On Fermat curves and maximal nodal curves

Mutsuo Oka
Tokyo University of Science

May 27, 2014, Institute of Mathematics, PAS
A nodal curve $C$ is an irreducible plane curve of degree $n$ which contains only nodes ($= A_1$ singularities). A nodal curve is called a maximal nodal curve if it is rational and nodal. By Plücker formula, it must contain $\frac{(n-1)(n-2)}{2}$ nodes to be maximal. In the space of polynomials of two variables, a maximal nodal curve can be understood as a generalization of a Chebycheff polynomial. In our paper [?], we constructed a maximal nodal curve of join type $f(x) + g(y) = 0$ using a Chebycheff polynomial $f(x)$ and a similar polynomial $g(y)$ that has one maximal value and two minimal values.
In this paper, we present another extremely simple way, for a given integer \( n > 2 \), to construct a maximal nodal curve \( D_{n-1} \) with a beautiful symmetry, as a bi-product of the geometry of the Fermat curve \( x^n + y^n + 1 = 0 \). A smooth point \( P \) of a plane curve \( C \) is called a flex point of flex-order \( k - 2 \), \( k \geq 3 \) if the tangent line \( T_P \) at \( P \) and \( C \) intersect with intersection multiplicity \( k \). The maximal nodal curve \( D_n \), which we construct in this paper, contains 3 flexes of flex-order \( n - 2 \) and it is symmetric with respect to the permutation of three variables \( U, V, W \).
By a special case of Zariski and Fulton ([?], [?]),
\[ \pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/n\mathbb{Z} \]
if \( C \) is a maximal nodal curve of degree \( n \).
The examples \( D_n \) provide an alternate proof. Zariski observed ([?])
that the fundamental group of the complement of an irreducible
curve \( C \) of degree \( n \) is abelian if \( C \) has a flex of flex-order either \( n \)
or \( n - 1 \). Since the moduli of maximal nodal curves of degree \( n \) is
irreducible by Harris ( [?]), the claim follows.
For the construction, we start from the Fermat curve \( \mathcal{F}_n : x^n + y^n + 1 = 0 \) and study singularities of the dual curve \( \tilde{\mathcal{F}}_n \). The Fermat curve and the dual curve \( \tilde{\mathcal{F}}_n \) have canonical \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) actions, thus the defining polynomial of \( \tilde{\mathcal{F}}_n \) is written as \( h(u^n, v^n) = 0 \) for a polynomial \( h(u, v) \) of degree \( n - 1 \). The curve \( h(u, v) = 0 \) defines our maximal nodal curve \( D_{n-1} \). Geometrically this is the quotient of the dual curve \( \tilde{\mathcal{F}}_n \) by the above action. Moreover the curve \( D_{n-1} \) is explicitly parametrized as

\[
D_{n-1} : \quad u(t) = t^{n-1}, \quad v(t) = (-1 - t)^{n-1}.
\]
We consider an irreducible plane curve $C$ of degree $n$, $C : f(x, y) = 0 \subset \mathbb{C}^2$. Its homogenization $F(X, Y, Z) = 0$ defines the projective curve $C$ of degree $n$ in $\mathbb{P}^2$ where $F(X, Y, Z) = f(X/Z, Y/Z)Z^n$. For a smooth point $P = (a, b, c) \in C$, the tangent line is defined by $F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0$ where $F_X, F_Y, F_Z$ are derivatives in the corresponding variables. The dual projective plane $\mathbb{P}^2$ has the dual coordinates $U, V, W$. In the dual projective plane $\mathbb{P}^2$, we usually work in the affine space $\{W \neq 0\}$ with the coordinates $(u, v)$ where $u = U/W$, $v = V/W$. The Gauss map associated with $C$ is defined by

$$G_F : C \to \mathbb{P}^2, \quad G_F(P) = (F_X(P) : F_Y(P) : F_Z(P)).$$

Thus in the affine coordinates $(x, y)$, $P = (x, y) \in C$ is mapped into $G_f(P) = (f_x(x, y) : f_y(x, y) : -xf_x(x, y) - yf_y(x, y)).$
**The class formula**

\[
\text{Class formula} 0 \hat{n} = n(n - 1) - \sum_{P \in \Sigma(C)} \left( \mu(C, P) + m(C, P) - 1 \right)
\]

If \( C \) is non-singular, we have \( \hat{n} = n(n - 1) \).

