On \textit{t}-filters on Residuated Lattices (AAA88)

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Definition of a Residuated Lattice

**Definition**

A *bounded pointed commutative integral residuated lattice* is a structure

$$L = (L, \& , \rightarrow, \land, \lor, \overline{0}, \overline{1})$$

of type $(2, 2, 2, 2, 0, 0)$ which satisfies the following conditions:

(i) $(L, \land, \lor, \overline{0}, \overline{1})$ is a bounded lattice.

(ii) $(L, \& , \overline{1})$ is a monoid.

(iii) $(\& , \rightarrow)$ form an adjoint pair, i.e. $x \& z \leq y$ if and only if $z \leq x \rightarrow y$ for all $x, y, z \in L$. 
Definition of a Filter

**Definition**

A non-empty subset $F$ of $L$ is called a *filter* on $L$ if following conditions hold for all $x, y \in L$:

(i) if $x, y \in F$, then $x \& y \in F$,

(ii) if $x \in F$, $x \leq y$, then $y \in F$. 

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Special Types of Filters

Definition

A nonempty subset $F$ of a BL-algebra $L$ called a *fantastic* filter if it satisfies:

1. $\overline{1} \in F$
2. $z \rightarrow (y \rightarrow x) \in F$ and $z \in F$ imply $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$ for all $x, y, z \in A$.

Other types of filters such as implicative, positive implicative, ... filters are defined similarly by replacing the second condition by some different one.
Summary of Some Existing Results - Example I.

Theorem (Haveshki, Eslami, Saeid (2006))

On BL-algebra $\mathbf{L}$, the following statements are equivalent:

1. $\{1\}$ is a fantastic filter.
2. Every filter on $\mathbf{L}$ is a fantastic filter.
3. $\mathbf{L}$ is an MV-algebra.

MV-algebras are just BL-algebras satisfying $\neg\neg x = x$. 
Theorem

On BL-algebra $L$, the following statements are equivalent:

1. $\overline{\{I\}}$ is an implicative filter.
2. Every filter on $L$ is an implicative filter.
3. $L$ is a Gödel algebra.

Gödel algebras are just BL-algebras satisfying $x \& x = x$. 
Motivation – Example II

Theorem

Let $F$, $G$ be filters on BL-algebra $L$ such that $F \subseteq G$. If $F$ is a fantastic filter, then $G$ is a fantastic filter.
Motivation – Example II’

Theorem

Let $F, G$ be filters on $\mathbf{BL}$-algebra $L$ such that $F \subseteq G$. If $F$ is an implicative filter, then $G$ is an implicative filter.
Motivation – Example III

Theorem

Let $F$ be a filter of (a BL-algebra) $L$. Then $F$ is a fantastic filter if and only if every filter of the quotient algebra $L/F$ is a fantastic filter.
Theorem

Let $F$ be a filter of (a BL-algebra) $L$. Then $F$ is an implicative filter if and only if every filter of the quotient algebra $L/F$ is an implicative filter.
Alternative Definitions of Special Types of Filters

Theorem (Kondo and Dudek (2008))

Let $L$ be a BL-algebra, $F \subseteq L$ a filter on $L$. Then $F$ is a fantastic filter iff for all $x \in L$, $\neg\neg x \rightarrow x \in F$ and $F$ is an implicative filter iff for all $x \in L$, $x \rightarrow x \& x \in F$.

Starting now, $L$ is a residuated lattice.
Generalization: $t$-filters

**Definition**

Let $t$ be an arbitrary term. A filter $F$ on $L$ is a $t$-filter if $t(\overline{x}) \in F$ for all $\overline{x} \in L$.

$\overline{x}$ is an abbreviation for a list $x, y, \ldots$. Since now, $t$ is a fixed term.
Theorem

Let $F$ and $G$ be filters on a residuated lattice $L$ such that $F \subseteq G$. If $F$ is a $t$-filter, then so is $G$. 
Theorem

Let $\mathcal{B}$ be a variety of residuated lattices and $L \in \mathcal{B}$. Moreover let $\mathcal{C}$ be a subvariety of $\mathcal{B}$ which we get by adding the equation in the form $t = \bar{1}$. Then the following statements are equivalent:

1. $\{\bar{1}\}$ is a $t$-filter.
2. Every filter on $L$ is a $t$-filter.
3. $L$ is in $\mathcal{C}$. 
Let $F$ be a filter on a residuated lattice $L$. Then $F$ is a t-filter if and only if every filter of the quotient algebra $A/F$ is a t-filter.
Simple Observations

- \( \overline{I} \)-filters are just filters on \( L \), \( x \)-filters are just trivial filters.
- If \( t_1(x) \leq t_2(x) \) for all \( x \in L \), then
  \[
  \{ F \subseteq L \mid F \text{ is a } t_1\text{-filter} \} \subseteq \{ F \subseteq L \mid F \text{ is a } t_2\text{-filter} \}.
  \]
**t-filters and Extended Filters**

**Definition (Kondo (2013))**

Let \( B \) be an arbitrary nonempty subset of \( L \), \( F \) filter on \( L \). The set \( E_F(B) = \{ x \in L \mid \forall b \in B(x \lor b \in F) \} \) is called **extended filter** associated with \( B \).

**Theorem (Kondo (2013))**

Let \( F \) be a filter on \( L \). Then:

- \( F \) is an implicative filter if and only if \( E_F(x \rightarrow x^2) = L \) for all \( x \in L \)
- \( F \) is a fantastic filter if and only if \( E_F(\neg \neg x \rightarrow x) = L \) for all \( x \in L \)
Theorem

Let $F$ be a filter on $L$, $t$ a term. Then $F$ is a $t$-filter if and only if $E_F(t(x)) = L$ for all $x \in L$.

Proof.

Let $x$ be an arbitrary element of $L$, $F$ be a $t$-filter. Since $F$ is a $t$-filter, then $t(x) \in F$, thus for every element $y$ of $L$ is $y \lor t(x) \in F$, thus $y \in E_F(t(x))$, i.e., $E_F(t(x)) = L$.

Conversely, if $E_F(t(x)) = L$, then $\bar{0} \in E_F(t(x))$, therefore $\bar{0} \lor t(x) = t(x) \in F$. QED
$l$-filters and Possible Generalizations

$l$-filters defined by Z. M. Ma and B. Q. Hu (2014) are just special cases of $t$-filters (…)

Possible Generalizations: replace the condition $t(\bar{x}) \in F$ by condition in form if $t_1(\bar{x}) \in F$ and $t_2(\bar{x}) \in F$ and …, then $t(\bar{x}) \in F$ and start dealing with quasivarieties.
Acknowledgement

Thank you for your attention!