Abstract. MV-algebras were introduced by Chang as an algebraic counterpart of the Lukasiewicz infinite-valued logic. D. Mundici proved that the category of MV-algebras is equivalent to the category of abelian ℓ-groups with strong unit. A. Di Nola and A. Lettieri established a categorical equivalence between the category of perfect MV-algebras and the category of abelian ℓ-groups. In this paper we investigate the convergence with a fixed regulator in perfect MV-algebras using Di Nola-Lettieri functors. The main result of the paper states that every locally Archimedean MV-algebra has a unique υ-Cauchy completion.

Introduction
MV-algebras were defined by C.C. Chang in 1958 as algebraic models for the Lukasiewicz infinite-valued logic ([7]). Due to D. Mundici, MV-algebras can be viewed as intervals of abelian ℓ-groups ([14]). A special subcategory of the category of MV-algebras is the class of perfect MV-algebras which are directly connected with the incompleteness of Lukasiewicz first order logic. A. Di Nola and A. Lettieri proved in [10] that the category of perfect MV-algebras is equivalent to the category of abelian ℓ-groups.

The order convergence in abelian ℓ-groups is studied in [15] and [16], while α-convergence is presented in [1]. Š. Černák studied the convergence with a fixed regulator for abelian ℓ-groups in [4] and for Archimedean ℓ-groups in [5]. In the case of MV-algebras, the order convergence is presented in [12], α-convergence was investigated in [13] and various kinds of Cauchy completions are studied in [2]. Using the Mundici functor Γ, Š. Černák extended the convergence with a fixed regulator from abelian ℓ-groups to MV-algebras ([6]). Order convergence in perfect MV-algebras has been presented in [11]. Using methods similar to those from [11], in this paper we...
investigate the convergence with a fixed regulator in the case of perfect MV-algebras. The main result states that every locally Archimedean MV-algebra has a unique \( \nu \)-Cauchy completion. The exposition there is based on the Di Nola-Lettieri functors \( D \) and \( \Delta \). Based on the isomorphism between an arbitrary MV-algebra and some subalgebra of a perfect MV-algebra established by L.P. Belluce and A. Di Nola in [3], we investigate how some results in perfect MV-algebras can be transferred to an arbitrary MV-algebra.

1. Preliminaries

In this section we recall some definitions and results regarding MV-algebras and the convergence with fixed regulator in \( \ell \)-groups. For more details on the subject we refer the reader to [8] and respectively [4].

On an MV-algebra \( A \), the distance function \( d : A \times A \to A \) is defined by:

\[
d(x, y) = (x \odot y^-) \oplus (x^- \odot y).
\]

Among the properties of the distance function (see [11]), we will use the following:

1. \( d(x, y) = 0 \) iff \( x = y \),
2. \( d(x, y) = d(y, x) \),
3. \( d(x, 0) = x \),
4. \( d(x, z) \leq d(x, y) \oplus d(y, z) \),
5. \( x \leq y \) implies \( y = x \oplus d(x, y) \).

An element \( x \) in an MV-algebra is said to be finitely small or infinitesimal if \( x \neq 0 \) and \( nx \leq x^- \) for all \( n \in \mathbb{N} \). The set of all infinitesimals in \( A \) is denoted by \( \text{Infinit}(A) \).

The radical \( \text{Rad}(A) \) of an MV-algebra \( A \) is the intersection of all maximal ideals of \( A \). The MV-algebra \( A \) is called perfect if \( A = \text{Rad}(A) \cup (\text{Rad}(A))^- \), where

\[
(\text{Rad}(A))^- = \{ x^- \mid x \in \text{Rad}(A) \}.
\]

For any MV-algebra \( A \), \( \text{Rad}(A) = \text{Infinit}(A) \cup \{0\} \).

An MV-algebra \( A \) is said to be Archimedean or semisimple if \( nx \leq x^- \) for all \( n \in \mathbb{N} \) implies \( x = 0 \) (see [8]).

According to [9], a perfect MV-algebra \( A \) is called locally Archimedean whenever \( x, y \in \text{Rad}(A) \) are such that \( nx \leq y \) for all \( n \in \mathbb{N} \), it follows that \( x = 0 \).

