Hypergraph functor and attachment

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Since the inception of the notion of fuzzy topology much attention was paid to the possible means of interaction between fuzzy and crisp topological settings, in order to see whether fuzzy topology was doing anything new. In particular, different functors relating the categories Top of topological spaces and L-Top of L-topological spaces (L being a suitable complete lattice) appeared in the literature, providing the desired machinery for comparing classical and fuzzy developments. One of the most important examples in the field is hypergraph functor. Initiated by R. Lowen [8] and E. S. Santos [13], the concept was studied by many researchers [4],[6],[7],[11], but still failed to gain much prominence in the fuzzy community. The main reasons were, firstly, the lack of information on functorial properties of the hypergraph functor and, secondly, remarkable differences in its definition by various authors (cf., e.g., those of U. Höhle [6] and S. E. Rodabaugh [11]).

There has been several attempts to amend the situation. The former of the above-mentioned deficiencies was partly removed by W. Kotzé, T. Kubiak [7] and U. Höhle [6] by considering the hypergraph functor from the categorical point of view. The second deficiency, however, appeared more resistant and was approached to only recently by C. Guido [5]. Motivated by the concept of quasi-coincidence [9] (which is an analogue of the intersection property for fuzzy sets), he introduced the notion of attachment on a complete lattice as follows.

Definition 1. An attachment family, or more simply an attachment on a complete lattice L is a family \(A = \{F_a | a \in L\}\) of subsets of L such that \(F_\bot = \emptyset\), and for every \(a \in L \setminus \{\bot\}\), \(F_a\) is a completely prime filter of \(L\) (\(\bigvee S \in F_a\) implies \(S \cap F_a \neq \emptyset\)).

The new concept gave rise to a functor \(L\text{-Top} \xrightarrow{\sim,\ast} \text{Top}\), which appeared to have striking similarities with the hypergraph functor. In particular, a slight modification of the notion of attachment (generalized attachment) done by C. Guido gave \(\sim,\ast\) the power to provide a common framework for many (if not all) approaches to the topic. It is important to notice, however, that the author does not consider any categorical property of his functor apart from that of being an embedding. It is the purpose of this talk to develop the categorical aspects of the attachment theory and its relationship to the hypergraph functor.

Based on our current research on topological properties which could be used in an arbitrary variety of algebras, we introduce the notion of variety-based attachment, replacing complete lattices of [5] by algebras (in an obvious sense, as sets with operations).

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Definition 2. Let $A$ be a variety of algebras and let $A \xrightarrow{(-)^*} \text{Set}^{op}$ be a functor such that $A^* = |A|$ ($|\cdot|$ is the underlying set of $A$). $\text{AttA}$ is the category, whose objects are triples $F = (\Omega F, \Sigma F, \models)$ (called $\Sigma F$-attachments on $\Omega F$), where $\Omega F$ and $\Sigma F$ are algebras, and $\Omega F \xrightarrow{\models} A(\Omega F, \Sigma F)$ is a map. Morphisms $F_1 \xrightarrow{f=(\Omega f, \Sigma f)} F_2$ are $A \times A$-morphisms $(\Omega F_1, \Sigma F_1) \xrightarrow{(\Omega f, \Sigma f)} (\Omega F_2, \Sigma F_2)$ such that for every $a_1 \in \Omega F_1$ and every $a_2 \in \Omega F_2$,

$$(\models_2(a_2))(\Omega f(a_1)) = (\Sigma f \circ \models_1((\Omega f)^{op}(a_2)))(a_1),$$

with the composition and identities being those of $A \times A$.

The notion gives rise to a new category for topology, which is a proper super-category of the currently dominating one for topological structures in the fuzzy community, introduced by S. E. Rodabaugh [12].

On the next step, we use the notion of variety-based topological system (developed in [15],[19], and motivated by the concept of topological system of S. Vickers [20], already modified by various authors [1], cite51.1I,[3],[10]) to provide a variety-based hypergraph functor, which incorporates the respective fixed-basis concepts of [8],[11] and partly those of [6],[7]. Moreover, the above-mentioned functor $(\cdot)^*$ of [5] is also included in the framework. Restricting the setting to the fixed-basis case, we give the sufficient conditions for the new functor to be an embedding, construct a right adjoint to it and show a relation of the obtained adjunction to that provided by U. Höhle [6] for the particular case of frames (complete lattices with the property that finite meets distribute over arbitrary joins).

The results of the talk not only make the nature of hypergraph functor more transparent, but also clearly show that its definition and many of its properties depend not on a particular lattice-theoretic peculiarity of the respective underlying structure for fuzziness, but rather on its categorically-algebraic aspects. This crucial fact makes a significant contribution to the new approach to topological structures introduced by us recently under the name of categorically-algebraic (catalg) topology [14],[16],[17],[18]. The new theory is based on both category theory and universal algebra (relying more on the former) that is reflected in its name. The main advantage of the catalg setting is the possibility of uniting (almost) all approaches to (fuzzy) topology, currently developed in mathematics, under one roof, ultimately erasing the border between crisp and fuzzy developments and postulating the slogan: algebra is at the bottom of everything.

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REFERENCES


