CHARACTERISTIC DISTRIBUTIONS
ON 4-DIMENSIONAL ALMOST COMPLEX MANIFOLDS

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Abstract. In this paper the Nijenhuis tensor characteristic distributions on a non-integrable four-dimensional almost complex manifold is investigated for integrability, singularities and equivalence.

1. Introduction. For a non-integrable four-dimensional almost complex manifold we will canonically define a distribution \( \Pi^2 \) by the Nijenhuis tensor \( N_J \). In Section 2 we complete the description [K1] of invariants of an almost complex structure in dimension four, using this distribution. In Sections 3–4 we describe singularities of \( \Pi^2 \). We show they are standard if our field of planes is considered as a distribution, but they become quite specific if it is considered as a differential system.

In Sections 5–6 we study moduli and hyperbolicity of the germ of a neighborhood of a pseudoholomorphic curve. Section 7 is devoted to a geometric meaning of the integrability of the Nijenhuis tensor characteristic distribution \( \Pi^2 \) and its relation to a question of V. Arnold.

In [HH] Hirzebruch and Hopf proved the following topological result: If a 4-dimensional manifolds admits a rank 2 distribution, it admits an almost complex structure as well. Moreover if the manifold admits two almost complex structures, defining opposite orientations, then it admits a rank 2 distribution.

We associate a rank 2 distribution to a non-integrable almost complex structure, realizing the above topological correspondence (to one side) canonically on the differential level. Note that any almost complex structure on a 4-dimensional manifold can be perturbed to be non-integrable outside a discrete set.

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2. Local classification of almost complex structures in dimension 4. Let $(M, J \in \text{Aut}(TM))$ be an almost complex manifold of dimension 4, $J^2 = -1$. Its Nijenhuis tensor is the following $(2,1)$-tensor

$$N_J \in \Lambda^2 T^* M \otimes TM, \quad N_J(\xi, \eta) = [J\xi, J\eta] - J[J\xi, \eta] - [\xi, J\eta] - [\xi, \eta].$$

Integrability of $J$ is expressed via it as $N_J = 0$ ([NW]).

This tensor satisfies the property $N_J(J\xi, \eta) = N_J(\xi, J\eta) = -JN_J(\xi, \eta)$ and so can be considered as an antilinear map $N_J : \Lambda^2 \mathbb{C}^2 \to \mathbb{C}^2$, $\mathbb{C}^2 = (T_x M^4, J)$. The image is invariant under $J$ and if $N_J \neq 0$ it is a complex line $\mathbb{C} \subset \mathbb{C}^2$.

Thus in the domain where the structure $J$ is non-integrable a canonical distribution is obtained:

**Definition 1.** We call $\Pi^2 = \text{Im} N_J \subset TM$ the *Nijenhuis tensor characteristic distribution* on a 4-dimensional almost complex manifold $(M^4, J)$.

This distribution $\Pi^2$ is in general situation non-integrable. Therefore it has a non-trivial derivative $\partial \Pi^2 = \partial \Pi^2$, which is defined as the differential system with $C^\infty (M)$-module of sections $\mathcal{P}_3 = C^\infty (\Pi^3)$ generated by the self-commutator of the submodule

$$\mathcal{P}_2 = C^\infty (\Pi^2) \subset D(M): \mathcal{P}_3 = [\mathcal{P}_2, \mathcal{P}_2].$$

$\Pi^3$ is not a distribution everywhere and its singularities form a stratified submanifold $\Sigma^2_1$ of codim = 2.

The distribution $\Pi^3$ on $M \setminus \Sigma^2_1$ is generically non-integrable, so that $\partial \Pi^3 = TM$ (or $[\mathcal{P}_2, \mathcal{P}_3] = D(M)$) outside a stratified submanifold $\Sigma^2_2$ of codim = 2.