Mundici proved in [14] that for any MV-algebra \( A \) there is an abelian \( \ell \)-group \( (G, +, u) \) with strong unit \( u \) such that \( A \) is isomorphic to \( \Gamma(G, u) = [0, u] \) endowed with a canonical structure of MV-algebra:

\[
x \oplus y = (x + y) \wedge u, \ x^- = u - x, \ x \odot y = (x + y - u) \vee 0.
\]

The Mundici functor \( \Gamma \) is a categorical equivalence between the category of abelian \( \ell \)-groups with strong unit and the category of MV-algebras.
In the case of perfect MV-algebras a crucial result is the categorical equivalence between the category of perfect MV-algebras and the category of abelian ℓ-groups established by A. Di Nola and A. Lettieri ([10]).

For each abelian ℓ-group \((G, +)\), consider the lexicographic product \(\mathbb{Z} \times_{lex} G\) and define the perfect MV-algebra \(\Delta(G) = \Gamma(\mathbb{Z} \times_{lex} G, (1, 0))\) with the operations:

\[
\begin{align*}
(x, y) \oplus (u, v) &= (1, 0) \land (x + u, y + v) \\
(x, y)^- &= (1, 0) - (x, y) = (1 - x, -y) \\
(x, y) \odot (u, v) &= (0, 0) \lor (x + u - 1, y + v).
\end{align*}
\]

An element of \(\Delta(G)\) has either the form \((0, g)\) with \(g \geq 0\) or the form \((1, g)\) with \(g \leq 0\) \((g \in G)\). According to [11]) we have:

(1) \((1, 0)\) is a strong unit of \(\mathbb{Z} \times_{lex} G\);
(2) If \(A\) is a perfect MV-algebra, then \((\text{Rad}(A), \oplus, 0)\) is a cancellative abelian monoid;
(3) \(\text{Rad}(\Delta(G)) = \{(0, x) \mid x \geq 0\}; (\text{Rad}(\Delta(G)))^- = \{(1, x) \mid x \leq 0\}.

On \(\text{Rad}(A) \times \text{Rad}(A)\) we define the congruence \(\approx\) by

\[
(x, y) \approx (u, v) \text{ if } x \oplus v = y \oplus u
\]

and denote by \([x, y]\) the congruence class of \((x, y) \in \text{Rad}(A) \times \text{Rad}(A)\).

Denote \(\mathcal{D}(A) = \text{Rad}(A) \times \text{Rad}(A)/\approx\) and define:

\[
[x, y] + [u, v] = [x \oplus u, y \oplus v] \\
[x, y] \leq [u, v] \text{ if } x \oplus v \leq y \oplus u.
\]

With these operations \(\mathcal{D}(A)\) becomes an abelian ℓ-group such that:

\[
[x, y] \land [u, v] = [(x \oplus v) \land (y \oplus u), y \oplus v] \\
[x, y] \lor [u, v] = [x \oplus u, (x \oplus v) \land (y \oplus u)].
\]

Di Nola-Lettieri functors \(\mathcal{D} : \mathcal{P} \to \mathcal{A}\) and \(\Delta : \mathcal{A} \to \mathcal{P}\) realize a categorical equivalence between the category \(\mathcal{P}\) of perfect MV-algebras and the category \(\mathcal{A}\) of abelian ℓ-groups([10]).

**Proposition 1.1** ([11]). In \(\mathcal{D}(A)\) we have:

(1) \(\mathcal{D}(A)^+ = \{(x, 0) \mid x \in \text{Rad}(A)\}\);
(2) \(-[x, y] = [y, x]\);
Proof. (1) Since $p \in x$, we have $x \in P$.

(2) If $x \in \ell$-subgroup of $H$.

(3) Every element of $H$ is an $v$-limit of some sequence in $G$.

If every $v$-Cauchy sequence is convergent in $G$, then $G$ is said to be $v$-Cauchy complete.

Definition 1.2 ([4]). If $G$ is Archimedean, then an Archimedean $\ell$-group $H$ is called a $v$-Cauchy completion of $G$ if the following conditions are satisfied:

(1) $G$ is an $\ell$-subgroup of $H$;

(2) $H$ is $v$-Cauchy complete;

(3) Every element of $H$ is a $v$-limit of some sequence in $G$.

The $v$-Cauchy completion for an arbitrary $\ell$-group $G$ is constructed in [4].

2. Convergence with a fixed regulator in perfect MV-algebras

The functor $\Gamma$ was used in [6] to obtain the $v$-convergence for MV-algebras from the theory of $v$-convergence in $\ell$-groups. Using the functors $D$ and $\Delta$ we will investigate the $v$-convergence in perfect MV-algebras.

