Measure rigidity for some transcendental meromorphic functions

Agnieszka Badeńska

Faculty of Mathematics and Information Science
Warsaw University of Technology
Pl. Politechniki 1
00-661 Warszawa
Poland
badenska@mini.pw.edu.pl

Abstract
We consider hyperbolic meromorphic functions of the form $f(z) = R \circ \exp(z)$, where $R$ is a non-constant rational function, satisfying so-called rapid derivative growth condition. We study several types of conjugacies in this class and prove a measure rigidity theorem in the case when $f$ has a logarithmic tract over $\infty$ and under some additional assumptions.

1 Introduction
Let $f: \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function of the form

$$f(z) = R(e^z), \quad (1.1)$$

where $R$ is a non-constant rational function. This family of functions was studied by many authors (see e.g. [7] for references). We are interested in the case, when $f$ is hyperbolic, $\infty$ is an asymptotic value and $f$ does not belong to an exceptional class which will be specified in Definition 1.2.

With these assumptions we prove the following rigidity result.

**Theorem 1.1** Suppose that $f$ and $g$ are two non-exceptional hyperbolic functions of the form $(1.1)$ with $\infty$ being an asymptotic value. Let $h: J(f) \to J(g)$ be a homeomorphism conjugating $f$ to $g$, namely $h \circ f = g \circ h$ on $J(f) \setminus f^{-1}(\infty)$. Then the following conditions are equivalent.

1. $h$ extends to an affine conjugacy from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ between $f: \mathbb{C} \to \overline{\mathbb{C}}$ and $g: \mathbb{C} \to \overline{\mathbb{C}}$.
2. $h$ extends to a conformal homeomorphism conjugating $f$ and $g$ on neighbourhoods of $J(f)$ and $J(g)$ in $\mathbb{C}$.
3. $h$ extends to a real-analytic diffeomorphism conjugating $f$ and $g$ on neighbourhoods of $J(f)$ and $J(g)$ in $\mathbb{C}$.
4. $h$ and $h^{-1}$ are Lipschitz continuous.
5. $h$ preserves the moduli of multipliers of periodic cycles in $J(f)$, i.e. if $f^p(z) = z$, then $|(f^p)'(z)| = |(g^p)'(h(z))|$.

In fact we will prove a stronger result, Theorem 1.3, so-called measure rigidity and obtain Theorem 1.1 as its straightforward corollary. In order to state it we need to clarify the context and present some elements of thermodynamical formalism. Before we do this, let us introduce the following definition of an exceptional function.

**Definition 1.2** We call a hyperbolic meromorphic function $f: \mathbb{C} \to \overline{\mathbb{C}}$ of the form $(1.1)$ exceptional if one of the following occurs.

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1. $f$ has poles and admits a continuous, $f$-invariant line field on the Julia set $J(f)$,

2. $f$ is of the form $f(z) = be^{dz} + a$ with $b, a \in \mathbb{C}$, $b \neq 0$, $d \geq 1$ and $\log|f'|$ is cohomologous to a constant on $\mathbb{C} \setminus Q$ via a function $u(z) = h(z) + c\text{Arg}(z-a)$, where $Q$ is a piecewise smooth ray emanating from the asymptotic value $a$, $h$ is a non-constant harmonic function in $\mathbb{C}$ and $c \in \mathbb{C} \setminus \{0\}$.

As we will see in the following sections, for exceptional functions we are not able to prove a measure rigidity Theorem 1.3. However, we conjecture that the exceptional class is actually empty (see the comment at the end of the article).

Note that functions of the form (1.1) are of finite order $\rho(f) = 1$ and belong to the class of maps considered by Mayer and Urbaniński (see [9, 10]), who introduced a new approach to develop the theory of thermodynamical formalism for hyperbolic transcendental functions having a rapid derivative growth.

They associate with a given meromorphic function $f$ a suitable metric of the form

$$d\tau(z) = \frac{|dz|}{|z|^\tau}, \quad \tau > 0,$$

assuming that $0$ belongs to the Fatou set $F(f)$ (cf. [10, (3.2)]), and use the derivatives of $f$ with respect to the metric $d\tau$ instead of the Euclidean or spherical metric. Next, using Nevanlinna’s theory, they obtain estimates for a transfer operator associated with so-called tame potential $\phi$ and a formula for a topological pressure $P(\phi)$. This in turn allows to construct a probability measure $m_f$, which is $e^{P(\phi) - \phi}$-conformal, and a probability $f$-invariant measure $\mu_f$ equivalent to $m_f$.

For a hyperbolic function $f$ of the form (1.1) we consider tame geometric potentials

$$\phi = -t_f \log|f'|\tau.$$

We set the parameter $t_f = \text{HD}(J_r(f))$ and the Bowen’s formula [10, Theorem 8.3] implies that the topological pressure $P(\phi)$ equals zero. Moreover, since $\infty$ is an asymptotic value, $f$ has a logarithmic tract over $\infty$, consequently $\text{HD}(J_r(f)) > 1$ (see [2]). If we now take $f$ and $g$ and potentials as just described, then we can choose a common parameter $\tau > 0$ such that $\max(\frac{1}{t_f}, \frac{1}{t_g}) < \tau < 1$ and apply results from [10] to obtain conformal measures $m_f$ and $m_g$ for $f$ and $g$ respectively. With this notion we prove the following measure rigidity result.

**Theorem 1.3** If $f$ and $g$ are non-exceptional hyperbolic functions of the form (1.1) with $R(\infty) = \infty$ and $m_f, m_g$ are conformal measures given by Theorem 2.6 for parameters $t_f = \text{HD}(J_r(f))$, $t_g = \text{HD}(J_r(g))$, then the conditions 1.–5. from Theorem 1.1 are equivalent to the following.

6. $h$ transports the measure class of $m_f$ to the measure class of $m_g$.

Let us mention, that a rigidity theorem, in the similar form to the above result, was proved by Sullivan in [16] for holomorphic expanding repellers and by Przytycki and Urbaniński in [12] for tame rational functions. Recently it was also extended by Kotus and Urbaniński in [6] to the class of regular pseudo non-recurrent elliptic functions. The concept of our paper is mainly based on their ideas but some substantial changes are necessary. First of all we consider derivatives of $f$ with respect to the metric $d\tau$. It leads to different estimates comparing with [6], where the conformal measure is defined with respect to the spherical metric. Secondly, the functions in our class may have finite asymptotic values what is not possible for elliptic functions. Finally, our class includes entire functions.

The plan of this paper is the following. In the second section we recall basic definitions and state some results, which we need later on. In the third section we state and prove a technical Theorem 3.2, which is an important tool in proving the main results of this paper. As a byproduct we obtain in Lemma 3.7 the absence of continuous invariant line fields (the precise definition comes in section 3) supported on the Julia set for transcendental entire functions from class $\mathcal{B}$. The fourth section contains the proof of Theorem 1.3. Finally, in the last section we give some comments concerning assumptions of Theorems 1.1 and 1.3 and state some questions.
2 Preliminaries

Throughout the entire paper by dist we denote the Euclidean distance on the complex plane \( \mathbb{C} \). For any set \( A \subset \mathbb{C} \) and a radius \( r > 0 \) we denote \( B(A, r) = \{ z \in \mathbb{C} : \text{dist}(A, z) < r \} \), while \( B(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \} \). HD(A) is used for the Hausdorff dimension of a set \( A \), and, as usually, \( \overline{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \). If a measure \( \mu_1 \) is absolutely continuous with respect to another measure \( \mu_2 \), it will be denoted by \( \mu_1 \ll \mu_2 \).

The Fatou set \( F(f) \) of a meromorphic function \( f : \mathbb{C} \rightarrow \overline{\mathbb{C}} \) is the set of points for which there exists a neighbourhood, where all iterates \( f^n \) are defined and form a normal family. The Julia set \( J(f) \) is the complement of \( F(f) \) in \( \overline{\mathbb{C}} \) while \( J(f) = J(f) \cap \mathbb{C} \). For a general description of the dynamics of meromorphic functions and more references see e.g. [3]. The Julia set splits into two dynamically different subsets, i.e. the radial Julia set which in our (hyperbolic) case is given by

\[
J_r(f) = \{ z \in J(f) : \liminf_{n \rightarrow \infty} |f^n(z)| < \infty \}
\]

and its complement which for the considered class of functions essentially coincides with the escaping set \( I_\infty(f) = J(f) \setminus J_r(f) \) consisting of points escaping to infinity under iterates of \( f \) (cf. [10, Remark 4.1]). We are conscious that in general the radial Julia set is defined differently, however we do not want to introduce unnecessary definitions and refer to [13] if one is interested in this topic.

We consider transcendental meromorphic functions \( f : \mathbb{C} \rightarrow \overline{\mathbb{C}} \) of the form (1.1). As we mentioned at the beginning these functions have finite order \( \rho(f) = 1 \). To see this, recall the characterization of the order in terms of Borel sums (cf. [5, 11], see also [10, Theorem 3.5]):

\[
\Sigma(u, a) = \sum_{z \in f^{-1}(a) \setminus \{0\}} |z|^{-u}.
\]

The function \( f \) is of finite order \( \rho(f) \) if for all \( a \in \overline{\mathbb{C}} \), except for (at most two) Picard exceptional values, \( \Sigma(u, a) \) converges for \( u > \rho(f) \) and diverges for \( u < \rho(f) \). Moreover, functions of the form (1.1) are of divergent type, since

\[
\Sigma(\rho(f), a) = \infty.
\]

By \( \text{Sing}(f^{-1}) \) we denote the set of singular values in \( \mathbb{C} \). This set consists of critical values \( f(C_\text{rit}(f)) \) and finite asymptotic values. The functions of the form (1.1) have at most two asymptotic values \( R(0), R(\infty) \) and we use notation \( \text{AS}(f) = \{ R(0), R(\infty) \} \cap \mathbb{C} \). Note that the functions under consideration belong to the so-called Speiser class

\[
S = \{ f : \text{Sing}(f^{-1}) \text{ is a finite set} \}.
\]

As a consequence of this fact, their Fatou set contains no wandering domain and no Baker domain (cf. [3]).

We assume about \( f \) that \( \infty \) is one of the asymptotic values. Conjugating if necessarily \( f \) by \( h(z) = -z \) we can actually assume that \( R(\infty) = \infty \) or equivalently

\[
\deg(P) > \deg(Q) \quad \text{where} \quad f(z) = R(e^z) \quad \text{with} \quad R = \frac{P}{Q}.
\]

Having this we easily see that \( f \) has the logarithmic singularity over \( \infty \), hence, as it was proved in [2],

\[
\text{HD}(J_r(f)) > 1.
\]

This result can be also obtained as a consequence of the Bowen’s formula [10, Theorem 8.3], since a hyperbolic function \( f \) satisfying (2.1) has a balanced derivative growth (cf. Lemma 2.3) with \( \min a_2 = 1 \).

The post-singular set \( \mathcal{P}(f) \) is defined as follows

\[
\mathcal{P}(f) = \bigcup_{n \geq 0} f^n(\text{Sing}(f^{-1})).
\]
We will also need the following set
\[ \mathcal{P}(f) = \mathcal{P}(f) \cup \text{Crit}(f) \cup f^{-1}(\infty). \]

In our paper we always assume that \( f \) is hyperbolic, i.e.
\[ \mathcal{P}(f) \subset F(f) \quad (2.3) \]
and let us denote
\[ \delta_f = \frac{1}{4} \text{dist} (J(f), \mathcal{P}(f)) > 0. \quad (2.4) \]
Note that it is also expanding, i.e. there are constants \( c > 0 \), \( \gamma > 1 \) such that
\[ |(f^n)'(z)| \geq c \gamma^n \quad \text{for all} \quad z \in J(f) \setminus \bigcup_{j=1}^{n} f^{-j}(\infty). \quad (2.5) \]

For functions of the form (1.1) with \( R(\infty) = \infty \) it follows directly from (2.3) and Lemma 2.3 as a consequence of [10, Proposition 4.4].

The Fatou set of a hyperbolic function is never empty, so without loss of generality we can assume that \( 0 \in F(f) \) (otherwise it suffices to conjugate \( f \) by a translation). In this case \( 0 \) is attracted by some attracting periodic cycle in \( F(f) \). There exists thus \( T_f > 0 \) such that
\[ B(0, T_f) \cap J(f) = \emptyset. \]
This allows us to formulate derivative growth conditions in the following convenient form (cf. [10, (2.4), (2.5)]).

**Definition 2.1** A meromorphic function \( f \) has a rapid derivative growth if there exist constants \( \alpha_2 > 0 \), \( \alpha_1 > -\alpha_2 \) and \( \kappa > 0 \) such that
\[ |f'(z)| \geq \kappa^{-1} |z|^{\alpha_1} |f(z)|^{\alpha_2} \quad (2.6) \]
for all \( z \in J(f) \setminus f^{-1}(\infty) \).

**Definition 2.2** A meromorphic function \( f \) has a balanced derivative growth if there exists a bounded function \( \alpha_2: J(f) \to [\inf \alpha_2, \sup \alpha_2] \subset (0, \infty) \) and constants \( \alpha_1 > -\inf \alpha_2 \), \( \kappa \geq 1 \) such that
\[ \kappa^{-1} |z|^{\alpha_1} |f(z)|^{\alpha_2(z)} \leq |f'(z)| \leq \kappa |z|^{\alpha_1} |f(z)|^{\alpha_2(z)} \quad (2.7) \]
for all \( z \in J(f) \setminus f^{-1}(\infty) \).

Note that the balanced derivative growth condition (2.7) implies rapid derivative growth with the exponent \( \alpha_2 = \inf \alpha_2(z) \).