If $x \notin \Sigma^2_1$ then $\Pi^2_x = \Pi^3_x$ has a transversal measure. In fact since the $J$-antilinear isomorphism $N_J(\cdot, \xi_3) : \Pi^2_x \to \Pi^2_x$ is orientation reversing, there exist vectors $\xi_1, \xi_2 \in \Pi^2_x$, $\xi_3 \in \Pi^3_x \setminus \Pi^2_x$ such that $N_J(\xi_1, \xi_3) = \xi_1$, $N_J(\xi_2, \xi_3) = -\xi_2$. These $\xi_1, \xi_2$ are defined up to multiplication by a constant, while $\xi_3 (\text{mod } \Pi^2_x)$ is defined up to multiplication by $\pm 1$. Therefore $\Pi^3/\Pi^2$ is normed. By a similar reason $T_x M/\Pi^3_x$ is normed outside $\Sigma^2_1$ via the vector $\xi_4 = J\xi_3$.

Note that $\Pi^3_x/\Pi^2_x$ is oriented. Actually $[\xi_1, \xi_2] (\text{mod } \Pi^2_x)$ depends only on the values of $\xi_1, \xi_2$ at the point $x$. It is a vector $f \xi_3 (\text{mod } \Pi^2_x)$ for some $f$. So if we require $\xi_2 = J\xi_1$ then $\xi_3$ can be chosen so that $f > 0$. This produces a coorientation on $\Pi^3_x \subset \Pi^2_x$ and then via $J$ a coorientation on $\Pi^2_x \subset T_x M$.

Moreover the requirement $f = 1$ determines canonically vector field $\xi_1$ (still however up to $\pm 1$) and hence $\xi_2 = J\xi_1$. Then we set $\xi_3 = [\xi_1, \xi_2]$ and $\xi_4 = J\xi_3$. So the pair $(\xi_1, \xi_2)$ is defined canonically up to a sign and the pair $(\xi_3, \xi_4)$ is absolutely canonical. The following statement generalizes Theorem 7 [K1]:

**Theorem 1.** Let an almost complex structure $J$ be of general position. Then at a generic point $x \in M^4$ the canonical frame $(\xi_1, \xi_2, \xi_3, \xi_4)$ is defined. It restores uniquely the almost complex operator $J$ and the tensor $N_J$ by the tables:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$JX$</th>
<th>$N_J(1, -)$</th>
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<th>$\xi_2$</th>
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The condition Im $\{e\}$-structure solves completely the equivalence problem. The idea is as follows. Consider the moduli of the problem, i.e. functions $c_{jk}^i$ given by the formula $[\xi_j, \xi_k] = \sum c_{jk}^i \xi_i$. Denote by $A = \{c_{jk}^i\}$ the space of all invariants and by $\Phi : M \to A$ the “momentum map” $x \mapsto \{c_{jk}^i(x)\}$. Then two equivalent structures have the same images and the equivalence follows. See [S] for more details.

3. Singularities of a Nijenhuis tensor characteristic distribution. A distribution $V = V_1$ is called completely non-holonomic if one of its successive derivatives $V_i = \partial V_{i-1}$ equals the whole tangent bundle $TM$ and the minimal such $i = r$ is called the degree of non-holonomy (can vary from point to point). The growth vector of a distribution at a point $x \in M$ is the sequence of the dimensions $(rk_1 V_1, \ldots, rk_r V_r(x))$.

Generically a Nijenhuis tensor characteristic distribution is completely non-holonomic outside a discrete subset in $M$. In an open dense set the growth vector is $(2, 3, 4)$. Then it is an Engel distribution, which has the following local normal form ([$E$]):

$$\Pi^2 = (\xi_1 = \partial_3, \xi_2 = \partial_4 - x_3 \partial_2 - x_2 \partial_1); \quad \partial_i := \partial/\partial x_i.$$ 

Locally this $\Pi^2$ can be realized as a Nijenhuis tensor characteristic distribution ([K2]). In fact, consider two transversal symmetries of the distribution: $\eta_1 = \partial_1, \eta_2 = \partial_2 - x_4 \partial_1$. Define the almost complex structure by the formula

$$J\xi_1 = \varphi \xi_2, \quad J\eta_1 = \eta_2; \quad \varphi \neq 0.$$ 

Then one easily checks that $\text{Im } N_J = \Pi^2$ whenever $(\partial_{\eta_1} \varphi)^2 + (\partial_{\eta_2} \varphi)^2 \neq 0$.

