Precise Tail Asymptotics of Fixed Points of the Smoothing Transform with General Weights.

Based on works by G. Alsmeyer, D. Buraczewski, E. Damek, K. Kolesko, S. Mentemeier, and J. Zienkiewicz

Let \( N > 1 \) be an integer, \( A_1, ..., A_N, B \) real valued random variables such that \( A_i \) are independent and identically distributed (i.i.d.). On the set \( P(\mathbb{R}) \) of probability measures on the real line the smoothing transform is defined as follows

\[
\mu \mapsto \mathcal{L}\left( \sum_{i=1}^{N} A_i R_i + B \right),
\]

where \( R_1, ..., R_N \), are i.i.d random variables with distribution \( \mu \), independent of \( (B, A_1, ..., A_N) \) and \( \mathcal{L}(R) \) denotes the law of the random variable \( R \). A fixed point of the smoothing transform is given by any \( \mu \in P(\mathbb{R}) \) such that, if \( R \) has distribution \( \mu \), the equation

\[
R = d \sum_{i=1}^{N} A_i R_i + B,
\]

holds true. If \( B = 0 \) a.s. the transform is called homogeneous, if not - nonhomogeneous.

The homogeneous equation (0.1) is used e.g to study interacting particle systems [5] or the branching random walk [6, 2]. In recent years, from very practical reasons, the inhomogeneous equation has gained importance. It appears e.g. in the stochastic analysis of the Pagerank algorithm (which is the heart of the Google engine) [7, 8, 13] as well as in the analysis of a large class of divide and conquer algorithms including the Quicksort algorithm [11, 12].

Properties of the fixed points of equation (0.1) are governed by the function

\[
m(s) = E\left[ \sum_{i=1}^{N} A_i^s \right] = NE[A_i^s]
\]

Suppose that \( s_1 = \sup\{s : m(s) < \infty \} \) is strictly positive. Clearly \( m \) is convex and differentiable on \((0, s_1)\). We assume that there are \( 0 < \gamma < \alpha < s_1 \) such that

\[
m(\gamma) = m(\alpha) = 1.
\]

Then

\[
0 < m'(\alpha) = E\left[ \sum_{i=1}^{N} A_i^\alpha \log |A_i| \right]
\]

and the latter quantity is finite. I am going to indicate the main steps of the proof of the following result

**Theorem 0.2** (D.Buraczewski, E.Damek, J.Zienkiewicz). Suppose that
- \( A_1 \) is nonlattice;
- \( s_1 > 0; \)
- there are \( 0 < \gamma < \alpha < s_1 \) such that \( m(\gamma) = m(\alpha) = 1; \)
- there is \( \varepsilon > 0 \) such that \( E|B|^\gamma + \varepsilon < \infty. \)
Suppose that $R$ is a solution to (0.1) such that $\mathbb{E}[|R|^{\gamma + \varepsilon}] < \infty$. Then
\[
\liminf_{t \to \infty} t^\alpha \mathbb{P}[|R| > t] > 0.
\]

Existence of such a solution implies $\gamma < 2$ for the nonhomogeneous case and $1 \leq \gamma < 2$ for the homogeneous one. Then the solution is basically unique (given the mean if it exists) and, if $\mathbb{E}[|B|^\alpha] < \infty$ then for every $s < \alpha$
\[
(0.3) \quad \mathbb{E}[|R|^s] < \infty.
\]
In view of the result of Jelenkovic and Olvera-Cravioto (Theorem 4.6 in [9]) Theorem 0.2 implies

**Corollary 0.4.** Suppose that the assumptions of Theorem 0.2 are satisfied and additionally let $\mathbb{E}[|B|^\alpha] < \infty$. Then
\[
(0.5) \quad \lim_{t \to \infty} t^\alpha \mathbb{P}[R > t] = K > 0.
\]

The existence of the limit in (0.5) for such $R$, in a more general case of random $N$, was proved by Jelenkovic and Olvera-Cravioto [9], Theorem 4.6, but from the expression for $K$, given by their renewal theorem, it is not possible to conclude its strict positivity except of the very particular case when $A_1, ..., A_N, B$ are positive and $\alpha \geq 1$.

**Some other results in this direction will be also mentioned:** [1, 3] and [4]. They correspond respectively to the situation of random $N$ [1], the case when $A_1, ..., A_N$ are matrices and $X_1, ..., X_N, B$ are vectors [3] and so called critical case $\gamma = \alpha, m'(\alpha) = 0, B \neq 0$ [4].

**References**