We want to show that hyperbolic functions of the form (1.1) with \( R(\infty) = \infty \), assuming that \( 0 \in F(f) \), satisfy the above conditions. Note that hyperbolicity and \( 0 \in F(f) \) imply
\[ |f'|_{J(f)} \geq C > 0, \quad |f|_{J(f)} \geq T_f > 0 \quad \text{and} \quad \text{AS}(f) \subset F(f). \]
Consider first an entire function \( f: \mathbb{C} \to \mathbb{C} \) of the form (1.1). Then actually \( f(z) = P(e^z) \), where \( P \) is a polynomial. Consequently for \( z \in J(f) \),
\[ |f'(z)| = \frac{|P'(e^z) e^z|}{|P(e^z)|} = |P'(e^z)| = |P(e^z)|, \]
thus \( f \) is balanced with \( \alpha_1 = 0 \) and \( \alpha_2 \equiv 1 \).

If we now take \( f: \mathbb{C} \to \mathbb{C} \) of the form (1.1) that has poles, then in a neighbourhood \( U_b \) of a pole \( b \) of multiplicity \( q \) we have for \( z \in J(f) \cap U_b \setminus \{ b \} \),
\[ |f(z)| \asymp |z - b|^{-q} \quad \Rightarrow \quad |f'(z)| \asymp |z - b|^{-q-1} \asymp |f(z)|^{1+\frac{1}{q}}. \]
Note that $f$ has a finite number of periodic families of poles. Let us denote $f(z) = \frac{P(e^z)}{Q(e^z)}$ for two polynomials $P$ and $Q$ with $\deg(P) > \deg(Q)$ and we see that for $z \in J(f) \setminus \bigcup_{b \in f^{-1}(\infty)} U_b$, 

$$|f'(z)| = \left| \frac{P'(e^z)Q(e^z) - P(e^z)Q'(e^z)}{P(e^z)Q(e^z)} \right| e^z |f(z)|,$$

since the degree of the polynomial $P'Q - PQ'$ necessarily equals $\deg(P) + \deg(Q) - 1$. So again $\alpha_1 = 0$, but this time $\alpha_2$ is no longer a constant function, it varies depending on the multiplicities of poles of $f$. Finally note that the latter case $(\alpha_2(z) = 1)$ must occur for some sequence $z_n \in J(f)$ tending to $\infty$, since $f$ has a logarithmic tract over $\infty$. Therefore in particular $\min \alpha_2(z) = 1$.

Thus we have proved the following fact.

**Lemma 2.3** Every hyperbolic function $f: \mathbb{C} \to \mathbb{T}$ of the form (1.1) with $R(\infty) = \infty$ and $0 \in F(f)$, satisfies the balanced derivative growth condition (2.7) with $\alpha_1 = 0$ and $1 \leq \alpha_2(z) \leq 1.5$.

For every $\tau > 0$ let

$$|f'(z)|_\tau = |f'(z)| \frac{|z|^\tau}{|f(z)|^\tau}$$

be the derivative of $f$ with respect to the metric $d\tau$ described in the introduction. We will need the following distortion lemma proved in [10, Lemma 4.8].

**Lemma 2.4** Let $f: \mathbb{C} \to \mathbb{T}$ be a hyperbolic meromorphic function and let $\delta_f > 0$ be defined by (2.4). For every $\tau > 0$ there exists a constant $K_\tau \geq 1$ such that for every integer $n \geq 0$, every $w \in J(f)$, every $z \in f^{-n}(w)$ and all $x, y \in B(w, \delta_f)$, we have that

$$K_\tau^{-1} \leq \frac{|(f^{-n})'(x)|_\tau}{|(f^{-n})'(y)|_\tau} \leq K_\tau,$$

where $f^{-n}$ denotes the inverse branch of $f^n$ defined near $f^n(z)$ and mapping $f^n(z)$ back to $z$.

For a hyperbolic function $f$, having the rapid derivative growth (2.6), we use terminology from [10] and consider geometric tame potentials $\phi: J(f) \setminus f^{-1}(\infty) \to \mathbb{R}$ of the form

$$\phi(z) = -\tau f \log |f'(z)|_\tau,$$

where $\tau \in (0, \alpha_2)$ is chosen so that

$$t_f > \frac{\rho(f)}{\alpha_1 + \tau} > \frac{\rho(f)}{\alpha_1 + \alpha_2}.$$  \hspace{1cm} (2.8)

Recall that in the considered case $\rho(f) = 1$, $\alpha_1 = 0$ and $\alpha_2 = \min \alpha_2(z) = 1$ so in fact $\tau$ could be any number greater than $\frac{1}{t_f}$ and smaller than $1$. In the following sections we will consider two functions $f$ and $g$ satisfying the assumptions described above. As we mentioned in the introduction, we take the parameters of the potential 

$$t_f = \text{HD}(J_r(f)) \quad \text{and} \quad t_g = \text{HD}(J_s(g)).$$

Since both the numbers are greater than $1$, we can choose a positive $\tau$ so that $\max(\frac{1}{t_f}, \frac{1}{t_g}) < \tau < 1$. Then (2.8) is satisfied for both $f$ and $g$ and we can apply the thermodynamical formalism developed in [10].

Recall the definition of a conformal measure.

**Definition 2.5** A probability measure $m_f$ is called $pe^{-\phi}$-conformal if for every Borel set $E \subset J(f)$ such that $f|_E$ is injective, we have

$$m_f(f(E)) = \int_E pe^{-\phi} dm_f.$$
One of the main results of [10, Theorem 5.15] is the following theorem.

**Theorem 2.6** If \( f : \mathbb{C} \to \mathbb{C} \) is a hyperbolic meromorphic function of finite order satisfying the rapid derivative growth condition (2.6), then for every geometric tame potential \( \phi \), with the parameters \( t, \tau \) specified in (2.8),

1. There exists a unique \( g \)-\( e^{-\phi} \)-conformal measure \( m_f \) and \( \rho = e^{P(\phi)} \), where \( P(\phi) \) is the topological pressure of \( \phi \). Also, there exists a unique probability measure \( \mu_f \) which is \( f \)-invariant and equivalent to \( m_f \).

2. Both measures \( m_\phi \) and \( \mu_\phi \) are ergodic and supported on the radial Julia set \( J_r(f) \).

3. The density \( \rho_\phi = \frac{d\mu_\phi}{dm_\phi} \) is a nowhere vanishing, continuous and bounded function on \( J(f) \).

Moreover, the hyperbolic dimension of \( f \), i.e. \( \text{HD}(J_r(f)) \), is the unique zero of the pressure function (cf. [10, Theorem 8.3]). Therefore, our choice of parameters guaranties that \( P(\phi) = 0 \) and consequently \( m_f \) is simply \( e^{-\phi} \)-conformal:

\[
m_f(f(E)) = \int_E |f'(z)|^t \, dm_f(z)
\]

for every \( E \subset J(f) \) such that \( f|_E \) is injective (and analogously for \( g \) and \( m_g \)).

We will need one more property of \( m_\phi \) from [10, Proposition 5.22].

**Lemma 2.7** There exists an \( M > 0 \) such that for \( m_\phi \)-a.e. \( z \in J(f) \) we have

\[
\liminf_{n \to \infty} |f^n(z)| \leq M.
\]

The following fact, proved in [1], will be used several times in the proof of Theorem 1.3.

**Theorem 2.8** If \( f : \mathbb{C} \to \mathbb{C} \) is a hyperbolic meromorphic function of finite order satisfying the rapid derivative growth condition such that \( \text{HD}(J(f)) > 1 \) and \( \phi \) is a geometric tame potential, then the Jacobian \( D_{\mu_\phi} = \frac{d\mu_\phi}{dm_\phi} \) has a real analytic extension on a neighbourhood of \( J(f) \setminus f^{-1}(\infty) \) in \( \mathbb{C} \).

Let us state some results, which will be important tools in our paper. Recall first the classical distortion theorem due to Koebe.

**Theorem 2.9 (Koebe’s Distortion Theorem)** For every point \( z_0 \in \mathbb{C}, r > 0, t \in [0,1) \) and any univalent analytic function \( H : B(z_0, r) \to \mathbb{C} \) we have

\[
\frac{1 - t}{(1 + t)^3} \leq \frac{|H'(z)|}{|H'(z_0)|} \leq \frac{1 + t}{(1 - t)^3},
\]

where \( z \in B(z_0, tr) \).

Thus for some \( k > 0 \) we have the following result.

**Lemma 2.10** Suppose \( D \subset \mathbb{D} \) is an open set, \( z \in D \) and \( H : D \to \mathbb{C} \) is an analytic map, which has an analytic inverse \( H_z^{-1} \) defined on \( B(H(z), 2R) \) for some \( R > 0 \). Then for every \( 0 \leq r \leq R \)

\[
B(z, k^{-1}r|H'(z)|^{-1}) \subset H_z^{-1}(B(H(z), r)) \subset B(z, kr|H'(z)|^{-1}).
\]

We will also need a geometrical version of the Koebe’s Distortion Theorem involving conformal modulus of an annulus

**Theorem 2.11** There exists a function \( w : (0, \infty) \to [1, \infty) \) such that for any two open topological disks \( Q_1 \subset Q_2 \) with \( \text{Mod}(Q_2 \setminus Q_1) \geq t \), and any univalent analytic function \( H : Q_2 \to \mathbb{C} \), we have:

\[
\sup\{|H'(\xi)| : \xi \in Q_1\} \leq w(t) \inf\{|H'(\xi)| : \xi \in Q_1\}.
\]
Finally, let us state the following result proved in \cite[Lemma 4.4]{12} and \cite[Lemma 1.5]{6}, which we will need in our proofs.

**Lemma 2.12** Let $\mu$ and $\nu$ be Borel probability measures on $Y$, a bounded subset of an Euclidean space. Suppose that there exists a constant $M > 0$ and for every point $x \in Y$ we can find a decreasing to zero sequence of positive radii \( \{r_j(x) : j \geq 1 \} \) such that for every $x \in Y$ and all $j \geq 1$,

\[
\mu(B(x, r_j(x))) \leq M \nu(B(x, r_j(x))).
\]

Then the measure $\mu$ is absolutely continuous with respect to $\nu$ and the Radon-Nikodym derivative $\frac{d\mu}{d\nu} \leq CM$, where $C$ is a universal constant depending only on the dimension of the Euclidean space under consideration.

### 3 Nonlinearity

The main goal of this section is to prove Theorem 3.2. The idea is based on the proof of \cite[Theorem 6.5]{6}, however some substantial changes have to be done, so let us present it with details. We begin with the following lemma.

**Lemma 3.1** Every function $f$ of the form (1.1) has infinitely many fixed points in the Julia set $J(f)$.

*Proof.* Suppose that $f$ has at most finitely many fixed points, which equivalently means that the function $g(z) = f(z) - z$ assumes 0 at most finitely many times. Then, by Iversen’s theorem, 0 is an asymptotic value for $g$. This is however impossible. Indeed, take the asymptotic path to $\infty$ along which $g$ tends to 0. Then obviously along the same path $f$ tends to $\infty$ which means that $\infty$ must be the asymptotic value of $f$. But the only asymptotic values of $f$ are $R(0)$ and $R(\infty)$, where $f = R \circ \exp$. Moreover, along each asymptotic path for $f$ the real part of $z$ tends to $\pm \infty$, thus $|f(z)|$ grows exponentially fast which means that $g(z)$ would still tend to $\infty$ on such a path and not to 0.

Thus, by Picard’s Theorem, $g$ assumes 0 infinitely many times, which means that $f$ has infinitely many fixed points. Since $f$ has only finite number of singular values, the classification of Fatou components (cf. \cite{3}) implies that at most finitely many fixed points belong to the Fatou set $F(f)$. \(\square\)

Recall that the parameters $\tau, t_f, t_g > 0$ are chosen according to the condition (2.8). Therefore, as a consequence of Theorem 2.6, there exist for $f$ and $g$ Borel probability measures $m_f$ and $m_g$ respectively, which are conformal in the sense of Definition 2.5 and additionally $P(\phi) = 0$. Moreover, there exist probability invariant measures $\mu_f$ and $\mu_g$ equivalent to $m_f$ and $m_g$ respectively.

Before we state and prove Theorem 3.2, let us recall after \cite{6} that $u : Y \to S^1$, $Y \subset \mathbb{C}$, is an $f$-invariant line field on $Y$ if for all $x \in Y \cap f^{-1}(Y)$,

\[
u(f(x)) = \left( \frac{f'(x)}{|f'(x)|} \right)^2 u(x).
\]

We will also use the following definition: log $|f'|$ is continuously cohomologous to a locally constant function on the Julia set if there exists a continuous function $u : J(f) \to \mathbb{R}$ and a locally constant function $c : J(f) \setminus f^{-1}(\infty) \to \mathbb{R}$ such that

\[
\log |f'(z)| = c(z) + u(z) - u(f(z))
\]

for all $z \in J(f) \setminus f^{-1}(\infty)$.

**Theorem 3.2** Let $f$ be of the form (1.1). Under the assumptions of Theorem 1.3 none of the following statements is true.