Definition 2.1 ([6]). Let $A$ be an arbitrary MV-algebra and $0 < v \in A$. The sequence $(x_n)_n$ in $A$ $v$-converges to an element $x \in A$ (or $x$ is a $v$-limit of $(x_n)_n$), denoted $x_n \rightarrow_v x$, if for every $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that $p|d(x_n, x) \leq v$ for each $n \in \mathbb{N}$, $n \geq n_0$.

Proposition 2.2 ([6]). If $(x_n)_n$ and $(y_n)_n$ are sequences in an arbitrary MV-algebra $A$ and $x, y \in A$ such that $x_n \rightarrow_v x$ and $y_n \rightarrow_v y$, then: $x_n \rightarrow_v x^$, $x_n \rightarrow_v y^$, $x_n \rightarrow_v x \oplus y$, $x_n \rightarrow_v x \odot y$, $x_n \rightarrow_v x \lor y$, $x_n \rightarrow_v x \land y$.

Proposition 2.3. In an arbitrary MV-algebra $A$ the following hold:

(1) If $(x_n)_n \subseteq \text{Rad}(A)$, $0 < v \in \text{Rad}(A)$ and $x_n \rightarrow_v x$, then $x \in \text{Rad}(A)$;

(2) If $(x_n)_n \subseteq (\text{Rad}(A))^-$, $v \in (\text{Rad}(A))^-$, $v < 1$ and $x_n \rightarrow_{v^-} x$, then $x \in (\text{Rad}(A))^-$.

Proof. (1) Since $x_n \rightarrow_v x$, for each $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that $pd(x_n, x) \leq v$ for each $n \in \mathbb{N}$, $n \geq n_0$. Using the properties of the distance...
function on $A$ we have:
\[ x = d(x, 0) \leq d(x, x_n) + d(x_n, 0) = d(x_n, x) + x_n \leq v + x_n \]
Because $\text{Rad}(A)$ is an ideal and $v, x_n \in \text{Rad}(A)$ it follows that $v + x_n \in \text{Rad}(A)$ and then $x \in \text{Rad}(A)$.

(2) We have $(x_n^-)_n \subseteq \text{Rad}(A), 0 < v^- \in \text{Rad}(A)$ and apply (1). \hfill \Box

**Proposition 2.4.** Let $A$ be a locally Archimedean MV-algebra. Then:

(1) A sequence $(x_n)_n \subseteq \text{Rad}(A)$ has a unique $v$-limit for any $0 < v \in \text{Rad}(A)$;

(2) If $(x_n)_n, (y_n)_n \subseteq \text{Rad}(A)$ and $0 < v \in \text{Rad}(A)$ such that $x_n \rightarrow_v x$, $y_n \rightarrow_v y$ and $x_n \leq y_n$ for any $n \in \mathbb{N}$, then $x \leq y$.

**Proof.** (1) Consider $x_1, x_2 \in A$ such that $x_n \rightarrow_v x_1$ and $x_n \rightarrow_v x_2$. Then, by the above proposition we have $x_1, x_2 \in \text{Rad}(A)$ and by the properties of distance:

\[ \text{pd}(x_1, x_2) \leq \text{pd}(x_1, x_n) + \text{d}(x_n, x_2) \leq 2v \text{ for all } p \in \mathbb{N}. \]

Since $A$ is locally Archimedean, we get $d(x_1, x_2) = 0$, hence $x_1 = x_2$.

(2) Since $x_n \leq y_n$, we have $x_n^- \oplus y_n = 1 \rightarrow_v 1$. By Proposition 2.2 it follows that $x_n^- \oplus y \rightarrow_v x^- \oplus y$ and by (1) we get $x^- \oplus y = 1$. Thus $x \leq y$. \hfill \Box

**Proposition 2.5.** If $A$ is a perfect MV-algebra, then the following are equivalent:

(i) $A$ is locally Archimedean;

(ii) $\mathcal{D}(A)$ is an Archimedean $\ell$-group.