**Cyclic action** We assume that there exists a polynomial \( g(x, y) \) such that \( f(x, y) = g(x^m, y^s) \) \( m \geq s \) for some positive integers \( m, s \geq 2 \). Under this assumption, we consider the action on \( \mathbb{P}^2 \) of the product of cyclic groups \( \mathbb{Z}_{m,s} := \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/s\mathbb{Z} \), which is defined as follows. Let \( \omega_{\ell} := \exp(2\pi i/\ell) \) and we identify the cyclic group \( \mathbb{Z}/\ell\mathbb{Z} \) with the multiplicative subgroup of \( \mathbb{C}^* \) generated by \( \omega_{\ell} \). The action is defined by

\[
\psi : \mathbb{Z}_{m,s} \times \mathbb{P}^2 \to \mathbb{P}^2, \quad (\gamma, (x, y)) \mapsto (x \omega_{m}^j, y \omega_{s}^k), \text{ where } \gamma = (\omega_{m}^j, \omega_{s}^k)
\]

In the homogeneous coordinates, this action is written as

\[
(\gamma, (X : Y : Z)) \mapsto (X \omega_{n}^j : Y \omega_{s}^k : Z)
\]
Define the action of $\mathbb{Z}_{m,s}$ on the dual projective plane similarly:

$$\tilde{\psi}(\gamma, (u, v)) = (\omega^j_m u, \omega^k_s v) \text{ or,}$$

$$\tilde{\psi}(\gamma, (U : V : W)) = (\omega^j_m U : \omega^k_s V : W).$$

Then by an easy computation,

$$G_f(P) = (mx^{m-1}g_x(x^m, y^s) : sy^{s-1}g_y(x^m, y^s) :$$

$$-mx^mg_x(x^m, y^s) - sy^sg_y(x^m, y^s))$$

$$G_f(P^\gamma) = (m(\omega^j_m x)^{m-1}g_x(x^m, y^s) : s(\omega^k_s y)^{s-1}g_y(x^m, y^s) :$$

$$-mx^mg_x(x^m, y^s) - sy^sg_y(x^m, y^s))$$

$$= G_f(P)^{1/\gamma}$$
Proposition: The dual curve is invariant by the $\mathbb{Z}_{m,s}$-action. This implies that $\tilde{f}(u, v)$ can be written as $h(u^m, v^s)$ using some polynomial $h(u, v)$. Note that $h(u, v)$ is not the defining polynomial of the dual curve of $g(x, y) = 0$ in general. However we have the following fundamental result.

Theorem Let $C(g) := \{(x, y); g(x, y) = 0\} \subset \mathbb{P}^2$ and $D := \{(u, v); h(u, v) = 0\} \subset \tilde{\mathbb{P}}^2$. Then there exists a canonical birational mapping $\Phi_{m,s} : C(g) \to C(h)$. 

Mutsuo Oka Tokyo University of Science

On Fermat curves and maximal nodal curves
Proof

Let $\pi_{m,s}: \mathbb{P}^2 \rightarrow \mathbb{P}^2$, $\tilde{\pi}_{m,s}: \tilde{\mathbb{P}}^2 \rightarrow \tilde{\mathbb{P}}^2$ be the branched covering map defined by

$$
\pi_{m,s}(X : Y : Z) = (X^m : Y^s Z^{m-s} : Z^m), \quad \pi_{m,s}(x, y) = (x^m, y^s)
$$

$$
\tilde{\pi}_{m,s}(U : V : W) = (U^m : V^s W^{m-s} : W^m), \quad \tilde{\pi}_{m,s}(u, v) = (u^m, v^s)
$$

$\pi_{m,s}: C \rightarrow C(g)$ and $\tilde{\pi}_{m,s}: \tilde{C} \rightarrow C(h)$.

$$
\begin{align*}
C : f(x, y) = 0 & \quad \xrightarrow{G_f} \quad \tilde{C} : \tilde{f}(u, v) = 0 \\
\downarrow \pi_{m,s} & \quad & \downarrow \tilde{\pi}_{m,s} \\
C(g) : g(x, y) = 0 & \quad \xrightarrow{\Phi_{m,s}} \quad C(h) : h(u, v) = 0
\end{align*}
$$
Let us consider the multi-valued section of $\pi_{m,s}$:

$$
\lambda : \mathcal{C}(g) \to \mathcal{C}, \quad \lambda(x, y) = (x^{1/m}, y^{1/s}).
$$