(a) The Jacobian $D_{\mu_f} : J(f) \setminus f^{-1}(\infty) \to (0, \infty)$ is locally constant.
(b) The function \( \log |f'|: J(f) \setminus f^{-1}(\infty) \to \mathbb{R} \) is cohomologous to a locally constant function on \( J(f) \setminus f^{-1}(\infty) \) in the class of continuous functions on \( J(f) \).

c) There exists a continuous \( f \)-invariant line field on \( J(f) \).

d) For every \( n \geq 1 \) and every point \( z \in J(f) \setminus \bigcup_{j=1}^{n+1} f^{-j}(\infty) \)
\[
\det (\nabla (D_{\mu_f} \circ f^n)(z), \nabla (D_{\mu_f})(z)) = 0
\]

The structure of the proof is as in [6, Theorem 6.5]. In a sequence of lemmas (some of them in a more general context) we establish the following implications \((a) \Rightarrow (b)\) and \((d) \Rightarrow (a) \lor (c)\) and we show that both \((b)\) and \((c)\) lead to exceptional cases stated in Definition 1.2.

**Lemma 3.3** For any function \( f \) satisfying the rapid derivative growth condition (2.6) and measures \( m_f, \mu_f \) as in Theorem 2.6, the following implication is true. If the Jacobian \( D_{\mu_f} \) is locally constant, then \( \log |f'| \) is cohomologous to a locally constant function on \( J(f) \setminus f^{-1}(\infty) \) in the class of continuous functions on \( J(f) \).

**Proof.** Recall that, by Theorem 2.6, \( \rho_f = \frac{d\mu_f}{dm_f} \) is a continuous nonvanishing function on \( J(f) \). Moreover
\[
D_{\mu_f} = \rho_f \circ f \circ e^{P(t_f)}|f'| t_f^{\rho_f} \mu_f^{-1},
\]
where by \( P(t_f) \) we denote the topological pressure of the potential \( \phi = -t_f \log |f'| \). Therefore
\[
\log D_{\mu_f} = \log \rho_f \circ f + P(t_f) + t_f \log |f'| \tau - \log \rho_f.
\]
Hence for \( z \in J(f) \setminus f^{-1}(\infty) \)
\[
\log D_{\mu_f}(z) = \log \rho_f(f(z)) + P(t_f) + t_f \log |f'(z)| + \log |z| \tau - \log |f(z)| \tau - \log \rho_f(z)
\]
and modifying it we get
\[
\log |f'(z)| = \frac{\log D_{\mu_f}(z) - P(t_f)}{t_f} + \left( \frac{\log \rho_f(z)}{t_f} - \log |z| \tau \right) - \left( \frac{\log \rho_f(f(z))}{t_f} - \log |f(z)| \tau \right).
\]
If we denote \( c(z) = \frac{\log D_{\mu_f}(z) - P(t_f)}{t_f} \) and \( u(z) = \frac{\log \rho_f(z)}{t_f} - \log |z| \tau \), we obtain
\[
\log |f'(z)| = c(z) + u(z) - u(f(z)),
\]
where \( u \) is continuous on \( J(f) \) and, by \((a)\), \( c \) is a locally constant function on \( J(f) \setminus f^{-1}(\infty) \). \( \square \)

Note that this is exactly the implication \((a) \Rightarrow (b)\). Moreover, in our case \( P(t_f) = 0 \), so the calculations are even easier.

**Lemma 3.4** Under the assumptions of Theorem 1.3, if \( \log |f'| \) is cohomologous to a locally constant function on \( J(f) \setminus f^{-1}(\infty) \) in the class of continuous functions on \( J(f) \), then \( f \) is necessarily of the form \( f(z) = be^{az} + a \) and it belongs to the exceptional class 2. in the sense of Definition 1.2.

**Proof.** The proof of this part is rather long and technical. First we introduce a geometrical “spider” \( Q \subset \mathbb{C} \) containing all singular values as its endpoints. Next, we describe the construction of a harmonic function \( \tilde{u}_Q : \mathbb{C} \setminus Q \to \mathbb{R} \), which realizes a cohomology between \( \log |f'| \) and a locally constant function on \( \mathbb{C} \setminus (Q \cup f^{-1}(Q)) \). Finally, in a sequence of claims we show that the assertions of the lemma can be satisfy only in the exceptional case.

Let \( N := 2 \mathbb{Z}(\text{Sing}(f^{-1})) \) and denote singular values by \( s_1, \ldots, s_N \). We introduce the following definition in order to simplify the construction of the proof.
Definition 3.5 A spider $Q$ based on a point $x \in \mathbb{C}$ is a sum of closed polygonal arcs $\gamma_1, \ldots, \gamma_{N+1}$, consisting of finitely many line segments, with the following properties:

(i) For every $i = 1, \ldots, N$, the arc $\gamma_i$ is compact in $\mathbb{C}$ with endpoints $x$ and $s_i$.

(ii) The arc $\gamma_{N+1}$ is unbounded with one endpoint $x$.

(iii) For all $i \neq j$, $1 \leq i, j \leq N + 1$, we have $\gamma_i \cap \gamma_j = \{x\}$.

Choose a point $x \in \mathbb{C} \setminus \bigcup_{n \geq 0} f^{-n}(\mathcal{P}(f))$ and consider a spider $Q = \bigcup_{n=1}^{N+1} \gamma_n$ based on $x$. By Lemma 3.1 the function $f$ has infinitely many fixed points in $J(f)$. Take one of them and denote it by $w$. Fix also an arbitrary point

$$\xi_0 \in J(f) \setminus (f^{-1}(\infty) \cup Q).$$

Note that $\mathbb{C} \setminus Q$ is an open, simply connected set, containing no singular values, therefore, for every $\xi_1 \in f^{-1}(\xi_0)$, there exists a unique holomorphic inverse branch $f_{\xi_1}^{-1}: \mathbb{C} \setminus Q \to \mathbb{C}$ of $f$, mapping $\xi_0$ to $\xi_1$. Since the function $f$ is $2\pi i$-periodic, $f_{\xi_1}^{-1}(\mathbb{C} \setminus Q)$ is contained in some fundamental domain $G$ for the exponential function (i.e., $G$ is closed with $\exp(G) = \mathbb{C}$ and $\exp \mathrm{Int}(G)$ is injective). Moreover, for every $k \in \mathbb{Z}$, $\xi_1 + 2k\pi i$ is also a preimage of $\xi_0$ and $f_{\xi_1+2k\pi i}^{-1}(\mathbb{C} \setminus Q) \subset G + 2k\pi i$. Therefore, since the postsingular set $\mathcal{P}(f)$ is bounded, we can choose $\xi_1 \in f^{-1}(\xi_0)$ so that the corresponding fundamental domain $G$ has positive distance from the postsingular set.

For $R > 0$ let

$$A_R := \begin{cases} B(\xi_0, R) \setminus (Q \cup B(a, 1/R)) & \text{if } f \text{ has a finite asymptotic value } a, \\ B(\xi_0, R) \setminus Q & \text{otherwise} \end{cases}$$

(recall that $f$ has at most one finite asymptotic value). Now, let $R$ be so large, say $R \geq R_0$, that $A_R$ is an open topological disk and consider the set $f_{\xi_1}^{-1}(A_R)$. Since $A_R$ is disjoint from a neighbourhood of infinity and asymptotic values, therefore $f_{\xi_1}^{-1}(A_R)$ is a bounded set. In every fundamental domain $G$ there are finitely many poles, so we have that

$$\text{dist}(f_{\xi_1}^{-1}(A_R), f^{-1}(\infty)) > 0.$$

Moreover, by our choice of $\xi_1$, $f_{\xi_1}^{-1}(A_R)$ has a positive distance from the postsingular set $\mathcal{P}(f)$.

Denote by $V$ the unbounded connected component of $\mathbb{C} \setminus f_{\xi_1}^{-1}(A_R)$. It is an open, connected set (simply connected in $\mathbb{C}$), whose boundary is a piecewise smooth Jordan curve contained in $\partial(f_{\xi_1}^{-1}(A_R))$. There also exists an $r > 0$ such that

$$B(\mathcal{P}(f), r) \subset V.$$

Thus, for every $s \in (0, r]$, if we denote

$$W_s := B(\mathbb{C} \setminus V, s) \supset B(f_{\xi_1}^{-1}(A_R), s),$$

then $W_s$ is a topological disk disjoint from $\mathcal{P}(f)$. On $W_s$ we can therefore define all inverse branches of $f^k$, $k \geq 1$. Choose a sequence $\{\xi_n\}_{n \geq 1}$ of consecutive preimages of $\xi_0$ such that $\xi_n \in f^{-n}(\xi_0)$, $f(\xi_{n+1}) = \xi_n$ for $n \geq 1$ and

$$\lim_{n \to \infty} \xi_n = w,$$

where $w$ is the fixed point of $f$ chosen at the beginning of the proof. For every $n \geq 2$, denote by $\psi_n: W_s \to \mathbb{C}$ the unique holomorphic inverse branch of $f^{n-1}$ mapping $\xi_1$ to $\xi_n$. We will show that the following fact is true.

Claim 1. The series

$$\sum_{n=2}^{\infty} \log |f'(\xi_n)| - \log |f'(\psi_n(z))| \quad (3.2)$$

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converges uniformly on $W_2$ to a harmonic function.

**Proof.** Since $\xi_0 \in J(f)$, also $\xi_n \in J(f)$ for every $n \geq 1$, therefore, using the expanding property of $f$, we get that

$$|\psi'_n(\xi_1)| \leq C \gamma^{-(n-1)}$$

for constants $C > 0$, $\gamma > 1$ and all $n \geq 1$. Note that $W_2 \subset W_3$ are open topological disks with $\text{Mod}(W_3 \setminus W_2) > 0$. It follows from Theorem 2.11 that there is a constant $K > 0$ such that

$$|\psi'_n(z)| \leq K C \gamma^{-(n-1)} \quad \text{for all } z \in W_2.$$ (3.3)

We conclude that the sequence of functions $\psi_n$ converges uniformly on $W_2$ to the constant function equal $w$.

Take now arbitrary $l > k \geq 1$. By the Chain Rule

$$\sum_{j=k+1}^l \log |f'(\xi_j)| - \log |f'(\psi_j(z))| = \left| \log \frac{|\psi_k'(\xi_k)|}{|\psi_k'(\psi_k(z))|} \right|,$$

where $\psi_k: \psi_k(W_0) \to \mathbb{C}$ is the holomorphic inverse branch of $f^{k-1}$ mapping $\xi_k$ to $\xi_l$. Since $|\psi_k(z) - \xi_k|$ converges uniformly to 0 for $z \in W_2$, therefore Theorem 2.9 implies that the quotient of derivatives of $\psi_k$ is arbitrarily close to 1 if only $k$ and $l$ are big enough. But this means that partial sums of the series (3.2) form a Cauchy sequence and it therefore converges uniformly to a harmonic function. This finishes the proof of Claim 1. \(\square\)

By the assumption (b) there exists a continuous function $u: J(f) \to \mathbb{R}$ and a locally constant function $c: J(f) \setminus f^{-1}(\infty) \to \mathbb{R}$ such that the condition (3.1) holds. For $z \in A_R$ let us define

$$u_R(z) = u(\xi_0) + \log |f'(\xi_1)| - \log |f'(f_{\xi_1}^{-1}(z))| + \sum_{n=2}^{\infty} \left( \log |f'(\xi_n)| - \log |f'(\psi_n(f_{\xi_1}^{-1}(z)))| \right).$$

It follows from Claim 1 that the function $u_R$ is well-defined and harmonic on $A_R$. Moreover, it coincides with the function $u$ on a neighbourhood of $\xi_0$ in $J(f) \setminus f^{-1}(\infty)$. To see this, take some small $r_w > 0$ such that the function $c$ is constant on $B(w, r_w) \cap J(f)$. If we now take a small neighbourhood of $\xi_1$ in $W_2$, then it follows from (3.3) that for $n$ big enough its $\psi_n$ image will be contained in $B(w, r_w)$. There are only finitely many iterates left, therefore we can choose a radius $r_0 > 0$ so small that $c$ is constant on the following sets: $f_{\xi_1}^{-1}(B(\xi_0, r_0)) \cap J(f)$ and $\psi_n(f_{\xi_1}^{-1}(B(\xi_0, r_0))) \cap J(f)$ for every $n \geq 1$. Now simply use the relation (3.1) in the formula defining $u_R$ to see that

$$u_R(z) = u(z) \quad \text{for } z \in B(\xi_0, r_0) \cap J(f).$$ (3.4)

Exactly in the same way, considering small neighbourhoods of points from $A_R$ in $J(f)$, we get that $u_R - u$ is locally constant on $A_R \cap J(f)$.

Take now two different radii $R_2 > R_1 \geq R_0$ and consider functions $u_{R_1}$ and $u_{R_2}$ defined as described above. Since both $u_{R_1}$, $u_{R_2}$ are harmonic and $\text{HD}(J(f)) > 1$ (actually every open subset of $J(f)$ has the Hausdorff dimension greater than one), it follows from (3.4) that $u_{R_2}$ restricted to $A_{R_1}$ coincides with $u_{R_1}$. Thus we can define a harmonic function $\tilde{u}_Q: \mathbb{C} \setminus Q \to \mathbb{R}$ by the formula

$$\tilde{u}_Q(z) = u_{|z|<\xi_0|+R_0}(z).$$

Moreover, as we have just shown:

**Claim 2.** The function $u - \tilde{u}_Q$ is locally constant on $(\mathbb{C} \setminus Q) \cap J(f)$ and $u = \tilde{u}_Q$ on a neighbourhood of $\xi_0$ in $J(f)$.