**Proof.** (i)⇒(ii) Consider $[x, y], [u, v] \in \mathcal{D}(A)$ such that $n[x, y] \leq [u, v]$ for any $n \in \mathbb{N}$. Using the operations defined in $\mathcal{D}(A)$ and properties of MV-algebras, for any $n \in \mathbb{N}$ we have $n[x, y] \leq [u, v] \iff n(x, ny) \leq [u, v] \iff nx \oplus v \leq ny \oplus u \iff nx \oplus (ny)^- \leq u \oplus v^-$. Hence, $nx \leq nx \oplus (ny)^- \leq u \oplus v^-$ for any $n \in \mathbb{N}$. Since $A$ is locally Archimedean, it follows that $x = 0$, and therefore $\mathcal{D}(A)$ is an Archimedean $\ell$-group.

(ii)⇒(i) Consider $x, y \in A$ such that $nx \leq y$ for any $n \in \mathbb{N}$. It follows that $n[x, 0] \leq [y, 0]$ for any $n \in \mathbb{N}$. Since $\mathcal{D}(A)$ is Archimedean, it follows that $x = 0$, hence $A$ is locally Archimedean. \hfill \Box

**Proposition 2.6.** If $A$ is a perfect MV-algebra, $(x_n)_n \subseteq \text{Rad}(A)$ and $0 < v \in \text{Rad}(A)$ then the following are equivalent:

(i) $x_n \rightarrow_v x$ in $A$;

(ii) $[x_n, 0] \rightarrow_{[v, 0]} [x, 0]$ in $\mathcal{D}(A)$.

**Proof.** (i)⇒(ii) Assume that for each $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that $\text{pd}(x_n, x) \leq v$ for each $n \in \mathbb{N}, n \geq n_0$. Then, for each $p \in \mathbb{N}$ and $n \in \mathbb{N}, n \geq n_0$.

\[ \text{pd}(x_n, x) \leq v \]

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Proof. Because \( (x_n)_n \) is a Cauchy sequence in \( A \) and 0 < \( v \in A \). The sequence \( (x_n)_n \) in \( A \) is said to be \( v \)-fundamental or \( v \)-Cauchy if for each \( p \in \mathbb{N} \) there is \( n_0 \in \mathbb{N} \) such that \( p d(x_n, x_m) \leq v \) for each \( m, n \in \mathbb{N} \), \( m \geq n \geq n_0 \).

Definition 2.7 ([6]). Let \( A \) be an arbitrary MV-algebra and 0 < \( v \in A \). The sequence \( (x_n)_n \) in \( A \) is said to be \( v \)-fundamental or \( v \)-Cauchy if for each \( p \in \mathbb{N} \) there is \( n_0 \in \mathbb{N} \) such that \( pd(x_n, x_m) \leq v \) for each \( m, n \in \mathbb{N} \), \( m \geq n \geq n_0 \).

Proposition 2.8 ([6]). Let \( A \) be an arbitrary MV-algebra and 0 < \( v \in A \). If the sequence \( (x_n)_n \) in \( A \) is \( v \)-convergent in \( A \), then \( (x_n)_n \) is \( v \)-Cauchy in \( A \).

Proposition 2.9 ([6]). Let \( A \) be an arbitrary MV-algebra and 0 < \( v \in A \). If the sequence \( (x_n)_n \) is \( v \)-Cauchy in \( A \), then the sequences \( x_n \oplus y_n \), \( x_n \odot y_n \), \( x_n \lor y_n \), \( x_n \land y_n \), \( x_n \) are \( v \)-Cauchy in \( A \).

Corollary 2.10. Let \( A \) be a perfect MV-algebra, \( (x_n)_n \subseteq \text{Rad}(A) \) and 0 < \( v \in \text{Rad}(A) \). If \( ([x_n, y_n])_n \) is a \([v, 0]\)-Cauchy sequence in \( D(A) \), then \( ([x_n, y_n])_n \) and \( ([x_n, y_n])_n \) are also \([v, 0]\)-Cauchy sequences in \( D(A) \).

Proposition 2.11. Let \( (x_n)_n \) be a \( v \)-Cauchy sequence in the perfect MV-algebra \( A \) with 0 < \( v \in \text{Rad}(A) \). Then there is \( n_0 \in \mathbb{N} \) such that \( \{x_n \mid n \geq n_0\} \subseteq \text{Rad}(A) \) or \( \{x_n \mid n \geq n_0\} \subseteq (\text{Rad}(A))^\perp \).