The composition $\Phi_{m,s} := \tilde{\pi}_{m,s} \circ G_f \circ \lambda : \mathcal{C}(g) \to \mathcal{C}(h)$ is a well-defined single valued rational mapping and it does not depend on the choice of $\lambda$. In fact, it is given by:

$$
\Phi_{m,s}(x, y) = \left( \frac{m^m x^{m-1} g_x(x, y)^m}{(-m x g_x(x, y) - s y g_y(x, y))^m}, \frac{s^s y^{s-1} g_y(x, y)^s}{(-m x g_x(x, y) - s y g_y(x, y))^s} \right)
$$

Similarly we consider a (multi-valued) section $\tilde{\lambda} : \mathbb{C}^2 \to \mathbb{C}^2$, $\tilde{\lambda}(u, v) = (u^{1/m}, v^{1/s})$, of $\tilde{\pi}_{m,s}$ and the composition $\Psi_{m,s} = \pi_{m,s} \circ G \tilde{\lambda} : \mathcal{C}(h) \to \mathcal{C}(g)$,

$$
\Psi_{m,s}(u, v) = \left( \frac{m^m u^{m-1}(h_u(u, v))^m}{(-m u h_u(u, v) - s v h_v(u, v))^m}, \frac{s^s v^{s-1}(h_v(u, v))^s}{(-m u h_u(u, v) - s v h_v(u, v))^s} \right).
$$
It is easy to observe that $\Phi_{m,s}$, $\Psi_{m,s}$ satisfies $\Psi_{m,s} \circ \Phi_{m,s} = \text{id}_{C(g)}$ and $\Phi_{m,s} \circ \Psi_{m,s} = \text{id}_{C(h)}$ as the $G_{\tilde{f}}$ and $G_f$ are mutually inverse. For example, the equality $\Psi_{m,s} \circ \Phi_{m,s} = \text{id}_{C(h)}$ is shown as follows. Put $(x', y') := \lambda(x, y)$ and $(u, v) := G_f(x', y')$. Then

\[
\Psi_{m,s} \circ \Phi_{m,s}(x, y) = \pi_{m,s} \circ (G_{\tilde{f}} \circ \tilde{\lambda} \circ \tilde{\pi}_{m,s})(u, v)
\]

\[
= \pi_{m,s} \circ G_{\tilde{f}}((u, v)^\gamma) \quad \exists \gamma \in \mathbb{Z}_{m,s}
\]

\[
= \pi_{m,s} \circ (G_{\tilde{f}}(u, v))^{1/\gamma}
\]

\[
= \pi_{m,s}((x', y')^{1/\gamma})
\]

\[
= \pi_{m,s}(x', y') = (x, y)
\]
Singularities of the dual curves

We recall basic properties for the dual curve which we use later.

First case: Singularity from the singular points of $C$. Suppose that $P$ is a singular point of $C$. Then $G_f(P)$ is a singular point of $\tilde{C}$. The exceptional case is when the topological equivalence class of $(C, P)$ is $B_{k,k-1}$, $k \geq 3$. The Gauss image of $P$ is a flex point of flex-order $k - 2$. 

The locus of the flex points are described by
$\text{Hess}(F)(X, Y, Z) = F(X, Y, Z) = 0$ where $\text{Hess}(F)(X, Y, Z)$ is the hessian of $F$:

$$
\text{Hess}(F)(X, Y, Z) = \begin{vmatrix}
F_{XX} & F_{XY} & F_{XZ} \\
F_{YX} & F_{YY} & F_{YZ} \\
F_{ZX} & F_{ZY} & F_{ZZ}
\end{vmatrix}
$$

Thus by Bézout theorem, we have

$$
\#(\text{flex points}) = 3n(n - 2)
$$

where the number is counted with multiplicity.
There is another singularity which is produced from a special point of \( C \). There are two such special points: flex points (already described above) and points with multi-tangent lines. A smooth point \( P \in C \) gives a *multi-tangent line* if the tangent line \( T_P \) is also tangent to \( C \) at some other point \( Q \in C \), so \( T_Q = T_P \). The most common one is a *bi-tangent line*. If \( P \) is a bi-tangent point (so there is another point \( Q \in C \) so that \( I(C, T_P; Q) = 2 \) and any other intersections \( C \cap T_P \) are transverse), its image by the Gauss map is a node (i.e., \( A_1 \)). If it has \( q \)-tangent points, the image is topologically equivalent to a Brieskorn singularity \( B_{q,q} \). This singularity has \( q \) smooth local branches intersecting transversely.
The degree $\tilde{n}$ of $\tilde{C}$ is