Moreover the following statement holds:

**Proposition 2.** Let $\Pi$ be an analytic distribution of rank 2 in $\mathbb{R}^4$. Then it can be locally realized as a Nijenhuis tensor characteristic distribution.

**Proof.** Let $\Pi^2$ be generated by $\xi_1 = \partial_3$ and $\xi_2 = \partial_4 + h_1 \partial_1 + h_2 \partial_2$. A pair of generators can always be chosen in such a form. Consider $\xi_2$ as a vector field in $\mathbb{R}^3(x_1, x_2, x_4)$ depending on a parameter $x_3$. It has two independent symmetries $\eta_1, \eta_2 \in D(\mathbb{R}^3)[\eta_1, \xi_2] = 0$. Let us differentiate these fields by the parameter: $\partial_3 \eta_i = (\partial_3, \eta_i) = a_i^j \eta_j + b_i \xi_2$.

Define the almost complex structure by the formula

$$J\xi_1 = \varphi \xi_2, \quad J\eta_1 = \alpha \eta_1 + \beta \eta_2; \quad \beta, \varphi \neq 0.$$ 

The condition $\text{Im } N_J = \Pi^2$ is equivalent to the system

$$\begin{align*}
\varphi \partial_{\xi_1} \alpha &= \alpha \partial_{\xi_1} \alpha - \frac{1 + \alpha^2}{\beta} \partial_{\xi_1} \beta + \left[ a_1^i (1 + \alpha^2) - a_1^2 \frac{1 + \alpha^2}{\beta} + a_2^i \alpha \beta - a_2^2 (1 + \alpha^2) \right] \\
\varphi \partial_{\xi_2} \beta &= \beta \partial_{\xi_1} \alpha - \alpha \partial_{\xi_1} \beta + \left[ a_1^i \alpha \beta + a_1^2 (1 - \alpha^2) + a_2^i \beta^2 - a_2^2 \alpha \beta \right]
\end{align*}$$

and the inequality $(\partial_{\eta_1} \varphi - b_1 \alpha - b_2 \beta)^2 + (\partial_{\eta_2} \varphi - b_1 \frac{1 - \alpha^2}{\beta} + b_2 \alpha)^2 > 0$. The system is in the Cauchy-Kovalevskaya form and so possesses a local solution. After this the inequality is arranged to hold.

**Theorem 3.** Nijenhuis tensor characteristic distributions in the domain of non-integrability for $J$ have the same singularities as the usual two-dimensional distributions in $\mathbb{R}^4$. 

Note that reducing a geometric structure to a frame ($\{e\}$-structure) solves completely the equivalence problem. The idea is as follows. Consider the moduli of the problem, i.e. functions $c_{jk}^i$ given by the formula $[\xi_j, \xi_k] = \sum c_{jk}^i \xi_i$. Denote by $A = \{c_{jk}^i\}$ the space of all invariants and by $\Phi : M \to A$ the “momentum map” $x \mapsto \{c_{jk}^i(x)\}$. Then two equivalent structures have the same images and the equivalence follows. See [S] for more details.
Proof. Let us first define the degeneration locus of a distribution. Introduce the partial order on the growth vectors: \((m_1, \ldots, m_s) \leq (n_1, \ldots, n_r)\) iff \(s \geq r\) and \(m_i \leq n_i\) for \(i = 1, \ldots, r\). Fix one growth vector \(I\). Then the degeneration locus \(\Sigma_I \subset \mathcal{M}\) is the set of points with the growth vector less than or equal to \(I\). Proposition 2 (it holds formally as well—on the jets of the structure) and the Thom transversality theorem imply that for a typical \(J\) the sets \(\Sigma_I\) are nice subvarieties, stratifying the manifold \(\mathcal{M}\). The statement follows.