Proof. Because \( (x_n)_n \) is a \( v \)-Cauchy sequence, for each \( p \in \mathbb{N} \) there is \( n_0 \in \mathbb{N} \) such that \( p d(x_n, x_{n+k}) \leq v \) for each \( n, k \in \mathbb{N}, n \geq n_0 \). Thus \( d(x_n, x_{n+k}) \in \text{Rad}(A) \). Assume there are \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \) such that \( x_n \in (\text{Rad}(A))^\perp \) and \( x_{n+k} \in \text{Rad}(A) \), so \( x_{n+k} \leq x_n \). It follows that \( x_n = x_{n+k} \oplus d(x_n, x_{n+k}) \), with \( x_{n+k}, d(x_n, x_{n+k}) \in \text{Rad}(A) \). It follows that \( x_n \in \text{Rad}(A) \), which is a contradiction. Similarly, if \( x_n \in \text{Rad}(A) \) and \( x_{n+k} \in (\text{Rad}(A))^\perp \), then \( x_{n+k} \leq x_n \) and \( x_n = x_{n+k} \oplus d(x_n, x_{n+k}) \), with \( x_n, d(x_n, x_{n+k}) \in \text{Rad}(A) \). It follows that \( x_n \in \text{Rad}(A) \), which is again a contradiction.

Generally, a \( v \)-Cauchy sequence in \( A \) is not convergent (see [6]). If every \( v \)-Cauchy sequence in \( A \) is convergent, then \( A \) is said to be \( v \)-Cauchy complete.

Similar to the proof of Proposition 2.6 we can prove the following result.

Proposition 2.12. If \( A \) is perfect MV-algebra, \( (x_n)_n \subseteq \text{Rad}(A) \) and 0 < \( v \in \text{Rad}(A) \), then the following are equivalent:

(i) \( (x_n)_n \) is a \( v \)-Cauchy sequence in \( A \);
(ii) \( ([x_n, 0])_n \) is a \([v, 0]\)-Cauchy sequence in \( D(A) \).

Theorem 2.13. If \( A \) is a perfect MV-algebra and 0 < \( v \in \text{Rad}(A) \), then the following are equivalent:
Proof. (i) Suppose that \( ([x_n, y_n])_n \) is a \([v, 0]\)-Cauchy sequence in \( \mathcal{D}(A) \). It follows that \( ([x_n, y_n]^+)_n \) and \( ([x_n, y_n]^−)_n \) are also \([v, 0]\)-Cauchy sequences. By Proposition 1.1 and Proposition 2.12, \( x_n ⊕ y_n^− \) and \( x_n^− ⊕ y_n \) are \( v \)-Cauchy sequences in \( A \). Since \( A \) is \( v \)-Cauchy complete, it follows that \( x_n \circ y_n^− \to_v z_1 \) and \( x_n^− \circ y_n \to_v z_2 \), with \( z_1, z_2 \in A \). By Proposition 2.3 we have \( z_1, z_2 \in \text{Rad}(A) \). By Proposition 1.1 and Proposition 2.6 we get \( [x_n, y_n]_1 \to_v [v, 0] [z_1, 0] \) and \( [x_n, y_n]_2 \to_v [v, 0] [z_2, 0] \). Since \( [z_1, z_2] = [z_1, 0] − [z_2, 0] \), we get \( [x_n, y_n] \to_v [v, 0] [z_1, z_2] \). Thus \( \mathcal{D}(A) \) is a \([v, 0]\)-Cauchy complete group.

(ii)⇒(i) Consider the \( v \)-Cauchy sequence \( (x_n)_n \) in \( A \). Since \( (x_n)_n \subseteq \text{Rad}(A) \) and \( 0 < v \in \text{Rad}(A) \), the sequence \( ([x_n, 0])_n \) is \([v, 0]\)-Cauchy in \( \mathcal{D}(A) \). Therefore there is \( x \in \text{Rad}(A) \) such that \( [x_n, 0] \to_v [v, 0] [x, 0] \). From Proposition 2.12 it follows that \( x_n \to_v x \) in \( A \), so \( A \) is \( v \)-Cauchy complete.

**Definition 2.14.** Let \( A \) be a locally Archimedean MV-algebra and \( 0 < v \in \text{Rad}(A) \). A locally Archimedean MV-algebra \( B \) is called a \( v \)-Cauchy completion of \( A \) if the following are satisfied:

1. \( A \) is a subalgebra of \( B \);
2. \( B \) is \( v \)-Cauchy complete;
3. Every element of \( \text{Rad}(B) \) is a \( v \)-limit of some sequence in \( \text{Rad}(A) \).