Now, we want to show that the function $\tilde{u}_Q$ is actually bounded in a neighbourhood of every point except for singular values of $f$. Take two spiders $Q = \bigcup J_\gamma$ and $\hat{Q} = \bigcup \hat{J}_\gamma$ with this property that every connected component of $\mathbb{C} \setminus (Q \cup \hat{Q})$ intersects $J(f)$. Choose a point $\xi_0 \in J(f) \setminus (Q \cup \hat{Q})$. \(\square\)
and construct harmonic functions \( \tilde{u}_Q \) and \( \tilde{u}_{Q} \) defined as above on \( \mathbb{C} \setminus Q \) and \( \mathbb{C} \setminus \hat{Q} \) respectively. Then, again since \( \text{HD}(J(f)) > 1 \), using Claim 2 we get:

**Claim 3.** The function \( \tilde{u}_Q - \tilde{u}_{Q} \) is constant on every connected component of \( \mathbb{C} \setminus (Q \cup \hat{Q}) \).

Consider the function \( \tilde{u}_Q - \tilde{u}_Q \circ f - \log |f'| \) defined on \( \mathbb{C} \setminus (Q \cup f^{-1}(Q)) \). It is harmonic and, by (3.1) and Claim 2, it is locally constant on 
\[ (\mathbb{C} \setminus (Q \cup f^{-1}(Q))) \cap (J(f) \setminus f^{-1}(\infty)). \]
This implies that \( \tilde{u}_Q - \tilde{u}_Q \circ f - \log |f'| \) is constant on each connected component of \( \mathbb{C} \setminus (Q \cup f^{-1}(Q)) \) that intersects \( J(f) \). We will use Claim 3 to show that actually:

**Claim 4.** The function \( \tilde{u}_Q - \tilde{u}_Q \circ f - \log |f'| \) is constant on every connected component of \( \mathbb{C} \setminus (Q \cup f^{-1}(Q)) \).

**Proof.** Consider \( S, \) a connected component of \( \mathbb{C} \setminus f^{-1}(Q) = f^{-1}(\mathbb{C} \setminus Q) \). For any pair \( S_1, S_2 \) of connected components of \( \mathbb{C} \setminus (Q \cup f^{-1}(Q)) \) contained in \( S, S_1 \cap S_2 \) can be either empty, or a singleton, or finally a non-degenerate segment of \( Q \) (since \( S \) is simply connected). In the latter case let us call \( S_1 \) and \( S_2 \) neighbours. Note that \( S \) intersects the Julia set \( J(f) \), so at least one connected component of \( \mathbb{C} \setminus (Q \cup f^{-1}(Q)) \) contained in \( S \) intersects it also, thus \( \tilde{u}_Q - \tilde{u}_Q \circ f - \log |f'| \) is constant on it. We will show that if \( S_1, S_2 \subset S \) are neighbours and \( \tilde{u}_Q - \tilde{u}_Q \circ f - \log |f'| \) is constant on \( S_1 \) then it is also constant on \( S_2 \). It will finish the proof of Claim 4, since any two connected components \( S' \) and \( S'' \) of \( \mathbb{C} \setminus (Q \cup f^{-1}(Q)) \) contained in \( S \) can be connected by a sequence \( S' = S_1, \ldots, S_k = S'' \) of connected components of \( \mathbb{C} \setminus (Q \cup f^{-1}(Q)) \) such that any two consecutive are neighbours.

Take neighbours \( S_1, S_2 \subset S \) and assume that \( \tilde{u}_Q - \tilde{u}_Q \circ f - \log |f'| \) is constant on \( S_1 \). Let \( \Delta = S_1 \cap S_2 \subset Q \). We perturb the spider \( Q \) to a spider \( Q' \), based on the same point as \( Q \), replacing \( \Delta \) by a closed subset \( \Delta' \) with the same endpoints, chosen so that \( \text{Inf}(\Delta') \cap S_2 \neq \emptyset \) and moreover, there exists \( S_3 \), a connected component of \( \mathbb{C} \setminus (Q' \cup f^{-1}(Q')) \) which intersects both \( S_1 \) and \( S_2 \). Take now another spider \( Q \) with this property that all connected components of \( \mathbb{C} \setminus (Q \cup \hat{Q}) \) and \( \mathbb{C} \setminus (Q' \cup \hat{Q}) \) intersect \( J(f) \) (it is possible since perturbation can be arbitrarily small and \( \text{HD}(J(f)) > 1 \)). Then, using Claim 3 for pairs of spiders \( Q, Q' \), we conclude that \( u_Q - u_{Q'} \) is constant on every connected component of \( \mathbb{C} \setminus (Q \cup Q') \), while \( u_{Q} \circ f - u_{Q'} \circ f \) is constant on every component of \( \mathbb{C} \setminus f^{-1}(Q \cup Q') \).

Let \( S_{3,1} \) be a connected component of \( S_1 \cap S_3 \). It is a subset of \( S_3 \), therefore \( u_Q - u_Q \circ f - \log |f'| \) is constant on it. Since also \( u_Q - u_{Q'} \) and \( u_{Q} \circ f - u_{Q'} \circ f \) are constant on \( S_{3,1} \), we conclude that \( \tilde{u}_{Q} - u_{Q} \circ f - \log |f'| \) is constant on \( S_{3,1} \) and thus also on the whole \( S_3 \) (recall that all functions under consideration are harmonic where defined). Now similarly, let \( S_{3,2} \) be a connected component of \( S_1 \cap S_2 \). Again, \( u_Q - u_{Q'} \) and \( u_{Q} \circ f - u_{Q'} \circ f \) are constant on \( S_{3,2} \), so is \( \tilde{u}_{Q} - u_{Q} \circ f - \log |f'| \), therefore also \( \tilde{u}_{Q} - u_{Q} \circ f - \log |f'| \) is constant on \( S_{3,2} \) hence on \( S_2 \). This finishes the proof of Claim 4. □

We introduce the following sets of singularities of \( \tilde{u}_Q \).

\[
\text{Sing}_+(\tilde{u}_Q) = \{ w \in \mathbb{C} : \limsup_{z \to w} \tilde{u}_Q(z) = +\infty \}
\]
\[
\text{Sing}_-(\tilde{u}_Q) = \{ w \in \mathbb{C} : \liminf_{z \to w} \tilde{u}_Q(z) = -\infty \}.
\]

It follows from Claim 3 that
\[
\text{Sing}_+(\tilde{u}_Q) \cup \text{Sing}_-(\tilde{u}_Q) \subset \text{Sing}(f^{-1}).
\]

The family of closures of connected components of \( \mathbb{C} \setminus (Q \cup f^{-1}(Q)) \) is locally finite, thus we conclude from Claim 4 that
\[
f(\text{Sing}_+(\tilde{u}_Q) \cup (\text{Crit}(f) \setminus \text{Sing}_-(\tilde{u}_Q))) \subset \text{Sing}_+(\tilde{u}_Q)
\]
and
\[
f^{-1}(\text{Sing}_+(\tilde{u}_Q)) \subset \text{Sing}_+(\tilde{u}_Q) \cup \text{Crit}(f).
\]
Since the set $\text{Sing}(f^{-1})$ is finite and every critical value $v$ (if there is any), has infinitely many critical preimages (by periodicity of $f$), thus there exists

$$z \in (\text{Crit}(f) \setminus \text{Sing}(f^{-1})) \cap f^{-1}(v).$$

But then $z \notin \text{Sing}_-(\tilde{u}_Q)$ thus, by (3.5), $v = f(z) \in \text{Sing}_+(\tilde{u}_Q)$ and we conclude that

$$f(\text{Crit}(f)) \subset \text{Sing}_+(\tilde{u}_Q).$$

(3.6)

In view of (3.5), an easy consequence of the above fact is that the set $\bigcup_{n=0}^{\infty} f^n(\text{Crit}(f))$ is finite.

We need to consider two cases: first when critical points exist and second when $\text{Crit}(f) = \emptyset$. Begin with the case $\text{Crit}(f) \neq \emptyset$. Take any critical value $v \in f(\text{Crit}(f))$, then by (3.6) we know that $v \in \text{Sing}_+(\tilde{u}_Q)$. Since $\text{Sing}_+(\tilde{u}_Q) \cup \text{Sing}_-(\tilde{u}_Q)$ is finite, thus for some $k \in \mathbb{Z}$ we have that

$$z = v - 2k\pi i \notin \text{Sing}_+(\tilde{u}_Q) \cup \text{Sing}_-(\tilde{u}_Q).$$

In view of Claim 4 we conclude that there exist constants $c_1, c_2 \in \mathbb{R}$ and a sequence $\{z_n\}_{n \geq 0}$ with the following properties.

- $\forall n \geq 1 \ z_n \notin \text{Sing}_+(\tilde{u}_Q) \cup \text{Sing}_-(\tilde{u}_Q)$
- $\lim_{n \to \infty} z_n = z$ and $(z_n + 2k\pi i) \to v$
- $\tilde{u}_Q(z_n) - \tilde{u}_Q(f(z_n)) - \log |f'(z_n)| = c_1$
- $\tilde{u}_Q(z_n + 2k\pi i) - \tilde{u}_Q(f(z_n + 2k\pi i)) - \log |f'(z_n + 2k\pi i)| = c_2$
- $\lim_{n \to \infty} \tilde{u}_Q(z_n + 2k\pi i) = +\infty$

Note that, since $f$ is $2\pi$-periodic, $f(z_n + 2k\pi i) = f(z_n)$ and $f'(z_n + 2k\pi i) = f'(z_n)$, therefore $\tilde{u}_Q(z_n + 2k\pi i) = \tilde{u}_Q(z_n) + c_2 - c_1$. And because $z \notin \text{Sing}_+(\tilde{u}_Q)$, we conclude that

$$\limsup_{n \to \infty} \tilde{u}_Q(z_n + 2k\pi i) = c_2 - c_1 + \limsup_{n \to \infty} \tilde{u}_Q(z_n) < +\infty,$$

counter to the last property in the above list.

Now consider the case when $\text{Crit}(f) = \emptyset$. Then necessarily $\text{AS}(f) \neq \emptyset$ thus we have exactly one finite asymptotic value $a$ (recall that by our assumption $R(\infty) = \infty$). This implies that $f$ has to be of the form

$$f(z) = be^{dz} + a,$$

where $a$ is the asymptotic value, $b \in \mathbb{C} \setminus \{0\}$ and $d \geq 1$. Note that in this case, since we have only one singular value $a \in \mathbb{C}$, any spider $Q$ is an unbounded polygonal curve homeomorphic with a half-line. Moreover, by (3.5) and Claim 4, the set $\text{Sing}_+(\tilde{u}_Q) \cup \text{Sing}_-(\tilde{u}_Q)$ is completely invariant and contained in $\{a\}$. If it equals $\{a\}$, then we have $f(a) = a$ and $f^{-1}(a) = \{a\}$. But this is a contradiction since $a$ is omitted for $f$. Therefore we are left with the case $\text{Sing}_+(\tilde{u}_Q) \cup \text{Sing}_-(\tilde{u}_Q) = \emptyset$.

By now we know that $\tilde{u}_Q$ is bounded in a neighbourhood of every point in $\mathbb{C}$. From Claim 3 we conclude that there exists a constant $c \in \mathbb{C}$ such that the difference between limits of $\tilde{u}_Q(z)$ as $z$ approaches a point of $Q$ from two sides of $Q$ always equals $c$. This means that it can be expressed as follows:

$$\tilde{u}_Q(z) = \tilde{h}(z) + \frac{c}{2\pi} \text{Arg}(z - a)$$

(3.7)

for $z \in \mathbb{C} \setminus Q$, where $\tilde{h}$ is a harmonic function in $\mathbb{C}$ and $\text{Arg}$ is the principle argument of a complex number.

Suppose that $c = 0$. Then actually $\tilde{u}_Q$ extends to the whole $\mathbb{C}$ and by the continuity and since there are no critical points, Claim 4 implies that

$$\tilde{u}_Q - \tilde{u}_Q \circ f - \log |f'| = \text{const}$$
on \( \mathbb{C} \). But this is clearly impossible. To see this recall that \( f \) is hyperbolic so there is some attracting cycle in the Fatou set which implies that the constant should be positive. On the other hand, we have a lot of repelling periodic cycles in the Julia set so the constant should be negative and we get a contradiction.

We conclude, that \( \tilde{a}_Q \) have to be of the form (3.7) with \( c \neq 0 \). Moreover, as a consequence of the form of function \( f \) and Claim 4, the harmonic function \( \tilde{h} \) must be unbounded, hence it cannot be constant, which means that \( f \) is in the exceptional class 2 from Definition 1.2. \( \square \)

**Lemma 3.6** For any function \( f \) satisfying the rapid derivative growth condition (2.6) and measures \( m_f, \mu_f \) as in Theorem 2.6, the condition (d) from Theorem 3.2 implies (a) or (c).

**Proof.** Suppose first that \( \nabla D_{\mu_f} \equiv 0 \) on \( J(f) \setminus f^{-1}(\infty) \). This equivalently means that \( D_{\mu_f} \) is locally constant on \( J(f) \setminus f^{-1}(\infty) \), giving (a).

Now, assume that there exists \( v \in J(f) \setminus f^{-1}(\infty) \) such that \( \nabla D_{\mu_f}(v) \neq 0 \). Since, by Theorem 2.8, \( \nabla D_{\mu_f} \) is locally real-analytic, there exists a neighbourhood \( W \subset \mathbb{C} \setminus \mathcal{P}(f) \) of \( v \), where the gradient does not vanish. Thus, we can find a continuous function \( \tilde{l} : W \to \mathbb{R}^1 \) such that \( \tilde{l}(z) \) is the direction orthogonal to \( \nabla D_{\mu_f}(z) \) at every point \( z \in W \). The projective space \( \mathbb{R}P^1 \) can be seen as the quotient space \( S^1/\| \| \) so let us denote \( \tilde{l}(z) = [l(z)]_{\|} \) for some representative \( l(z) \) of the equivalence class \( \tilde{l}(z) \). We want to extend the definition of \( \tilde{l} \) to the whole set \( J(f) \). Since all exceptional points are in the Fatou set, for any \( z \in J(f) \) there exists an \( n \geq 0 \) and a point \( \xi \in W \cap f^{-n}(z) \), so let us define \( \tilde{l}(z) = [l(z)]_{\|} \) for

\[
l(z) = \frac{(f^n)'(\xi)}{|(f^n)'(\xi)|} l(\xi),
\]

where \( l(\xi) = [l(\xi)]_{\|} \).