$$\tilde{d} = n(n - 1) - \sum_P (\mu(P) + m(C, P) - 1)$$

Assuming that $C$ is smooth and flex points of $C$ are generic i.e., their flex-order are 3, and that $C$ has only bi-tangent lines, the classical formula tells that

$$\sharp(\text{bi-tangents}) = \frac{(\tilde{n} - 1)(\tilde{n} - 2)}{2} - 3n(n - 2) - \frac{(n - 1)(n - 2)}{2},$$

$$\tilde{n} = n(n - 1).$$

For more general situation

$$\frac{(\tilde{n} - 1)(\tilde{n} - 2)}{2} - \sum_{Q \in \Sigma(\tilde{C})} \delta(\tilde{C}, Q) = \frac{(n - 1)(n - 2)}{2} - \sum_{P \in \Sigma(C)} \delta(C, P)$$
We assume that $C$ is locally irreducible at $P$ and $C$ is parametrized as

$$x = x(t), \quad y = y(t), \quad |t| \leq 1$$

where $x, y$ are the affine coordinate $x = X/Z, y = Y/Z$. Then the local branch that is the image of the local irreducible germ $(C, P)$ has the parametrization at $G_f(P)$ (see [?], for example)

$$U(t) = y'(t), \quad V(t) = -x'(t), \quad W(t) = x'(t)y(t) - x(t)y'(t) \quad (1)$$

If $\tilde{C}$ is locally irreducible at $G_f(P)$, the above parametrization describes the local germ $(\tilde{C}, G_f(P))$. Equivalently in the affine coordinates $(u, v) = (U/W, V/W)$, the parametrization is given as

$$u(t) = \frac{y'(t)}{x'(t)y(t) - x(t)y'(t)}, \quad v(t) = \frac{-x'(t)}{x'(t)y(t) - x(t)y'(t)} \quad (2)$$
Geometry of Fermat curves

In this section, we study the Fermat curve of degree $n$:

$$\mathcal{F}_n : F(X, Y, Z) = X^n + Y^n + Z^n = 0.$$  

We denote the degree of the dual curve $\tilde{\mathcal{F}}_n$ by $\tilde{n}$. Note that $\tilde{n} = n(n - 1)$. There is an obvious $\mathbb{Z}_n$ acts on $\mathcal{F}_n$ and $\tilde{\mathcal{F}}_n$. Note that $\mathcal{F}_n$ has $3n$ flexes of flex-order $n - 2$ at

$$P_{1,j} := (0 : \xi_j : 1), \quad P_{2,j} := (\xi_j : 0 : 1), \quad P_{3,j} := (1 : \xi_j : 0),$$

$$j = 0, \ldots, n - 1$$

where $\xi_j = \exp((2j + 1)\sqrt{-1}/n)$

The tangent line at $P_{1,j}$ is defined by $y = \xi_j$ and it produces a $B_{n,n-1}$ singularity on $\tilde{\mathcal{F}}_n$ at $(U : V : W) = (0 : 1 : -\xi_j)$. The situation is exactly the same for other flexes through a permutation of coordinates.
Now we consider bi-tangent (or multi-tangent) lines on $\mathcal{F}_n$. The dual curve $\breve{\mathcal{F}}_n$ has genus $\frac{(n-1)(n-2)}{2}$ and $3n B_{n,n-1}$ singularities coming from flex points. Then by the formula (??), the number of the bi-tangent lines $\tau$ should be

$$\tau = \frac{(n-1)(n-2)}{2} - 3n \times \frac{(n-1)(n-2)}{2} - \frac{(n-1)(n-2)}{2}$$

$$= \frac{n^2(n-2)(n-3)}{2}$$
The Fermat curve $\mathcal{F}_n$ has $\frac{n^2(n-2)(n-3)}{2}$ bi-tangent lines.
Proof.

Let \( \omega := \exp(2\pi \sqrt{-1}/(n - 1)) \).
Suppose that \( P = (a, b), Q = (a', b') \in \mathcal{F}_n \) are bi-tangent points.
The tangent line at \( P \) is given by \( a^{n-1}x + b^{n-1}y + 1 = 0 \). Thus \( G_f(P) = (a^{n-1} : b^{n-1} : 1) \) and the assumption implies

\[
a^n + b^n + 1 = (a')^n + (b')^n + 1 = 0, \quad a^{n-1} = (a')^{n-1}, \quad b^{n-1} = (b')^{n-1}
\]

