An algebra \( A \) having a majority term will be called a *majority algebra*. If \( A \) is any algebra then the set \( S_2(A) \) of all subuniverses of \( A \times A \) has a natural algebraic structure: the binary operations \( \circ \) (relational product) and \( \cap \), the unary operation \( \sim \) (taking inverse relation) and the nullary operations \( \Delta \) (diagonal) and \( \nabla = A \times A \). C. Bergman denoted in [1] this structure by \( S_2(A) \). He proved that the categorical equivalence class of any finite majority algebra \( A \) is determined by \( S_2(A) \). For technical reasons he assumed that \( A \) had no nullary fundamental operations and thus in his approach \( S_2(A) \) always contained \( \emptyset \). This restriction was not a principal one because one can always replace nullary operations by unary constant operations.

We prefer a slightly different approach. Our \( S_2(A) \) is a set of all subalgebras of \( A \times A \), so it never contains \( \emptyset \). Since we do not require the existence of nullary fundamental operations, now the intersection is not an algebraic operation on \( S_2(A) \), in general. Therefore in our approach \( S_2(A) \) is the ordered structure \( \langle S_2(A); \circ, \sim, \Delta, \nabla, \subseteq \rangle \).

C. Bergman’s basic result in [1] was that two finite majority algebras \( A \) and \( B \) are categorically equivalent iff the algebraic structures \( S_2(A) \) and \( S_2(B) \) are isomorphic. The latter is clearly equivalent to the isomorphism of our ordered structures.

J. Snow posed in [5] the problem: characterize the structures that appear as \( S_2(A) \) for some finite majority algebra. Here we solve the problem for *minimal* algebras, that is, for algebras with no proper subalgebras.

Clearly the structures \( S \) we are interested in must be ordered monoids with involution and zero element. Thus, \( S = \langle S; \cdot, 1, 0, \leq \rangle \) and the following conditions must be satisfied:

- \( \langle S; \cdot, 1, 0 \rangle \) is a monoid with zero;
- \( (xy)^{-1} = y^{-1}x^{-1} \) for every \( x, y \in S \);
- \( (x^{-1})^{-1} = x \) for every \( x \in S \);
- \( \leq \) is a partial order relation;
- for every \( x, y, u, v \in S \), if \( x \leq y \) and \( u \leq v \) then \( xu \leq yv \);
- for every \( x, y \in S \), if \( x \leq y \) then \( x^{-1} \leq y^{-1} \).
It is clear that the operation symbols $\cdot, ^{-1}, 1, 0$ correspond in $S_2(A)$ to $\circ, \cdot, \Delta, \nabla$, respectively. The relation symbol $\leq$, however, corresponds to $\supseteq$, not to $\subseteq$, as one might think. This seemingly strange agreement has several reasons. In particular, it agrees with standard definition of the natural order in inverse semigroups. This immediately implies that our structure $S$ is a meet semilattice but not a join semilattice, in general. Our main result is the following (the notions $E$-reflexivity, completeness, distributivity) have been taken from the theory of inverse monoids.

**Theorem 1.** Let $S$ be a meet semilattice ordered monoid with involution and the zero element. Then the following are equivalent:

1. there exists a finite minimal majority algebra $A$ such that $S$ is isomorphic to $S_2(A)$;
2. $S$ satisfies the following conditions:
   - (a) $(E$-reflexivity) for every $x, y \in S$, if $x^{-1}y \leq 1$ then $xy^{-1} \leq 1$;
   - (b) (completeness) for every $x, y \in S$, the join $x \lor y$ exists if and only if $x^{-1}y \leq 1$;
   - (c) (distributivity) for every $x, y, z \in S$, if $y \lor z$ exists then also $xy \lor xz$ exists and the equality $x(y \lor z) = xy \lor xz$ holds.

There are two important special cases. First consider the case when $\langle S; \cdot, ^{-1}, 1 \rangle$ is an inverse monoid. Then it is easy to check that the order relation $\leq$ of $S$ is precisely the natural order of that inverse monoid. It has been proved in [3] and [4] that finite, $E$-reflexive, complete, distributive, inverse monoids with zero are precisely (up to isomorphism) the structures $S_2(A)$ for finite minimal algebras $A$ generating arithmetical varieties. Now this result easily follows from Theorem 1.

The other important special case is given by condition $x \leq 1$ for every $x \in S$. In terms of algebra $A$ this means that every subalgebra of $A^2$ contains $\Delta$. Note that in presence of majority term the latter is equivalent to the condition that all constants are terms. In this case $\langle S; \leq \rangle$ turns out to be a lattice and the whole structure $S$ becomes a lattice ordered semiring (cf. [2]). Thus, lattice ordered semirings with involution are precisely (up to isomorphism) the structures $S_2(A)$ where $A$ is a finite majority algebra with all constants being terms.

**References**


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