First we have to show that the line field \( \tilde{l} \) is well-defined, which means that if \( \xi \in W \cap f^{-n}(z) \) and \( \zeta \in W \cap f^{-m}(z) \), then

\[
\frac{(f^n)'(\xi)}{|(f^n)'(\xi)|} l(\xi) = \frac{(f^m)'(\zeta)}{|(f^m)'(\zeta)|} l(\zeta).
\]  

Suppose on the contrary that for some \( z, \xi, \zeta \) condition (3.8) fails. Then there exists a point \( x \in J(f) \cap W \), an integer \( k \geq 0 \) and \( w \in f^{-k}(x) \), so close to \( z \), that we can find two points \( y_1 \in f^{-n}(w) \) and \( y_2 \in f^{-m}(w) \) close to \( \xi \) and \( \zeta \) respectively, so that

\[
\frac{(f^n)'(y_1)}{|(f^n)'(y_1)|} l(y_1) \neq \pm \frac{(f^m)'(y_2)}{|(f^m)'(y_2)|} l(y_2).
\]

Hence, because \( f^n(y_1) = f^m(y_2) \), the Chain Rule implies

\[
\frac{(f^{k+n})'(y_1)}{|(f^{k+n})'(y_1)|} l(y_1) \neq \pm \frac{(f^{k+m})'(y_2)}{|(f^{k+m})'(y_2)|} l(y_2).
\]

So, either

\[
\frac{(f^{k+n})'(y_1)}{|(f^{k+n})'(y_1)|} l(y_1) \neq \pm l(x) \quad \text{or} \quad \frac{(f^{k+m})'(y_2)}{|(f^{k+m})'(y_2)|} l(y_2) \neq \pm l(x).
\]  

Without loss of generality we can assume that the first condition holds.

Consider now gradients as horizontal vectors and \( l(\cdot) \) as vertical ones. The standard inner product becomes the product of matrices. By the Chain Rule we have

\[
\nabla(D_{\mu_f} \circ f^{k+n})(y_1) = \nabla D_{\mu_f}(f^{k+n}(y_1))(f^{k+n})'(y_1) = \nabla D_{\mu_f}(x)(f^{k+n})'(y_1),
\]

thus, since the matrix \([f^{k+n}](y_1)]^{-1}\) is square and \( x \in W \), we get

\[
\langle \nabla(D_{\mu_f} \circ f^{k+n})(y_1), [(f^{k+n})'(y_1)]^{-1}l(x) \rangle = \langle \nabla D_{\mu_f}(x)(f^{k+n})'(y_1), [(f^{k+n})'(y_1)]^{-1}l(x) \rangle
\]

\[
= \langle \nabla D_{\mu_f}(x), l(x) \rangle = 0.
\]
Using the condition (3.9) we conclude that \( l(y_1) \) is not perpendicular to \( \nabla(D_{\mu_f} \circ f^{k+n})(y_1) \), which means that \( \nabla(D_{\mu_f} \circ f^{k+n})(y_1) \) and \( \nabla(D_{\mu_f} \circ f^{k+n})(y_1) \) are not parallel or equivalently
\[
\det (\nabla(D_{\mu_f} \circ f^{k+n})(y_1)), \nabla(D_{\mu_f}(y_1)) \neq 0.
\]
This is however a contradiction with (d), thus \( \tilde{l} \) is well-defined. Now it suffices to take \( u(z) = (l(z))^2 \) for any representative \( l(z) \) of \( \tilde{l}(z) \) to obtain a continuous \( f \)-invariant line field. □

Finally, let us show that the assumption (c) of Theorem 3.2 implies that \( f \) belongs to the exceptional class \( I \) from Definition 1.2.

**Lemma 3.7** Let \( f : \mathbb{C} \to \mathbb{C} \) be a hyperbolic entire transcendental function from class \( \mathcal{B} \), i.e. the set of singular values \( \text{Sing}(f) \) is bounded. Then the Julia set \( J(f) \) supports no continuous invariant line fields.

**Proof.** Suppose there is a continuous \( f \)-invariant line field on \( J(f) \), i.e. we can find a continuous function \( u : J(f) \to S^1 \) such that
\[
u(f(z)) = \left( \frac{f'(z)}{|f'(z)|} \right)^2 u(z), \quad (3.10)
\]
for any \( z \in J(f) \). Since the Julia set \( J(f) \) is the closure of repelling periodic points and (3.10) stays true for any iterate of \( f \), without loss of generality we can assume that there is a fixed point in the Julia set (passing to an appropriate iterate if necessary). Take the fixed point \( z_0 \in J(f) \) and denote \( \lambda = f'(z_0) \). By (3.10) we immediately get that \( \lambda \in \mathbb{R} \) and also \( \lambda > 1 \) since \( z_0 \) is repelling.

Consider a linearization around the fixed point \( \psi : B(0, r) \to U_{z_0} \) for some \( r > 0 \) and a neighbourhood \( U_{z_0} \) of \( z_0 \). The function \( \psi \) is a conformal homeomorphism on \( B(z_0, r) \) and it satisfies
\[
\psi(\lambda z) = f(\psi(z)) \quad (3.11)
\]
for every \( z \in B(0, r/\lambda) \). Moreover, we can choose \( \psi \) so that \( \psi'(0) = 1 \). Now, extend \( \psi \) to the whole \( \mathbb{C} \) using the equation (3.11). The extended function, denoted also by \( \psi \), is entire and satisfies (3.11) for every \( z \in \mathbb{C} \).

We will show that the line field \( u \) is an image by \( \psi \) of a constant line field and the linearization function \( \psi \) is automorphic with respect to a discrete group of isometries of the plane \( \mathbb{C} \) (see [8, Definition 2.4]).

**Definition 3.8** A holomorphic map \( \Psi : \mathbb{C} \to \overline{\mathbb{C}} \) is called automorphic with respect to a discrete group of isometries \( \Gamma \subset \text{Isom}(\mathbb{C}) \) if:

1. \( \Psi \circ \gamma = \Psi \) for every \( \gamma \in \Gamma \) and
2. \( \Gamma \) acts transitively on fibers, i.e. whenever \( \Psi(z_1) = \Psi(z_2) \), there is \( \gamma \in \Gamma \) with \( \gamma(z_1) = z_2 \).

It was shown in [8, Lemma 2.5] that the transitivity condition (2) in the above definition can be replaced by the following weaker assertion:

(2') \( \Gamma \) acts transitively on one regular fibre, i.e. there is \( w \in \Psi(\mathbb{C}) \) which is not a critical value of \( \Psi \) such that for every \( z_1, z_2 \in \Psi^{-1}(w) \) we can find some \( \gamma \in \Gamma \) with \( \gamma(z_1) = z_2 \).

Take any \( w \in \Psi^{-1}(J(f)) \). Using (3.10) and continuity of \( u \) we obtain the following:
\[
u(\psi(w)) = u \left( f^n \left( \psi \left( \frac{w}{\lambda^n} \right) \right) \right) = u \left( \psi \left( \frac{w}{\lambda^n} \right) \right) \left( \frac{(f^n)'(\psi(\frac{w}{\lambda^n}))}{|(f^n)'(\psi(\frac{w}{\lambda^n}))|} \right)^2 =
\]
\[
u(\psi(w)) = \left( \frac{\lambda^n \psi'(w)}{\psi'(\frac{w}{\lambda^n})} \right)^2 \xrightarrow{n \to \infty} u(\psi(0)) \left( \frac{\psi'(w)}{|\psi'(w)|} \right)^2 \psi'(0)^2.
\]
Assuming that \( u(\psi(0)) = u(z_0) = 1 \) (multiplying the line field \( u \) by \( \overline{u(z_0)} \) if necessarily) and using the fact that \( \psi'(0) = 1, \lambda \in \mathbb{R} \) we get that

\[
u(w) = \left( \frac{\psi'(w)}{|\psi'(w)|} \right)^2
\]

for every \( w \in \psi^{-1}(J(f)) \).

Consider the fiber \( \psi^{-1}(z_0) \) over the fixed point \( z_0 \). Recall that in a neighbourhood \( B(0, r) \) of \( 0 \) the linearization function \( \psi \) is univalent, therefore its derivative does not vanish. Take any \( w \in \psi^{-1}(z_0) \) different from 0. By the definition of \( \psi \) there is some \( n \geq 1 \) for which \( \frac{n}{n}w \in B(0, r) \) and \( \psi(w) = f^n(\psi(\frac{n}{n}w)) \). Since \( z_0 \in J(f) \), \( f \) is hyperbolic and \( \psi' \neq 0 \) on \( B(0, r) \), therefore

\[
\psi'(w) = (f^n)'(\psi(\frac{n}{n}w)) \psi'(\frac{n}{n}w) \frac{1}{\lambda^n}
\]

is different from zero, thus \( z_0 \) is not a critical value of \( \psi \), hence \( \psi^{-1}(z_0) \) is a regular fibre.

Take now neighbourhoods \( U_0 \) and \( U_w \) of 0 and \( w \) respectively on which \( \psi \) is univalent and such that \( \psi(U_0) = \psi(U_w) \). We can define a univalent map \( \gamma : U_w \to U_0 \) for which \( \psi \circ \gamma = \psi \) on \( U_w \) and such that \( \gamma(w) = 0 \) (by the Implicit Function Theorem). Denote \( K = \psi^{-1}(J(f)) \) and take any \( v \in K \cap U_w \). Since \( \psi(v) = \psi(\gamma(v))\gamma'(v) \), therefore

\[
\left( \frac{\psi'(v)}{|\psi'(v)|} \right)^2 = u(\psi(v)) = u(\psi(\gamma(v))) = \left( \frac{\psi'(\gamma(v))}{|\psi'(\gamma(v))|} \right)^2 = \left( \frac{\psi'(v)}{|\psi'(v)|} \frac{|\gamma'(v)|}{|\gamma'(v)|} \right)^2
\]

and consequently

\[
\gamma'(v) \in \mathbb{R} \quad \text{for every} \quad v \in K \cap U_w.
\] (3.12)

Consider first the case \( \gamma' \neq \text{const} \). Since the Julia set \( J(f) \) is \( f \)-invariant, thus we also have \( \lambda K \subseteq K \) which means that \( K \) is contained in a union of straight lines passing through the origin. From the condition (3.12) we conclude that \( K \cap U_w \) is a real analytic set. Shrinking \( U_w \) is necessarily we can assume that \( K \cap U_w \) is contained in an interval. But this means that \( K \cap U_0 = \gamma(K \cap U_w) \) is also contained in an interval, thus the whole set \( K \) is contained in a line \( e^{\theta+i} \mathbb{R} \) and \( J(f) \subset \psi(e^{i\theta} \mathbb{R}) \) This is however impossible since it was shown by Stallard in [15] that the Hausdorff dimension of the Julia set of every transcendental entire function from class \( B \) is strictly greater than one.

We are therefore left with the case \( \gamma'(v) = c \in \mathbb{R} \) for every \( v \in U_w \). This means that \( \gamma \) is an affine map of the form \( cv + b \) where \( c \in \mathbb{R} \) and we can extend the equation \( \psi \circ \gamma = \psi \) to the whole plane \( \mathbb{C} \). Moreover, it is easy to see that actually \( |c| = 1 \) (otherwise \( \psi \) would have to be constant). Thus we have shown that for any \( w_1, w_2 \in \psi^{-1}(0) \) we can find an affine map of the form \( \gamma(z) = \pm z + b \) such that \( \psi \circ \gamma = \psi \) on \( \mathbb{C} \) and \( \gamma(w_1) = w_2 \). Since \( \psi^{-1}(z_0) \) is a regular fibre we conclude from Definition 3.8 that \( \psi \) is automorphic with respect to a discrete group of isometries

\[
\Gamma \subset \{ \gamma : \gamma(z) = \pm z + b \}.
\]

This allows us to define a measurable line field on \( \mathbb{C} \) by the formula

\[
\tilde{u}(z) = \left( \frac{\psi'(z)}{|\psi'(z)|} \right)^2 \quad \text{for any} \quad z \in \psi^{-1}(w).
\]

Note that, since \( f \) is automorphic with respect to \( \Gamma \), the line field \( \tilde{u} \) is well defined almost everywhere (if \( z_1, z_2 \in \psi^{-1}(w) \), then \( \psi'(z_1) = \psi'(z_2) \)). Moreover, (3.11) immediately implies that \( \tilde{u} \) is \( f \)-invariant. Thus we have defined a measurable \( f \)-invariant line field \( \tilde{u} \) on \( \mathbb{C} \) which is univalent in a neighbourhood of the Julia set \( J(f) \) (cf. [14]). This is however a contradiction with Theorem 6.1 in [14] saying that it is impossible for a transcendental function to have such an invariant line field. This finishes the proof. □
4 Proof of the measure rigidity

In this section we prove Theorem 1.3. First note that the conjugacy $h$ from the theorem, which is defined on $J(f) \subset \mathbb{C}$, extends to $\hat{J}(f) \subset \hat{\mathbb{C}}$ in the obvious way, i.e.

$$h(z) \xrightarrow{z \to \infty} \infty. \quad (4.1)$$

Suppose it is not the case, which means that there exists in $J(f)$ a sequence $\{z_n\}_{n \in \mathbb{N}}$ tending to $\infty$ for which $\{h(z_n)\}_{n \in \mathbb{N}}$ is bounded. Then we can choose a subsequence $\{h(z_{n_k})\}_{k \in \mathbb{N}}$ converging to some point $w \in J(g)$. But, by the continuity of $h^{-1}$,

$$z_{n_k} = h^{-1}(h(z_{n_k})) \xrightarrow{k \to \infty} h^{-1}(w) \in J(f) \subset \mathbb{C},$$

contrary to the choice of $\{z_n\}_{n \in \mathbb{N}}$, so (4.1) is true.

