the *semigroup of all order-preserving transformations* (see [5]).

The set \( M_n := \{ \alpha \in T_n : \alpha \text{ is order-preserving or order-reversing} \} \) forms a subsemigroup of \( T_n \), which is called the *semigroup of all order-preserving or order-reversing transformations* (see [4]).

It is clear that the semigroups \( O_n \) and \( M_n \) are regular subsemigroups of \( T_n \). Moreover, \( O_n \subseteq M_n \subseteq T_n \). Let \( \alpha \in M_n \). Then the \( \text{(ker} \alpha \text{)} \)-classes are convex subsets \( C \) of \( X_n \), in the sense that \( x, y \in C \) and \( x \leq z \leq y \) together imply that \( z \in C \).

Let \( 1 \leq k \leq n \in N \) and \( \varepsilon \in E(T_n) \cap J_k \) with
\[
\varepsilon = \begin{pmatrix} \pi_1 & \ldots & \pi_k \\ a_1 & \ldots & a_k \end{pmatrix} \text{ and } a_1 < \cdots < a_k. \text{ Then we put }
\varepsilon^a := \begin{pmatrix} \pi_1 & \ldots & \pi_k \\ a_k & \ldots & a_1 \end{pmatrix}, \text{ i.e. } \pi_i \varepsilon^a = a_{k-i+1} \text{ for } 1 \leq i \leq k. \text{ Clearly, } (\varepsilon^a)^2 = \varepsilon.
\]

**Proposition 4.** Let \( 1 \leq k \leq n \in N \) and \( S \subseteq J_k \subseteq M_n \). Then \( S \) is an anti-inverse semigroup iff there is a semigroup \( \tilde{S} \) in \( E(T_n) \cap J_k \) such that \( S = \tilde{S} \) or \( S = \bigcup \{ \varepsilon, \varepsilon^a : \varepsilon \in \tilde{S} \} \).
Corollary 5. Let \( 1 \leq k \leq n \in N \) and \( S \subseteq J_k \subseteq O_n \). Then \( S \) is an anti-inverse semigroup iff \( S \) is a semigroup in \( E(T_n) \cap J_k \).

Now, let \( a = (a_1, a_2, \ldots, a_t) \) be a sequence of \( t(t \geq 1) \) elements from the chain \( X_n \). We say that \( a \) is cyclic (respectively, anti-cyclic) if there exists no more than one index \( i \in \{1, \ldots, t\} \) such that \( a_i > a_{i+1} \) (respectively, \( a_i < a_{i+1} \)), where \( a_{t+1} \) denotes \( a_1 \). A transformation \( \alpha \in T_n \) is called orientation-preserving (respectively, orientation-reversing) if the sequence of its images is cyclic (respectively, anti-cyclic).

The set \( OP_n := \{ \alpha \in T_n : \alpha \) is orientation-preserving \} forms a subsemigroup of \( T_n \), which is called the semigroup of all orientation-preserving transformations (see [3]).

The set \( OPR_n := \{ \alpha \in T_n : \alpha \) is orientation-preserving or orientation-reversing \} forms a subsemigroup of \( T_n \), which is called the semigroup of all orientation-preserving or orientation-reversing transformations (see [4]). Moreover, we put \( OR_n := OPR_n \setminus OP_n \).

Let \( \alpha \in OPR_n \). Then for all \( \alpha \in X_n / \ker \alpha \), the set \( \alpha \) or \( X_n \setminus \alpha \) is convex. Let \( 1 \leq k \leq n \in N \) and \( \varepsilon \in E(T_n) \cap J_k \) with \( \varepsilon = (\alpha_1 \ldots \alpha_k) \) where \( a_1 < \ldots < a_k \).

If \( k \in 2N \) then we put \( \varepsilon' \) as the orientation-preserving transformation in \( H_\varepsilon \) with

\[
\bar{x}_{\varepsilon'\varepsilon} = \begin{cases} 
  a_{p+\frac{k}{2}} & \text{if } 1 \leq p \leq \frac{k}{2} \\
  a_{p-\frac{k}{2}} & \text{if } \frac{k}{2} < p \leq k.
\end{cases}
\]

Clearly, \((\varepsilon')^2 = \varepsilon\). If \( k \) is odd then \( H_\varepsilon^* = \{\varepsilon\} \). If \( k \in 2N \) then \( H_\varepsilon^* = \{\varepsilon, \varepsilon'\} \).

Proposition 6. Let \( 1 \leq k \leq n \in N \) and \( S \subseteq J_k \subseteq OPR_n \). Then \( S \) is an anti-inverse semigroup iff there is a semigroup \( \tilde{S} \subseteq E(T_n) \cap J_k \) such that

(i) \( S = \tilde{S} \) or
(ii) \( S = \bigcup\{\{\varepsilon, \varepsilon \alpha \varepsilon\} : \varepsilon \in \tilde{S}\} \) for some \( \alpha \in OR_n \cap H_{\varepsilon_\alpha}^* \) with \( \varepsilon_\alpha \in \tilde{S} \) or
(iii) \( S = \bigcup\{\{\varepsilon, \varepsilon'\} : \varepsilon \in \tilde{S}\} \) and \( k \) is even or
(iv) \( S = \bigcup\{\{\varepsilon, \varepsilon', \varepsilon \alpha \varepsilon, \varepsilon \alpha \varepsilon_\alpha \varepsilon\} : \varepsilon \in \tilde{S}\} \) for some \( \alpha \in OR_n \cap H_{\varepsilon_\alpha}^* \) where \( \varepsilon_\alpha \in \tilde{S} \) and \( k \) is even.

Corollary 7. Let \( 1 \leq k \leq n \in N \) and \( S \subseteq J_k \subseteq OP_n \). Then \( S \) is an anti-inverse semigroup iff there is a semigroup \( \tilde{S} \subseteq E(T_n) \cap J_k \) such that

(i) \( S = \tilde{S} \) or
(ii) \( S = \bigcup\{\{\varepsilon, \varepsilon'\} : \varepsilon \in \tilde{S}\} \) and \( k \) is even.

References