Thus we can write \( a' = a\omega^k, \quad b' = b\omega^j \) for some integers
0 < \( j, k < n - 1 \) and \( a^n(\omega^k - \omega^j) = (\omega^j - 1) \). As we assume that
\( P \neq Q \) and \( P, Q \in \mathbb{C}^2 \), we may assume that \( j \neq k \) and \( k, j \neq 0 \). Thus putting \( \beta_{j,k} := \frac{\omega^j - 1}{\omega^k - \omega^j} \), we get:

\[
a^n = \beta_{j,k}, \quad b^n = -1 - \beta_{j,k}, \quad a' = a\omega^k, \quad b' = b\omega^j
\]

for some \( 1 \leq j, k \leq n - 1, k \neq j \).
For any $0 < j < n - 1$, put $j_c = n - 1 - j$. Observe that $\omega^{-j} = \omega^{j_c}$. Put $\alpha_{j,k} := \frac{\omega^k - 1}{1 - \omega^j}$. Then we have

$$\beta_{j,k} = \frac{\omega^j - 1}{\omega^k - \omega^j} = \alpha_{(j-k)c,jc}$$

The complex number $\beta_{j,k}$, or equivalently $\alpha_{j,k}$, $1 \leq j, k \leq n - 1$ and $j \neq k$ are all distinct.
Let $\tilde{f}(u, v) = 0$ (and put $\tilde{F}(U, V, W)$ be its homogenization) be the defining affine (resp. homogeneous) polynomial of the dual curve where $u, v$ are affine coordinates defined by $u = U/W, v = V/W$. As $F_n$ is a symmetric polynomial with $\mathbb{Z}_{n,n}$ action, $\tilde{F}(U, V, W)$ is a symmetric polynomial of degree $n(n-1)$ with $\mathbb{Z}_{n,n}$ action. (Recall that $\mathbb{Z}_{n,n} = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.) Namely we can write that $\tilde{f}(u, v) = h(u^n, v^n)$ for some symmetric polynomial $h(u, v)$ of degree $n-1$. We have already observed that $\tilde{n} = n(n-1)$ and the singularities are $3n \ B_{n,n-1}$ singularities, and $n^2(n-2)(n-3)/2$ nodes i.e., $A_1$-singularities. On each coordinate axis $U = 0, V = 0$ and $W = 0$, there are exactly $n \ B_{n,n-1}$ singularities. The tangent line at $P_1j$ is defined by $y = \xi_j$ and its Gauss image is a $B_{n,n-1}$ singularity at $(U : V : W) = (0 : 1 : -\xi_j)$. 

Mutsuo Oka Tokyo University of Science On Fermat curves and maximal nodal curves
We will give an explicit construction of such a maximal nodal curve as an application of the Fermat curve. Let $\tilde{F}(U, V, W)$ be the defining polynomial of $\tilde{F}_n$ and write it as $\tilde{F}(U, V, W) = H(U^n, V^n, W^n)$ with $H(U, V, W)$ is a polynomial of degree $n - 1$. Then we consider the curve of degree $n - 1$ defined by $H(U, V, W) = 0$. We denote it as $D_{n-1}$. We claim that $D_{n-1}$ is a maximal nodal curve of degree $n - 1$. In fact, the rationality follows from the rationality of the line $L : x + y + 1 = 0$ and by Theorem ???. As $\tilde{F}_n$ has $n^2(n - 1)(n - 2)/2$ nodes outside of the union of coordinate axis $UVW = 0$ and they are invariant by the $(\mathbb{Z}/n\mathbb{Z})^2$ action. We consider $n^2$-fold branched covering $\tilde{\pi}_{n,n} : \mathbb{P}^2 \to \mathbb{P}^2$ as before. The image of $n^2(n - 2)(n - 3)/2$ nodes is now $(n - 2)(n - 3)/2$ nodes on $D_{n-1}$. Thus $D_{n-1}$ is maximal nodal.
The curve $D_{n-1}$ is a maximal nodal curve and is parametrized as follows.

$$D_{n-1} : u(t) = t^{n-1}, \quad v(t) = (-1 - t)^{n-1}$$

It has 3 flexes of flex-order $n - 1$ on each coordinate axis whose tangent lines are the coordinate axes. The defining polynomial $h(u, v)$ of $D_{n-1}$ is given by:

$$h(u, v) = \text{Resultant} (u - u(t), v - v(t), t).$$
References

J. Harris.
On the Severi problem.

M. Namba.
*Geometry of projective algebraic curves.*

M. Oka.
Geometry of cuspidal sextics and their dual curves.

R. J. Walker.
*Algebraic Curves.*