**Theorem 2.15.** Let \( A, B \) be two locally Archimedean MV-algebras, \( A \subseteq B \) and \( 0 < v \in \text{Rad}(A) \). The following are equivalent:

(i) \( B \) is a \( v \)-Cauchy completion of \( A \);
(ii) \( \mathcal{D}(B) \) is a \([v, 0]\)-Cauchy completion of \( \mathcal{D}(A) \).

**Proof.** (i)⇒(ii) We prove the conditions (1)-(3) from Definition 1.2:

1. \( A \subseteq B \Rightarrow \mathcal{D}(A) \subseteq \mathcal{D}(B) \);
2. follows by Theorem 2.13;
3. Take \( [x, y] \in \mathcal{D}(B) \). Then there are two sequences \( (x_n)_n, (y_n)_n \subseteq \text{Rad}(A) \) such that \( x_n \to_v x \) and \( y_n \to_v y \). Thus, \( [x_n, y_n] \to_v [v, 0] [x, y] \), hence \( \mathcal{D}(B) \) is a \([v, 0]\)-Cauchy completion of \( \mathcal{D}(A) \).

(ii)⇒(i) We show that conditions (1)-(3) from Definition 2.14 hold:

1. holds by hypothesis and (2) holds by Theorem 2.13;
2. Take \( x \in \text{Rad}(B) \). There is a sequence \( ([x_n, 0])_n \) in \( \mathcal{D}(A) \) such that \( [x_n, 0] \to_v [v, 0] [x, 0] \). Thus \( x_n \to_v x \) and therefore \( B \) is a \( v \)-Cauchy completion of \( A \).

**Theorem 2.16.** Any locally Archimedean MV-algebra has a unique \( v \)-Cauchy completion.
Proof. Let $A$ be a locally Archimedean MV-algebra. By Proposition 2.5, $D(A)$ is an Archimedean $\ell$-group. By Theorems 3.16 and 3.17 from [4], there is a unique $v$-Cauchy completion $G$ of the abelian $\ell$-group $D(A)$. But $G = D(B)$ for some $B = \Delta(G)$, so $D(B)$ is the unique $v$-Cauchy completion of $D(A)$. By Theorem 2.15 it follows that $B$ is the unique $v$-Cauchy completion of $D(A)$.

\[\Box\]

3. The connection with interval MV-algebras

L.P. Belluce and A. Di Nola [3] established an isomorphism between an arbitrary MV-algebra and some subalgebra of a perfect MV-algebra, so it is interesting to investigate how some results in perfect MV-algebras can be transferred to an arbitrary MV-algebra.

Consider an arbitrary MV-algebra $(A, \oplus, \odot, -, 0, 1)$, $a \in A$ and the set $A_a = \{x \in A \mid 0 \leq x \leq a\}$. On $A_a$ we define the operations

\[
x \odot_a y = a \land (x \oplus y)
\]

\[
x^{-a} = a \lor x^-
\]

\[
x \odot_a y = (x^{-a} \oplus_a y^-)^-
\]

The structure $(A_a, \oplus_a, \odot_a, -, 0, 1)$ becomes an MV-algebra called an interval MV-algebra. In [3] it is proven that for any MV-algebra $B$ there is a perfect MV-algebra $A$ and an element $a \in A$ such that $a$ is a generator for $Rad(A)$ and $B$ is isomorphic to $A_a$.

We also define the join and the meet on $A_a$ as follows:

\[
x \lor_a y = (x \odot_a y^-) \oplus_a y
\]

\[
x \land_a y = (x \oplus_a y^-) \odot_a y
\]

In this section we will investigate the connections between the $v$-convergence on $A$ and $v$-convergence on $A_a$.

Proposition 3.1. [11] If $A$ is an arbitrary MV-algebra, then the following hold in $A_a$:

1. $x \odot_a y = a \odot (a^- \oplus x) \odot (a^- \oplus y)$;
2. $x \lor_a y = x \lor y$; $x \land_a y = x \land y$;
3. $x \leq_a y$ iff $x \leq y$;
4. $d_a(x, y) = d(x, y)$ where $d$ is the distance on $A$ and $d_a$ is the distance on $A_a$.

Corollary 3.2. In the MV-algebra $A$ the following hold:

1. $Rad(A_a) \subseteq Rad(A)$;
2. If $A$ is locally Archimedean, then $A_a$ is locally Archimedean.
Denote by \( \rightarrow_v^{\oplus} \) the \( v \)-convergence in \( A_a \). Applying the fact that \( d_a(x, y) = d(x, y) \), it follows that the inequalities \( pd(x, y) \leq v \) and \( pd_a(x, y) \leq v \) are equivalent.