The generic degenerations of two-plane fields in \(\mathbb{R}^4\), up to codimension 3, were classified by Zhitomirskii [Z]. Let us show how generic codimension 2 singularities are realized as a Nijenhuis tensor characteristic distribution.

There are two different types of such singularities, defined by the growth vectors \(I_1 = (2, 2, 4)\) and \(I_2 = (2, 3, 3, 4)\). All other growth vectors are subordinated to these two and hence the singular set is

\[
\Sigma = \Sigma_1^2 \cup \Sigma_2^2, \quad \Sigma_i^2 = \Sigma_i.
\]

Generically the loci \(\Sigma_i^2\) are smooth 2-dimensional submanifolds \([Z]\), which intersect non-transversally along a curve \(\Sigma_1^1\). There is also a curve \(\Sigma_2^1 \subset \Sigma_2^2\) separating the locus into the elliptic/hyperbolic parts \(\Sigma_2^2\).

The codimension 2 loci of \(\Pi^2 = (\xi_1, \xi_2)\) have the following normal forms:

\[
\Sigma_1^2 \setminus \Sigma_1^1: \quad \xi_1 = \partial_3, \quad \xi_2 = \partial_4 - x_3 x_4 \partial_2 - x_3^3 \partial_1;
\]
\[
\Sigma_2^2_{+}: \quad \xi_1 = \partial_3, \quad \xi_2 = \partial_4 - \left(\frac{1}{3} x_3^3 + x_3 x_4^2\right) \partial_2 - x_3 \partial_1;
\]
\[
\Sigma_2^2_{-}: \quad \xi_1 = \partial_3, \quad \xi_2 = \partial_4 - x_3 x_4 \partial_2 - x_3 \partial_1.
\]

In each of these cases the choice \(n_1 = \partial_1, n_2 = \partial_2\) and formula (2) will lead to realization \(\Pi^2 = \text{Im} \, N_J\). The cases of higher degenerations are studied similarly.

4. Singularities of \(\Pi = \text{Im} \, N_J\) as of a differential system. As differential systems Nijenhuis tensor characteristic distributions have singularities different from those of the usual differential systems in \(\mathbb{R}^4\); The rank of a Nijenhuis tensor characteristic distribution is even and so is 2 or 0.

**Proposition 4.** For a generic structure \(J\) the set where \(N_J = 0\) (the rank of \(\Pi\) falls to zero) is a discrete set \(\Sigma^0 \subset \mathcal{M}^4\). For each point of \(\Sigma^0\) there is a centered coordinate neighborhood \((x_1, y_1, x_2, y_2)\) around it such that the almost complex structure is given by the formula

\[
J \partial_{x_i} = \alpha_i \partial_{x_i} + (1 + \beta_i) \partial_{y_i}, \quad J \partial_{y_i} = \frac{-1 + \alpha_i^2}{1 + \beta_i} \partial_{x_i} - \alpha_i \partial_{y_i}, \quad i = 1, 2,
\]

where the functions \(\alpha_i, \beta_i\) are of the second order of smallness at the origin.

**Proof.** Singularities of the differential system \(\Pi = \text{Im} \, N_J\) are given by the vector equation \(N_J(\xi, \eta) = 0\) for some \(J\)-independent vector fields \(\xi, \eta\), and so are generically isolated points given by the integrability condition \(N_J = 0\).

To get the other claim recall ([K1]) that an almost complex structure can be approximated by a complex structure to the second order of smallness at the integrability points.
Let \((w_1, w_2)\) be the corresponding complex coordinates. By a theorem of Nijenhuis and Woolf [NW] (see also Proposition 9 below) there are two \(J\)-holomorphic foliations by disks \(C^1\)-close to the foliations \(\{w_i = \text{const.}\}\) at the origin, \(i = 1, 2\). Let \(z_1\) be a complex coordinate on the disk of the first family passing the origin and \(z_2\)—on the second. They define the complex coordinate system \((z_1, z_2)\) in a neighborhood of the origin with the required properties. ■