**Proof of Theorem 1.3.**

The implications 1.$\Rightarrow$2., 2.$\Rightarrow$3. and 3.$\Rightarrow$4. are obvious. The proof of the implication 4.$\Rightarrow$5. is very easy and can be found in [6, p. 116]. However, we repeat it here for the completeness of the exposition. The crucial and difficult part of the proof is to establish the implications 5.$\Rightarrow$6. and 6.$\Rightarrow$1.

4.$\Rightarrow$5.

Suppose the condition 5. is not satisfied, i.e. there exists a repelling periodic point $z_0 \in J(f)$ such that

$$|(f^p)'(z_0)| \neq |(g^p)'(h(z_0))|.$$  

Without loss of generality we can assume that $|(f^p)'(z_0)| > |(g^p)'(h(z_0))|$ and we can choose constants $\lambda, \gamma > 0$ so that

$$|(f^p)'(z_0)| > \lambda > \gamma > |(g^p)'(h(z_0))|.$$  

Let $U$ be a neighbourhood of $z_0$ such that we can define on $U$ an inverse branch $f_{z_0}^{-p}$ of $f^p$ mapping $z_0$ to $z_0$ and on $h(U)$ we can define an inverse branch $g_{h(z_0)}^p$ of $g^p$ mapping $h(z_0)$ to $h(z_0)$. Note that if $U$ is small enough, then

$$|f_{z_0}^{-p}(z) - z_0| \leq \lambda^{-n} \quad \text{and} \quad |g_{h(z_0)}^{-p}(w) - h(z_0)| \geq \gamma^{-n}$$

for all $n \geq 1$, $z \in J(f) \cap U$ and $w \in J(g) \cap h(U)$. But this immediately implies that

$$\frac{|h(f_{z_0}^{-p}(z)) - h(z_0)|}{f_{z_0}^{-p}(z) - z_0} \geq \frac{\gamma^{-n}}{\lambda^{-n}} = \left(\frac{\lambda}{\gamma}\right)^n \xrightarrow{n \to \infty} \infty$$

which means that $h$ cannot be Lipschitz continuous. \qed

5.$\Rightarrow$6.

Until the end of the proof of this implication we assume that the condition 5. of Theorem 1.1 is satisfied, i.e. $h$ preserves the moduli of multipliers of periodic cycles in $J(f)$. Let us begin with the following version of the shadowing lemma. Recall after (2.4) that

$$\delta_f = \frac{1}{4} \text{dist}(J(f), \mathcal{P}(f)) > 0.$$

**Lemma 4.1** For every $\rho_2 \in (0, \delta_f)$ there exist $\rho_1 > 0$ and $n_1 \geq 1$ such that for every $n \geq n_1$ if $f^n(x) \in J(f) \cap B(x, \rho_1)$ for some $x \in J(f)$, then there exists $y \in J(f)$ such that $f^n(y) = y$, $|f^j(x) - f^j(y)| < \rho_2$ for $0 \leq j \leq n - n_1$, and $|y - f^n(x)| < \rho_2$.
Proof. Take any \( z \in J(f) \). Since \( \rho_2 < \delta_f \), Theorem 2.9 implies that every holomorphic inverse branch \( f^{-n}_z \) of \( f^n \), for any \( n \geq 1 \), has bounded distortion on \( B(z, \rho_2) \) by some constant \( K > 0 \).

Using (2.5), the expanding property of \( f \), we get that

\[
\text{diam} \left( f^{-n}_z \left( B(z, \rho_2) \right) \right) \leq K(C\gamma^n)^{-1}2\rho_2.
\]

Thus, we can find \( n_1 \) so large that for \( n > n_1 \) and for all \( z \in J(f) \) the diameter of every connected component of \( f^{-n}(B(z, \rho_2)) \) is smaller than \( \frac{1}{2}\rho_2 \). Let \( \rho_1 < \frac{1}{2}\rho_2 \). For each \( n \geq n_1 \) denote by \( B_n \) the connected component of \( f^{-n}(B(f^n(x), \rho_2)) \) containing \( x \) and by \( f^{-n}_z \colon B(f^n(x), \rho_2) \to B_n \) the unique holomorphic inverse branch of \( f^n \) mapping \( f^n(x) \) to \( x \). Then we have the following inclusions

\[
f^{-n}_z \left( B(f^n(x), \rho_2) \right) \subset B(x, \rho_2/2) \subset B(f^n(x), \rho_1 + \rho_2/2) \subset B(f^n(x), \rho_2).
\]

Hence, by the Brower fixed point theorem, there exists \( y \in B(f^n(x), \rho_2) \) such that \( f^{-n}_z(y) = y \). Therefore \( f^n(y) = y \) and it is a repelling periodic point. Moreover, we get the estimates

\[
|f^n(x) - f^n(y)| = |f^{-n-j}(f^n(x)) - f^{-n-j}(f^n(y))| \leq \rho_2/2 < \rho_2
\]

for all \( j \leq n - n_1 \). □

Choose an arbitrary point \( x \in J(f) \) with a dense trajectory in \( J(f) \). For every \( z \in J(f) \setminus f^{-1}(\infty) \) define

\[
\eta(z) = \log |g'(h(z))| - \log |f'(z)|
\]

and for \( n \geq 1 \) let

\[
u(f^n(x)) = \sum_{j=0}^{n-1} \eta(f^j(x)).
\]

Lemma 4.2 The function \( u \) is continuous on the set \( \{f^n(x)\}_{n \geq 0} \).

Proof. Fix small \( \delta < \frac{1}{2} \min(\delta_f, \delta_\eta) \). Function \( h \) is continuous on \( J(f) \), therefore it is uniformly continuous on every ball \( B(z, 2R) \cap J(f), \) \( R > 0 \). Thus we can find a positive \( \rho_2 < \delta \) such that for any \( z, w \in B(0, 2R) \cap J(f) \)

\[|z - w| < \rho_2 \Rightarrow |h(z) - h(w)| < \delta.
\]

Choose \( \rho_1 \) and \( n_1 \) according to Lemma 4.1 applied to the function \( f \) and \( \rho_2 \). Consider any two points \( f^n(x), f^m(x) \in B(0, R) \cap J(f) \), with \( n > m \), such that \( |f^n(x) - f^m(x)| < \rho_1 \). Then, by Lemma 4.1, there exists \( y \in J(f) \cap B(0, R + \rho_1) \) such that \( f^{n-m}(f^m(y)) = f^m(y) \),

\[|f^n(x) - f^m(y)| < \rho_2 \quad \text{and} \quad |f^{m+j}(x) - f^{m+j}(y)| < \rho_2
\]

for \( 0 \leq j \leq n - m - n_1 \). Using condition 5. from Theorem 1.1 we immediately get that

\[\sum_{j=m}^{n-1} \eta(f^j(y)) = 0\]

and therefore

\[
u(f^n(x)) - \nu(f^m(x)) = \sum_{j=m}^{n-1} \eta(f^j(x)) = \sum_{j=m}^{n-1} \left( \eta(f^j(x)) - \eta(f^j(y)) \right)
\]

\[= \sum_{j=m}^{n-1} \log \left( \frac{|g'(h(f^j(x)))|}{|g'(h(f^j(y)))|} \right) - \log \left( \frac{|f'(f^j(x))|}{|f'(f^j(y))|} \right)
\]

\[= \log \left( \frac{|g^{n-m}(h(f^m(x)))|}{|g^{n-m}(h(f^m(y)))|} \right) - \log \left( \frac{|f^{n-m}(f^m(x))|}{|f^{n-m}(f^m(y))|} \right).
\]
Now, both of the terms are arbitrarily small. It follows from Theorem 2.9, since
\[ |f^m(x) - f^m(y)| < \rho_2 < \delta \]
and by the choice of \( \rho_2 \) also
\[ |h(f^m(x)) - h(f^m(y))| < \delta, \]
while the required holomorphic inverse branches of \( f^{n-m} \) and \( g^{n-m} \) are defined on balls with radii \( \delta_f \) and \( \delta_g \) respectively. This finishes the proof of continuity of \( u \) on the trajectory of \( x \). \( \square \)

Last lemma allows us to extend the function \( u \) continuously to the whole set \( J(f) \).

**Lemma 4.3** The functions \( \log |f'(z)| \) and \( \log |g'(h(z))| \) are cohomologous in the class of continuous functions on \( J(f) \), i.e. there exists a continuous function \( u: J(f) \to \mathbb{R} \) such that
\[ \log |g'(h(z))| - \log |f'(z)| = u(f(z)) - u(z) \] (4.2)
for all \( z \in J(f) \setminus f^{-1}(\infty) \).

This is a straightforward consequence of the construction and continuity of the function \( u \). By the definition of \( u \), (4.2) is satisfied on the set \( \{ f^n(x) \}_{n \geq 0} \) and by its density in \( J(f) \), the equation extends to \( J(f) \setminus f^{-1}(\infty) \).

We want to use Lemma 2.12 to show that the measure \( m_{\rho} \circ h \) is absolutely continuous with respect to \( m_f \) on \( J(f) \). However to do this, we need to consider these measures on a bounded set. That is why we will express \( J(f) \) as a union of its appropriate bounded subsets.

By Lemma 2.7, there exists an \( M > 0 \) such that for \( m_f \)-a.e. \( z \in J(f) \) we can find a subsequence \( (n_j)_{j \geq 0} \) with the following property
\[ |f^{n_j}(z)| < 2M \quad \text{for every} \quad j \geq 0. \]

Take any \( R > 2M \) and denote by \( J_R \) the set of points \( z \in J(f) \cap D(0, R) \) for which there exists a subsequence \( (n_j)_{j \geq 0} \) such that
\[ |f^{n_j}(z)| < R \quad \text{for every} \quad j \geq 0. \] (4.3)

Note that
\[ J_r(f) \subset \bigcup_{R > 2M} J_R, \] (4.4)
where recall that \( J_r(f) \) is the radial (non-escaping) part of the Julia set.

Now, for any fixed \( R > 2M \) and every radius \( r > 0 \), as a consequence of the regularity of the measure \( m_f \), we get that
\[ \inf_{z \in J_R} m_f(B(z, r)) > 0. \]
Moreover, since every \( J_R \) is a bounded subset of the Julia set, the function \( h \) is uniformly continuous on it, thus there exists a \( \delta_R < \delta_f \) such that
\[ h(J(f) \cap B(z, \delta_R)) \subset B(h(z), \delta_g) \quad \text{for every} \quad z \in J_R. \] (4.5)

Notice also that if \( z \in J_R \) and the subsequence \( (n_j)_{j \geq 0} \) is as in the condition (4.3), then \( f^{n_j}(z) \in J_R \) for every \( j \geq 0 \).

Fix \( z \in J_R \) and \( (n_j)_{j \geq 0} \) such that (4.3) is satisfied. Set \( K = \max(K_r, k) \), where the numbers come from Lemma 2.4 and Lemma 2.10, and define the following sequence of radii
\[ r_j(z) = K^{-1}(f_z^{-n_j})'(f^{n_j}(z)) \delta_R, \]
where \( f_z^{-n_j} \) is the unique holomorphic inverse branch of \( f^{n_j} \) mapping \( f^{n_j}(z) \) back to \( z \). Then by Lemma 2.10 we have for every \( j \geq 0 \) the following inclusions
\[ B(z, r_j(z)) \subset f_z^{-n_j}(B(f^{n_j}(z), \delta_R)) \] (4.6)
Thus, for $r_j(z)$ we have the following estimates of derivatives with respect to the metric $d\tau$.