**Corollary 3.3.** If \((x_n)_n \subseteq A_a\) and \(x, y \in A_a\) the following are equivalent:

(i) \( x_n \rightarrow_v x \);
(ii) \( x \rightarrow_v^{\oplus} x \).

**Corollary 3.4.** If \((x_n)_n \subseteq A\) and \(v \in A_a\) the following are equivalent:

(i) \((x_n)_n\) is a \( v \)-Cauchy sequence in \( A \);
(ii) \((x_n)_n\) is a \( v \)-Cauchy sequence in \( A_a \).

**Corollary 3.5.** If the locally Archimedean MV-algebra \( A \) is \( v \)-Cauchy complete, then \( A_a \) is \( v \)-Cauchy complete.

**Theorem 3.6.** Let \( A \) be a locally Archimedean MV-algebra, \( a \in A \) and \( v \in \text{Rad}(A) \). If \( B \) is the \( v \)-Cauchy completion of \( A \), then \( B_a \) is the \( v \)-Cauchy completion of \( A_a \).

**Proof.** We check the conditions (1)-(3) from Definition 2.14:

1. \( A \subseteq B \) implies \( A_a \subseteq B_a \);
2. Since \( B \) is \( v \)-Cauchy complete, by Corollary 3.5 it follows that \( B_a \) is \( v \)-Cauchy complete;
3. Let \( x \in B_a \). Since \( x \in B \) and \( B \) is the \( v \)-Cauchy completion of \( A \), there is a sequence \((x_n)_n \subseteq \text{Rad}(A)\) such that \( x_n \rightarrow_v x \). Consider \( x'_n = x_n \land a \) for every \( n \in \mathbb{N}, n \geq 1 \). We prove that \((x'_n)_n \subseteq \text{Rad}(A_a)\). Indeed, since \((x_n)_n \subseteq \text{Rad}(A) = \text{Infinit}(A) \cup \{0\}\), we have \( nx_n \leq x_n' \) for any \( n \in \mathbb{N} \). Thus \( nx'_n = n(x_n \land a) \leq nx_n \leq x_n \leq (x_n \land a)^{\rightarrow} \) for any \( n \in \mathbb{N} \), so \((x'_n)_n \subseteq \text{Rad}(A_a)\).

Finally, we prove that \( x'_n \rightarrow_v x \). As \( x \leq a \) we have \( a^{\rightarrow} \land x = 0 \) and then

\[
d(x'_n, x) = d(x_n \land a, x) = (x_n \land a)^{\rightarrow} \land x \lor (x_n \land a) \land x^{\rightarrow} =
\]

\[
= (x_n^{\rightarrow} \lor a^{\rightarrow}) \land x \lor (x_n \land a) \land x^{\rightarrow} =
\]

\[
= (x_n^{\rightarrow} \lor x \lor a^{\rightarrow} \lor x) \lor (x_n \land a \land a \land x^{\rightarrow} =
\]

\[
= (x_n^{\rightarrow} \lor x \lor x_n \land x^{\rightarrow} \lor (x_n^{\rightarrow} \lor x \lor a \land x^{\rightarrow} =
\]

\[
= d(x_n, x) \land (x_n^{\rightarrow} \lor x \lor a \land x^{\rightarrow}.
\]

Since \( x_n^{\rightarrow} \land x \leq x \leq a \) and \( a \land x^{\rightarrow} \leq a \), we get \( d(x'_n, x) \leq d(x_n, x) \land a \). But \( x_n \rightarrow_v x \) which means that for each \( p \in \mathbb{N} \) there is \( n_0 \in \mathbb{N} \) such that \( pd(x_n, x) \leq v \) for each \( n \in \mathbb{N}, n \geq n_0 \). Thus, \( pd(x'_n, x) \leq v \land a \leq v \) for each \( n \in \mathbb{N}, n \geq n_0 \), so \( x'_n \rightarrow_v x \). We conclude that \( B_a \) is the \( v \)-Cauchy completion of \( A_a \). \( \square \)
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References


DEPARTMENT OF MATHEMATICS, POLYTECHNICAL UNIVERSITY OF BUCHAREST
Splaiul Independenţei 313
BUCHAREST, ROMANIA
E-mail: lavinia.ciungu@math.pub.ro

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