**Remark 1.** For \(\dim M > 4\) the set where \(N_J = 0\) is generically empty.

Let \(\alpha^i_2, \beta^i_2\) be the quadratic parts of \(\alpha_i, \beta_i\). Using the coordinate system from Proposition 4 we calculate: \(\Pi^2 = \text{Im} N_J = \langle \xi_1, \xi_2 = J\xi_1 \rangle\), where linearizations of the generators at the origin are

\[
\xi_1^0 = \left( -\frac{\partial \beta^1_2}{\partial x_2} - \frac{\partial \alpha^2_1}{\partial y_2} \right) \partial x_1 + \left( \frac{\partial \alpha^1_2}{\partial x_2} + \frac{\partial \beta^2_1}{\partial y_2} \right) \partial y_1 + \left( \frac{\partial \beta^2_1}{\partial x_1} - \frac{\partial \alpha^1_2}{\partial y_1} \right) \partial x_2 + \left( \frac{\partial \beta^1_2}{\partial y_1} - \frac{\partial \alpha^2_1}{\partial x_1} \right) \partial y_2
\]

and \(\xi_2^0 = J_0 \xi_1^0\) (\(J_0\) is the constant coordinate extension of \(J\) from the origin).

Thus we see that the linearization of the considered differential system is special, not as for the usual differential systems. If we consider linear vector fields \(\xi_i^0\) as linear operators, we represent the first order approximation of \(\Pi\) by a two-dimensional subspace \(V^2 \subset \text{gl}(4)\). The condition \(V^2 = \langle X_1, X_2 = JX_1 \rangle\) for some \(J^2 = -1\) characterizes admissible 2-planes and thus linearizations. The higher order terms in \(\xi_1, \xi_2\) are special as well.

**5. Moduli of a PH-curve neighborhood.** Let \(C^2\) be a pseudoholomorphic (PH-) curve, i.e. a surface with \(J\)-invariant tangent bundle. At every point \(x \in C\) we have two \(J\)-invariant planes \(T_x C^2\) and \(\Pi^2_x\), which generically intersect by zero, except at a finite number of points \(\Sigma_0 \subset C\). The sets \(\Sigma'_1 = \Sigma_1 \cap C\) and \(\Sigma'_2 = \Sigma_2 \cap C\) are generically finite as well. The arrangement of all these points

\[
\Sigma' = \Sigma'_0 \cup \Sigma'_1 \cup \Sigma'_2 \subset C
\]

gives a (finite-dimensional) invariant of \(C\).

For points \(x \in C \setminus \Sigma'_1\) we define field of directions \(L^1 = TC \cap \Pi^3\). The integral curves of this 1-distribution foliate the set \(C \setminus \Sigma'_1\) and in general \(C\) foliates with only non-degenerate singular points. Denote the number of elliptic points by \(e(L^1)\) and the number of hyperbolic points by \(h(L^1)\). One can prove:

**Proposition 5.** Under \(C^1\)-small perturbation of the structure \(J\) the foliation \(L^1\) has minimal number of singularities: \(\min \{ e(L), h(L) \} = 0\), \(\max \{ e(L), h(L) \} = |\chi(C)|\). For instance if \(C = T^2\) we get a foliation without singularities.

Due to Section 2 the foliation \(L^1\) is oriented, cooriented and has parallel and transverse measures outside \(\Sigma'\). Thus there exist canonical vector fields \(v_1\) along \(L^1\) and \(v_2 = Jv_1\) transverse to it. Consequently the curve \(C\) has a lot of dynamical invariants like winding classes of \(v_1\) and \(v_2\). Moreover, decomposing

\[
[v_1, v_2] = \gamma_1 v_1 + \gamma_2 v_2,
\]

we obtain two invariant (under pseudoholomorphic isomorphisms) functions \(\gamma_1, \gamma_2\). These together with the germs of the functions \(c^{ij}_{jk}\) from Section 2 form *moduli* of the \(C\)-neigh-
neighborhoods germ. They solve the equivalence problem for PH-embeddings $C^2 \to M^4$ (of general position).