Next, using (4.6) and (4.5) we obtain the following sequence of inclusions
\[
\|\|\int_J f^{-n}(z)\|\| = |(f^{-n}(z))|^\frac{1}{t} m_f(B(f^n(z), K^{-2}\delta_R)) \geq M_R K^{-t_j} |(f^{-n})(z))|^\frac{1}{t_j}.
\]

Moreover, by (2.5), the expanding property of $f$, we get
\[
r_j(z) \xrightarrow{j \to \infty} 0.
\]

Denote $M_R = \inf_{z \in J_R} m_f(B(z, K^{-2}\delta_R)) > 0$. Using (4.7), conformity of $m_f$ and Lemma 2.4 we get
\[
m_f(B(z, r_j(z))) \geq m_f(f^{-n}(B(f^n(z), K^{-2}\delta_R))) \geq |(f^{-n})(z))|^\frac{1}{t_j} K^{-t_j} m_f(B(f^n(z), K^{-2}\delta_R)) \geq M_R K^{-t_j} |(f^{-n})(z))|^\frac{1}{t_j}.
\]

We need a few more estimates. First note that by (4.2) we get for any $n \geq 1$,
\[
\log |(f^n)'(h(z))| - \log |(f^n)'(z)| = u(f^n(z)) - u(z),
\]
where $u$ is continuous on the whole $J_f$. Thus, if we denote by $N_R$ the upper bound for $|u|$ on $J_R$, then
\[
\left| \log \left( \frac{|(g^n)'(h(z))|}{|(f^n)'(z)|} \right) \right| \leq 2N_R \Rightarrow e^{-2N_R} \leq \left| \frac{(g^n)'(h(z))}{(f^n)'(z)} \right| \leq e^{2N_R}
\]
for every $z \in J_R$ and $n_j$ from the subsequence associated with $z$. Recall that we can find $T_f, T_g \geq 0$ such that $D(0, T_f) \cap J_f = \emptyset$ and $D(0, T_g) \cap J_g = \emptyset$. For the iterates from our subsequence we have the following estimates of derivatives with respect to the metric $d\tau$ for $z \in J_R$.

\[
\left| \frac{|(g^n)'(h(z))|}{|(f^n)'(z)|} \left| \begin{array}{c} \left| h(z) \right| \left| f^n(z) \right| \left| f^n(z) \right| \left| g^n(h(z)) \right| \end{array} \right| \geq e^{-2N_R} \left( \frac{T_g}{R} \right) = Q_R^{-1}.
\]

Therefore, using (4.9), conformity of $m_g$, Lemma 2.4, (4.10) and (4.8), we get
\[
\begin{align*}
m_g \circ h(J_f \cap B(z, r_j(z))) & \leq m_g(g^{-n_j}_h(B(g^n(h(z)), \delta_g))) \\
& \leq K^{t^*_s} |(g^{-n_j}_h)'(g^n(h(z)))|^{t^*_s} m_g(B(g^n(h(z)), \delta_g)) \\
& \leq K^{t^*_s} |(g^n)'(h(z))|^{t^*_s} \\
& \leq K^{t^*_s} Q_R^{t^*_s} |(f^n)'(z)|^{t^*_s} m_f(B(z, r_j(z))) |(f^n)'(z)|^{t^*_s}.
\end{align*}
\]

Thus, for $z \in J_R$ and $j \geq 0$
\[
m_g \circ h(B(z, r_j(z))) \leq C_R m_f(B(z, r_j(z))) |(f^n)'(z)|^{t^*_s}.
\]
Note that for \( z \in J_R \), using the expanding property of \( f|_{J(f)} \), we have

\[ |(f^n)'(z)| \geq \left| \frac{|z|^r}{|J^n(f)(z)|} \right| \geq C \gamma^n \frac{T_j^f}{R} \to \infty. \]

Thus, if \( t_f < t_g \), then from Lemma 2.12 it would follow, that \( m_g(h(J_R)) = 0 \) and since \( R \) was chosen arbitrarily, this would imply, that \( m_g(h(J_r(f))) = 0 \). But it is a straightforward consequence of (4.1) that \( h(I_\infty(f)) = I_\infty(g) \), hence \( h(J_r(f)) = J_r(g) \), so we would get a contradiction with Theorem 2.6 (2). Therefore \( t_f \leq t_g \) and by the symmetry both exponents are actually equal. By the choice of parameters \( t_f \) and \( t_g \), it in particular implies that

\[ \text{HD}(J_r(f)) = \text{HD}(J_r(g)). \]

Finally, we get that for any \( R > 2M \), every \( z \in J_R \) and \( j \geq 0 \)

\[ m_g \circ h(B(z,r_j(z))) \leq C_R m_f(B(z,r_j(z))), \]

thus, by Lemma 2.12, \( m_g \circ h \ll m_f \) on \( J_R \). In view of (4.4), it implies that

\[ m_g \circ h \ll m_f \text{ on } J_r(f) \]

and by the symmetry also \( m_f \circ h^{-1} \ll m_g \). This finishes the proof of the implication 5.\( \Rightarrow \)6. \( \square \)

6.\( \Rightarrow \)1.

The proof of this part consists of a sequence of lemmas.

**Lemma 4.4** If the condition 6. of Theorem 1.1 is satisfied, then the conjugacy homeomorphism \( h: J(f) \to J(g) \) extends to a real-analytic endomorphism from a neighbourhood of \( J(f) \setminus f^{-1}(\infty) \) onto a neighbourhood of \( J(g) \setminus g^{-1}(\infty) \).

**Proof.** By Theorem 3.2 (d) there exists an \( n \geq 1 \) and a point \( z \in J(g) \setminus \bigcup_{j=1}^{n+1} g^{-j}(\infty) \) such that

\[ \det(\nabla(D_{\mu_g} \circ g^n)(z), \nabla(D_{\mu_g})(z)) \neq 0. \]

Theorem 2.8 implies that \( D_{\mu_g} \) extends real-analytically on a neighbourhood of \( J(g) \setminus g^{-1}(\infty) \), therefore we can find an open set \( W \subset \mathbb{C} \), disjoint from \( \bigcup_{j=0}^{n} g^{-j}(P^0(g)) \), such that \( z \in W \) and for every \( w \in W \)

\[ \det(\nabla(D_{\mu_g} \circ g^n)(w), \nabla(D_{\mu_g})(w)) \neq 0. \]

Since the measures \( m_f \circ h^{-1} \) and \( m_g \) are equivalent, also the measures \( \mu_f \circ h^{-1} \) and \( \mu_g \) are equivalent. As both of them are probability ergodic invariant measures sharing the same support, they actually coincide. Therefore, for any \( k \geq 0 \),

\[ D_{\mu_f} \circ f^k = D_{\mu_g} \circ g^k \circ h \quad (4.11) \]

on the set \( J(f) \setminus \bigcup_{j=1}^{k+1} f^{-j}(\infty) \). Thus, let us define functions

\[ F(x) = (D_{\mu_f}(x), D_{\mu_f} \circ f^k(x)) \]

and

\[ G(y) = (D_{\mu_g}(y), D_{\mu_g} \circ g^k(y)) \]

for \( x \in U \), an open neighbourhood of \( J(f) \setminus \bigcup_{j=1}^{k+1} f^{-j}(\infty) \), and \( y \in V \), an open neighbourhood of \( J(g) \setminus \bigcup_{j=1}^{k+1} g^{-j}(\infty) \). We may and do assume that \( W \subset U \). By the choice of \( W \) we see that for \( k = n \) the function \( G \) is invertible on \( W \). By (4.11) we get that \( F(z) = G(h(z)) \) for the point \( z \)
chosen at the beginning, thus there exists an open neighbourhood $U_z \subset U$ of $h^{-1}(z)$ such that $F(U_z) \subset G(W)$. Hence, the map $G^{-1} \circ F$ is well-defined on $U_z$ and again by (4.11)

$$G^{-1} \circ F(x) = h(x)$$

for $x \in J(f) \cap U_z$.

Consider now an arbitrary point $\xi \in J(f) \setminus f^{-1}(\infty)$. By the topological transitivity of $f$ on its Julia set, there exist $k \geq 0$ and $\xi \in U_z \cap f^{-k}(\xi)$. Choose a radius $r_\xi > 0$ so small, that $r_\xi < \delta_f/2$ and $f^{-k}(B(\xi, r_\xi)) \subset U_z$. Then the function

$$g^k \circ (G^{-1} \circ F) \circ f^{-k}_\xi$$

is well-defined and real-analytic on $B(\xi, r_\xi)$. Moreover, since $h$ conjugates $f$ and $g$ on $J(f)$,

$$g^k \circ (G^{-1} \circ F) \circ f^{-k}_\xi = h$$

on $J(f) \cap B(\xi, r_\xi)$. Repeating the above construction for an arbitrary point $\xi \in J(f) \setminus f^{-1}(\infty)$ we obtain extensions of $h$ on every ball $B(\xi, r_\xi)$, $r_\xi > 0$.

Now, since $\text{HD}(J(f)) > 1$, we can glue the functions $H_\xi := g^k \circ (G^{-1} \circ F) \circ f^{-k}_\xi$, where $k$ depends on $\xi$, on the balls $B(\xi, r_\xi/2)$. Indeed, suppose that

$$B(z_1, r_1/2) \cap B(z_2, r_2/2) \neq \emptyset$$

for some $z_1, z_2 \in J(f) \setminus f^{-1}(\infty)$ and the real-analytic extensions $H_{z_i}$ are defined on $B(z_i, r_i)$, $i = 0, 1$. Without loss of generality we can assume that $r_1 \leq r_2$, then

$$z_1 \in B(z_2, (r_1 + r_2)/2) \subset B(z_2, r_2)$$

and in particular

$$J(f) \cap B(z_1, r_1) \cap B(z_2, r_2) \neq \emptyset.$$ 

Since the Hausdorff dimension of this intersection is strictly larger than 1 and both $H_{z_1}$, $H_{z_2}$ coincide with $h$ on appropriate subsets of $J(f)$, we conclude that

$$H_{z_1|B(z_1, r_1) \cap B(z_2, r_2)} = H_{z_2|B(z_1, r_1) \cap B(z_2, r_2)}.$$ 

Thus, the formula

$$H(w) = H_\xi(w) \quad \text{for} \quad \xi \in J(f) \setminus f^{-1}(\infty), \ w \in B(\xi, r_\xi/2),$$

provides a well defined, real-analytic map from $\bigcup_{\xi \in J(f) \setminus f^{-1}(\infty)} B(\xi, r_\xi/2)$ on an open neighbourhood of $J(g) \setminus g^{-1}(\infty)$, which coincides with $h$ on $J(f)$.

**Lemma 4.5** If the topological conjugacy $h$: $J(f) \rightarrow J(g)$ has a real-analytic extension on a neighbourhood of $J(f) \setminus f^{-1}(\infty)$, then $h$ has a conformal extension on an open neighbourhood of $J(f) \setminus f^{-1}(\infty)$.

**Proof.** Note that a real-analytic extension of $h$ is in general a quasiconformal map whose complex dilatation $\mu$ is well defined and continuous in a neighbourhood of $J(f)$. If $\mu$ was not zero, then it would generate a continuous $f$-invariant line field on $J(f)$, which cannot exist by Theorem 3.2 (c). This means that $\mu \equiv 0$ thus the extension of $h$ is actually conformal. \(\square\)

Before stating the last lemma, recall our notation: $\mathcal{P}^0(f) = \mathcal{P}(f) \cup \text{Crit}(f) \cup f^{-1}(\infty)$.

**Lemma 4.6** Suppose that the topological conjugacy $h$: $J(f) \rightarrow J(g)$ has a conformal extension on a neighbourhood $U_f \subset \mathbb{C} \setminus \mathcal{P}^0(f)$ of $J(f) \setminus f^{-1}(\infty)$, where $U_f$ has the property that for each point $z \in U_f$ there exists a radius $r_z > 0$ such that $J(f) \cap B(z, r_z) \neq \emptyset$ and $B(z, r_z) \cap \mathcal{P}^0(f) = \emptyset$, and $h^{-1}$: $J(g) \rightarrow J(f)$ has an extension on $U_g \subset \mathbb{C} \setminus \mathcal{P}^0(g)$ with the analogous property. Then $h$ extends to an affine conjugacy between $f$: $\mathbb{C} \rightarrow \mathbb{C}$ and $g$: $\mathbb{C} \rightarrow \mathbb{C}$. 

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Proof. Denote by \( H : U_f \rightarrow \mathbb{C} \) the holomorphic extension of \( h : J(f) \rightarrow J(g) \). Without loss of generality we can assume that \( H \) maps bounded subsets of \( U_f \) onto bounded subsets of \( \mathbb{C} \) (shrinking \( U_f \) if necessary). First we prove the following fact.

Claim. If \( B \) is an open ball contained in \( U_f \) and \( f_\ast^{-n} \) is a holomorphic inverse branch of \( f^n \) such that \( f_\ast^{-n}(B) \subset U_f \), then the composition

\[
g^n \circ H \circ f_\ast^{-n} : B \rightarrow \overline{\mathbb{C}}
\]

coincides with \( H \).

Proof of the Claim. Fix a point \( z \in B \) and let \( r_z > 0 \) comes from the assumption of the lemma. Then \( B(z, r_z) \cap P^0(f) = \emptyset \) and if we take \( w = f_\ast^{-n}(z) \in U_f \), there exists a unique holomorphic inverse branch \( f_\ast^{-n} : B(z, r_z) \rightarrow \mathbb{C} \) of \( f^n \) mapping \( z \) back to \( w \). Since \( f_\ast^{-n} \) and \( f_\ast^{-n} \) coincide on \( B \cap B(z, r_z) \), the compositions \( g^n \circ H \circ f_\ast^{-n} \) and \( g^n \circ H \circ f_\ast^{-n} \) glue together to a holomorphic function \( F : B \cup B(z, r_z) \rightarrow \mathbb{C} \).

Since \( (J(f) \setminus f^{-1}(\infty)) \cap B(z, r_z) \neq \emptyset \), there exists an open, connected set \( \hat{B} \subset B(z, r_z) \cap U_f \) such that \( B \cap (J(f) \setminus f^{-1}(\infty)) \neq \emptyset \) and \( f_\ast^{-n}(\hat{B}) \subset U_f \). Then

\[
g^n \circ H \circ f_\ast^{-n} |_{\hat{B} \cap J(f)} = h |_{\hat{B} \cap J(f)},
\]

consequently \( g^n \circ H \circ f_\ast^{-n} |_{\hat{B}} = H |_{\hat{B}} \) and \( F |_{\hat{B}} = H |_{\hat{B}} \). Since

\[
\hat{B} \subset (B(z, r_z) \cap U_f) \subset (B(z, r_z) \cap U_f) \cup B \subset U_f,
\]

so we get that \( F = H \) on \( (B(z, r_z) \cap U_f) \cup B \) and finally we conclude, that

\[
g^n \circ H \circ f_\ast^{-n} = F |_{\hat{B}} = H |_{\hat{B}}. \quad \Box
\]

Now we want to extend \( H \) to the whole plane \( \mathbb{C} \). Consider the family \( \Phi \) consisting of all pairs \( (V, \phi) \) with the following properties:

(a) \( V \) is an open subset of \( \mathbb{C} \), \( U_f \subset V \) and \( \phi : V \rightarrow \mathbb{C} \) is a holomorphic function mapping bounded subsets of \( V \) onto bounded subsets of \( \mathbb{C} \).