**Example.** Let $M = T^2(\varphi, \psi) \times \mathbb{R}^2(x, y)$ be equipped with the structure

$$
J \partial_x = \partial_y; \quad J \partial_\varphi = \frac{2 - \rho y^2}{2} \partial_\psi + \frac{y^2}{2} \partial_\varphi + x \partial_x;
$$

$$
J \partial_y = -\partial_x; \quad J \partial_\psi = \frac{4 + y^4}{2\rho y^2 - 4} \partial_\varphi - \frac{y^2}{2} \partial_\psi + \frac{xy^2}{\rho y^2 - 2} \partial_x + \frac{2x}{\rho y^2 - 2} \partial_y.
$$

Then $\mathcal{C} = \{x = y = 0\}$ is a PH-torus and the winding number of $v_1$ is $\rho$. Similarly one shows the other considered invariants are non-trivial.

6. **Hyperbolicity of a PH-curve neighborhood.** In this section we consider the case of PH-tori $\mathcal{C} = T^2$. We assume for simplicity that the normal bundle is topologically trivial, though in general case the result is the same.

Recall that the Kobayashi pseudometric $d_M$ measures the distance between points via pseudoholomorphic disks ([Ko, KO]). An almost complex manifold is called Kobayashi hyperbolic if $d_M$ is a metric. Let $\| \cdot \|$ be a norm on $TM$.

**Proposition 6.** Let $\mathcal{O}$ be a small neighborhood of a pseudoholomorphic torus $T^2 \subset (M^4, J)$. Then the domain $\mathcal{O} \setminus T^2$ is not Kobayashi-hyperbolic.

Moreover, for some constant $C > 0$ and any $R > 0$ there exists a smooth family of PH-disks $f^R_\alpha : D_R \to \mathcal{O}$, with uniformly bounded norms $\|(f^R_\alpha)_*(z)\| \leq C$ satisfying $\|(f^R_\alpha)_*(0)\| = 1$, that fills some smaller neighborhood $\mathcal{O}' \subset \mathcal{O}$ of $T^2$:

$$
\mathcal{O}' \subset \bigcup_\alpha f^R_\alpha(D_R).
$$

**Proof.** Let us take the universal covering $\hat{\mathcal{O}} \simeq \mathbb{C} \times D^2$ of $\mathcal{O}$. The torus is covered by the entire line $\mathbb{C} \to T^2$. Changing the structure $J$ at infinity in $\hat{\mathcal{O}}$ and near the boundary to the integrable one we glue the manifold to the product $S^2 \times S^2$ with the line $\mathbb{C}$ being glued to the first factor $S^2_1$. Then the introduction of the taming symplectic product structure $\omega = \omega_1 \oplus \omega_2$ yields a foliation of $S^2_1 \times S^2_2$ by PH-spheres $S^2$ in the homology class of the first factor if we additionally demand that the homology class $[S^2_1]$ of the first sphere-factor is symplectically indecomposable (for example, if $\omega_1(S^2_1) = k \omega_2(S^2_2)$, $k \in \mathbb{N}$). Here we use the fact that the dimension is 4: due to positivity of intersections [M1] we actually have a foliation ([M2]).

This foliation of $S^2 \times S^2$ gives a family of big PH-disks on $\hat{\mathcal{O}}$ parametrized by the radius $R$ of disk in $\mathbb{C}$ out of which the almost complex structure is changed. The estimates follow from the Brody reparametrization lemma as in [KO]. Pulling-back we get the required family.

We now consider filling by pseudoholomorphic cylinders $\mathcal{C}_R = [-R; R] \times S^1 \subset \mathbb{C} \setminus \{0\}$, which is topologically different from the disk-filling (Fig. 1).

**Proposition 7.** In the statement of Proposition 6 we can change disks $D_R$ to the cylinders $\mathcal{C}_R$ and get for every $R > 0$ a filling family of PH-cylinders $f^R_\alpha : \mathcal{C}_R \to \mathcal{O}$ with
uniformly bounded norms and normalization \( \|(f^R_\alpha)_*(0)\| = 1 \):

\[
\mathcal{O}' \subset \bigcup \alpha f^R_\alpha(C_R).
\]

*Proof.* Actually take a covering of the neighborhood \( \mathcal{O} \) which corresponds to one cycle of the torus. The torus is covered by the entire cylinder \( C_\infty \to T^2 \). We can change the almost complex structure \( J \) at infinity so that it makes possible to “pinch” each end of the cylinder. This means we perturb the structure \( J \) so that it is standard integrable outside some \( C_{R_2} \subset C_\infty \) and the support is also a big cylinder \( C_{R_1} \). Then we glue the ends to the disks. This operation gives us a sphere \( S^2 \) instead of the cylinder \( C_\infty = \mathbb{R} \times S^1 \). We can also assume that neighborhoods of two cylinder ends are pinched (Fig. 2).

Thus we have a neighborhood \( U \) of the sphere \( S^2_0 \). It is foliated by PH-spheres close to \( S^2_0 \). Actually, we can change the structure \( J \) near the boundary of this neighborhood, glue and get the manifold product \( \tilde{M} = S^2 \times S^2 \). As before it is foliated by PH-spheres. Thus \( U \) is foliated by PH-spheres and in the preimage they give a PH-foliation by cylinders. ■
Remark 2. Neighborhoods of PH-spheres \(\mathcal{C} = S^2\) are also non-hyperbolic and if the normal bundle is topologically trivial can be foliated by close PH-spheres. Small neighborhood of PH-curves of higher genus \(\mathcal{C} = S^2_g, g > 1\), are Kobayashi hyperbolic.

Moreover the above non-hyperbolicity for the cases \(g = 0, 1\) can be strengthened as follows: Even the punctured neighborhoods \(\mathcal{O} \setminus \mathcal{C}\) are non-hyperbolic.

7. Arnold’s question. In [A2] (1993-25) Arnold asks about almost complex version for his Floquet-type theory of elliptic curves neighborhoods ([A1]) in the spirit of the Moser’s KAM-type theorem ([Mo]). Namely he asks if a germ of neighborhood \(\mathcal{O}\) of a PH-torus \(\mathcal{C} = T^2 \subset (M^4, J)\) is determined by its normal bundle \(N\mathcal{C} M\).

The following result is a direct consequence of the definition:

**Proposition 8.** If \(F : M^4 \to \mathbb{C}^2\) is a (local) PH-surjection and the structure \(J\) is non-integrable, then the Nijenhuis tensor characteristic distribution \(\Pi^2\) is integrable and is tangent to the fibers of \(F\).

Thus there is a functional obstruction to the equivalence of the \(C\)-germ in \(M^4\) and of the \(C\)-germ in the normal bundle (we do not discuss here the normal bundle: If \(\dim M = 4\), the almost complex structure on \(N\mathcal{C} M\) can be obtained via linearization along a family of transversal PH-disks; for the general case see [K3]). Integrability and transversality of \(\Pi^2\) to the torus \(C\) is a necessary, but by no means sufficient condition for the existence of an equivalence: There are other functional moduli.

In search of a proper generalization of Arnold’s result we notice that a neighborhood of an elliptic curve in a complex surface is foliated by half-infinite cylinders: They are given as \(|z| = \text{const.}\) in the representation of the neighborhood as \(\mathbb{C}^2/(z,w)/(z,w) \sim (z+2\pi, w) \sim (z+\nu, \lambda w)\), where \(\nu \in \mathbb{C} \setminus \mathbb{R}\) and \(\lambda \in \mathbb{C} \setminus \{0\}\) (see [A1] for the representation).

The hypothesis is then that for a non-integrable perturbation \(J\) of the complex structure \(J_0\) most of the cylinders persist (as in Moser’s theory).

Let us sketch how to prove existence of one such a half-cylinder. In Proposition 7 we have constructed a pre-compact family of finite cylinders \(f_R\) for different \(R\). If it winds up to the curve \(C\) (as in the holomorphic normal form with \(|\lambda| \neq 1\)), then one can extract a subsequence \(f_{R_k}\) with \(R_k \to \infty\) converging to a pseudoholomorphic curve due to the standard technique ([G, MS]). This is the required half-cylinder.