(b) \( \phi |_{U_f} = H \).

(c) For all \( z \in \mathbb{C}, r > 0 \) and integers \( n \geq 0 \), if \( B(f^n(z), r) \subset V \),

\[
B(f^n(z), r) \cap \bigcup_{j=0}^{n} f^j(Sing(f^{-1})) = \emptyset,
\]

and \( f_\ast^{-n}(B(f^n(z), r)) \subset U_f \), then \( \phi \) and \( g^n \circ H \circ f_\ast^{-n} \) coincide on \( B(f^n(z), r) \).

First note that \( (U_f, H) \in \Phi \). Indeed, conditions (a) and (b) are obvious, condition (c) follows directly from the Claim. In particular \( \Phi \neq \emptyset \). We introduce a partial order on \( \Phi \), namely for \( (V, \phi), (W, \psi) \in \Phi \),

\[
(V, \phi) \leq (W, \psi) \quad \text{if and only if} \quad V \subset W \quad \text{and} \quad \psi |_V = \phi.
\]

Now, if \( (V_i, \phi_i)_{i \in I} \) is a linearly ordered subset of \( \Phi \), then it has a natural upper bound \( (V, \phi) \in \Phi \), where \( V = \bigcup_{i \in I} V_i \) and \( \phi(z) = \phi_i(z) \) for \( z \in V_i \). Consequently, the Kuratowski-Zorn Lemma applies, so we can find in \( \Phi \) a maximal element, say \( (W, \psi) \).

Our aim is to show, that \( W \supset \mathbb{C} \setminus AS(f) \). Suppose on the contrary that there is a point \( y \in \partial W \setminus AS(f) \). Fix \( \delta > 0 \) so small, that \( B(y, \delta) \cap AS(f) = \emptyset \). Since \( U_f \cap J(f) \neq \emptyset \), there exists an \( n \geq 0 \) such that \( f^{-n}(y) \cap U_f \neq \emptyset \). Fix a point \( x \in f^{-n}(y) \cap U_f \) by choosing a “good” inverse
branch, i.e. omitting the trajectories of asymptotic values. Since the set \( \bigcup_{j=0}^{n} f^j(Sing(f^{-1})) \) is finite, we can find a positive \( R \leq \delta \) such that

\[
B(y, R) \cap \bigcup_{j=0}^{n} f^j(Sing(f^{-1})) \subset \{ y \}.
\]

Take \( r \in (0, R] \) so small that \( U_x \), the connected component of \( f^{-n}(B(y, r)) \) containing \( x \), is a subset of \( U_f \).

Note that \( y \) might be an iterate of a critical value, thus we need to be careful when defining inverse branches on \( B(y, r) \). Let \( \Delta_1 \) be a closed ray emanating from \( y \). Fix \( f_1^{-n} : B(y, r) \setminus \Delta_1 \to \mathbb{C} \), an arbitrary holomorphic inverse branch of \( f^n \), whose range is contained in \( U_x \). Then the composition

\[
g^n \circ H \circ f_1^{-n} : B(y, r) \setminus \Delta_1 \to \mathbb{C}
\]

is a meromorphic function, well-defined except maybe for a countable set being the closure of essential singularities. It follows from the condition (c) in the definition of \( \Phi \), that \( g^n \circ H \circ f_1^{-n} \) coincides with \( \psi \) on the connected components of \( W \cap B(y, r) \setminus \Delta_1 \). Repeat the above argument for a different closed ray \( \Delta_2 \) emanating from \( y \) and a corresponding inverse branch \( f_2^{-n} \). We get that \( g^n \circ H \circ f_1^{-n}, g^n \circ H \circ f_2^{-n} \) and \( \psi \) coincide on the connected components of \( W \cap B(y, r) \setminus (\Delta_1 \cup \Delta_2) \). So \( g^n \circ H \circ f_1^{-n} \) and \( g^n \circ H \circ f_2^{-n} \) glue together to a holomorphic function \( \psi_y \) on \( B(y, r) \setminus E \), which coincides with \( \psi \) on \( B(y, r) \cap W \), where \( E \) is a relatively closed countable subset of \( B(y, r) \) consisting of points \( b \) such that

\[
\limsup_{z \to b} |\psi_y(z)| = \infty , \ z \notin E.
\]

Suppose \( \partial W \cap B(y, r) \subset E \), then the set \( \partial W \cap B(y, r) \) would be countable and closed. We could therefore find an isolated point \( \xi_1 \in \partial W \cap B(y, r) \), i.e. there would exist \( r_2 > 0 \) such that

\[
A := B(\xi_1, r_2) \setminus \{ \xi_1 \} \subset W.
\]

But \( A \) is a bounded set and since \( A \subset W \), therefore \( \psi_y(A) \) would also be bounded. This is however a contradiction with the fact that \( \xi_1 \in E \). We conclude that \( \partial W \cap B(y, r) \setminus E \neq \emptyset \).

Take a point \( \xi \in \partial W \cap B(y, r) \setminus E \) and a radius \( r_1 > 0 \) so small, that \( B(\xi, 2r_1) \subset B(y, r) \) and \( B(\xi, 2r_1) \cap E = \emptyset \). Then \( \psi_y|_{B(\xi, 2r_1)} \) is holomorphic and \( \psi_y(B(\xi, r_1)) \) is bounded. Set \( G = B(\xi, r_1) \).

We obtain a holomorphic function \( \hat{\psi} \) defined on \( W \cup G \) such that \( \hat{\psi}|_W = \psi \) and \( \hat{\psi}|_G = \psi_y \). We claim that \( (W \cup G, \hat{\psi}) \in \Phi \). The conditions (a) and (b) are easy to check, since \( \hat{\psi}(G) = \psi_y(G) \) is a bounded set and \( \psi \) maps bounded subsets of \( W \) onto bounded subsets of \( \mathbb{C} \).

In order to prove that the condition (c) is also satisfied suppose that \( z \in U_f \) and \( r_3 > 0, n_3 \geq 0 \) are chosen so that

(i) \( B(\hat{f}^{n_3}(z), r_3) \subset W \cup G \),

(ii) \( \overline{B(\hat{f}^{n_3}(z), r_3)} \cap \bigcup_{j=0}^{n_3} f^j(Sing(f^{-1})) = \emptyset, \)

(iii) \( f_z^{-n_3}(B(\hat{f}^{n_3}(z), r_3)) \subset U_f \).

Without loss of generality we can assume that \( G \cap B(\hat{f}^{n_3}(z), r_3) \neq \emptyset \) (otherwise \( B(\hat{f}^{n_3}(z), r_3) \subset W \), while for \( (W, \psi) \) the condition (c) is satisfied). The set \( \bigcup_{j=0}^{n_3} f^j(AS(f)) \) is finite, so we can find a topological disc \( G_0 \subset G \) such that

\[
G_0 \cap B(\hat{f}^{n_3}(z), r_3) \neq \emptyset, \quad G_0 \cap W \neq 0 \quad \text{and} \quad \overline{G_0} \cap \bigcup_{j=0}^{n_3} f^j(AS(f)) = \emptyset.
\]

The choice of \( G_0 \) guarantees that a set \( A_z \), the connected component of \( f^{-n_3}(G_0 \cup B(\hat{f}^{n_3}(z), r_3)) \) containing \( z \), is bounded and disjoint from \( AS(f) \).
Take an arbitrary periodic source \( a \in J(f) \) and a small \( \varepsilon > 0 \) such that \( B(a, 2\varepsilon) \subset U_f \). There exists an integer \( k \geq 0 \) so big that \( f^k(B(a, \varepsilon)) \supset A_z \). Since the set \( \bigcup_{j=0}^{n_3+k} f^{j}(\text{Sing}(f^{-1})) \) is finite, there exists a simply connected open set \( B \subset G_0 \) such that

\[
B \cap W \neq \emptyset, \quad B \cap B(f^{n_3}(z), r_3) \neq \emptyset \quad \text{and} \quad \overline{B} \cap \bigcup_{j=0}^{n_3+k} f^{j}(\text{Sing}(f^{-1})) = \emptyset.
\]

Therefore, we can find \( f^k \circ f^{-(n_3+k)} \colon B \to \mathbb{C} \), a holomorphic inverse branch of \( f^{n_3+k} \), such that

\[
f^k \circ f^{-(n_3+k)}|_{B \cap B(f^{n_3}(z), r_3)} = f^{-n_3}|_{B \cap B(f^{n_3}(z), r_3)}
\]

and

\[
f^{-(n_3+k)}(B) \subset B(a, \varepsilon) \subset U_f.
\]

Now, it follows from the condition (c) applied to the pair \((W, \varphi)\) that

\[
g^{n_3+k} \circ H \circ f^{-(n_3+k)}|_{B \cap W} = \varphi|_B = \varphi|_{B \cap W}.
\]

Thus \( g^{n_3+k} \circ H \circ f^{-(n_3+k)} \) coincides with \( \varphi \) on \( B \). Since \( f^{-(n_3+k)} \circ B \cap B(f^{n_3}(z), r_3) \subset U_f \) and \( f^{-(n_3+k)}(B) \subset U_f \), we conclude from the Claim that

\[
g^k \circ H \circ f^{-(n_3+k)} \circ f^{n_3} = H
\]

on \( f^{-n_3}(B \cap B(f^{n_3}(z), r_3)) \). Consequently

\[
\hat{\varphi}|_{B \cap B(f^{n_3}(z), r_3)} = g^{n_3+k} \circ H \circ f^{-(n_3+k)}|_{B \cap B(f^{n_3}(z), r_3)} = g^{n_3} \circ g^k \circ H \circ f^{-(n_3+k)} \circ f^{n_3}|_{B \cap B(f^{n_3}(z), r_3)} = g^{n_3} \circ H \circ f^{-n_3}|_{B \cap B(f^{n_3}(z), r_3)}
\]

Hence

\[
\hat{\varphi}|_{B(f^{n_3}(z), r_3)} = g^{n_3} \circ H \circ f^{-n_3}|_{B(f^{n_3}(z), r_3)}.
\]

Consequently, the pair \((W \cup G, \hat{\varphi})\) satisfies the condition (c) and finally \((W \cup G, \hat{\varphi}) \in \Phi\). Since \((W, \varphi) \leq (W \cup G, \hat{\varphi}) \) and \((W, \varphi) \neq (W \cup G, \hat{\varphi})\) we get a contradiction with the maximality of \((W, \varphi)\). We conclude therefore that \( W \subset \mathbb{C} \setminus \text{AS}(f)\).

Since \( \varphi \) maps bounded sets onto bounded sets and \( \mathbb{C} \setminus W \) is finite, \( \varphi \) extends to a holomorphic function \( A_f \colon \mathbb{C} \to \mathbb{C} \) such that

\[
A_f|_{J(f)} = h.
\]

By the symmetry there also exists a holomorphic function \( A_g \colon \mathbb{C} \to \mathbb{C} \) such that

\[
A_g|_{J(g)} = h^{-1}.
\]

Thus \( A_f \circ A_g \) and \( A_g \circ A_f \) are holomorphic and coincide with the identity functions on \( J(g) \) and \( J(f) \) respectively, hence they are identities on \( \mathbb{C} \). Therefore both \( A_f \colon \mathbb{C} \to \mathbb{C} \) and \( A_g \colon \mathbb{C} \to \mathbb{C} \) are homeomorphisms. Moreover, since both map infinity to infinity, they are affine maps. \( \square \)

5 Comments

We would like to comment briefly some assumptions of Theorems 1.1 and 1.3. Note that the exceptional class specified in Definition 1.2 corresponds to the notion of linearity for rational functions (cf. [12]). Moreover, the only rational functions which may be linear, are actually critically finite with parabolic orbifolds. Note that for transcendental functions there is no general approach which would give a global classification of exceptional functions. However, there are nice results showing
that it happens rather rarely for a transcendental function to be exceptional in this or similar sense (see e.g. [6, 14]).

In Lemma 3.7 we proved that no hyperbolic entire function from class $B$ admits a continuous invariant line field supported on the Julia set. This is a general result giving no assumptions on the form of function under consideration. Note that we are not able to repeat the same construction in the meromorphic case, i.e. when $f$ has poles. Difficulties appear when we try to extend the local linearization $\psi$ to the whole plane $\mathbb{C}$, since the function obtained in this way has infinitely many essential singularities which accumulate. Moreover, there are examples of hyperbolic meromorphic functions whose Julia set is contained in the real line $\mathbb{R}$ (see [4]) and in this case the constant horizontal line field is invariant and obviously continuous. However, we conjecture that it is enough to assume that the Hausdorff dimension of the Julia set is greater than one to obtain the absence of continuous invariant line fields.

Finally we are left with the following question: do there exist any exceptional functions (in the sense of Definition 1.2) of the form (1.1) in the case when $\infty$ is an asymptotic value (hence the Hausdorff dimension of the Julia set it greater than one)? Note that these are the only functions for which we were not able to prove Theorem 3.2. As we have just mentioned, we think that in the considered class of functions it is impossible to fulfill the assertion of the point 1. of Definition 1.2. It seems moreover that the condition given in the point 2. is very restrictive and again we conjecture that there are no functions satisfying it, however we were not able to prove it so far.

References