There are no tools however to complete this construction to a PH-foliation (also a filling is problematic—a remark of V. Bangert). Note though that even if we construct a foliation, it is not necessarily so nice as its holomorphic original. To explain this let us notice the following fact, which is a corollary of a theorem by Nijenhuis and Woolf [NW]:

**Proposition 9.** A small neighborhood \(\mathcal{O}\) of a PH-curve \(\mathcal{C} \subset M^4\) can be foliated by transversal PH-disks \(D^2\).

Now consider a neighborhood of a PH-curve \(\mathcal{C}\) with topologically trivial normal bundle and suppose we have a foliating family \(f_\alpha : \mathcal{B} \to \mathcal{O}\) with unbounded or compact leaves in it. Let \(D_{\varphi}, \varphi \in \mathcal{C}\), be the family of normal disks from Proposition 9. Then every path \(\gamma(t)\) on \(\mathcal{C}\) with \(\gamma(0) = \varphi_0, \gamma(1) = \varphi_1\) gives a mapping \(\Phi_\gamma : D_{\varphi_0} \to D_{\varphi_1}\) of shift along the leaves of \(f_\alpha\). For a loop \(\gamma\) we have an automorphism of \(D_{\varphi}\). Since \(f_\alpha\) is a foliation there
is no local holonomy: $\Phi_\gamma = \text{id}$ for contractible loops $\gamma$. Thus we can consider the map $\pi_1(\mathcal{C}) \to \text{Aut}(D_\phi)$.

**Definition 2.** We call $\Phi_\gamma \in \text{Aut}(D_\phi)$ the **monodromy map** along $\gamma \in \pi_1(\mathcal{C})$ and $\Phi_\gamma : D_{\phi_0} \to D_{\phi_1}$ the **transport map**.

For example there is no monodromy for the sphere $\mathcal{C} = S^2$ and each choice of local coordinates in a normal disk $D_{\phi_0}$ gives coordinates for the others $D_{\phi_1}$.

Let now $\mathcal{C} = \mathbb{T}^2(2\pi, \nu)$ and we have a foliating family $f_\alpha$ of half-infinite cylinders. Since every leaf $\mathcal{B}$ is a cylinder, there is no monodromy along one generating cycle. Normalize it to be the cycle $\varphi \mapsto \varphi + 2\pi$. Denote by $\Phi_\nu$ the monodromy along the other cycle $\varphi \mapsto \varphi + \nu$.

Unlike the complex case, the almost complex monodromy can be non-holomorphic mapping of the fibers: It is possible to construct examples of PH-foliations with any prescribed monodromy $\Phi_\nu$.

Moreover even if the monodromy is complex, the transport maps $\Phi_\gamma : (D_{\phi_0}, J) \to (D_{\phi_1}, J)$ can be non-complex. In fact there are functional obstructions for the transports to be complex:

**Theorem 10.** Let $\mathcal{C}$ be a PH-curve in a 4-dimensional manifold $(M, J)$ and let $f_\alpha : \mathcal{B} \to \mathcal{O}$ be a local PH-foliating family in some neighborhood $\mathcal{O}$ of $\mathcal{C}$. Then if all transport maps $\Phi_\gamma$ are holomorphic, then the Nijenhuis tensor characteristic distribution $\Pi^2$ is integrable and is tangent to the leaves of $f_\alpha$.

**Proof.** Actually this is because the foliation provides a local bundle $\pi : \mathcal{O} \to D_\phi$ and so Proposition 8 applies. ■

Again the integrability is not a sufficient condition: There are other moduli.

So we see that the existence of foliating PH-family with complex transports (as in the original holomorphic case) is generically obstructed, and the obstructions are of the same nature as for the existence of equivalence between a germ of a neighborhood of a PH-curve $\mathcal{C}$ and its normal bundle (though in the first case the Nijenhuis tensor characteristic distribution is tangent to the curve $\mathcal{C}$, while in the second one it is transversal).

**References**


