Abstract. Weak states on posets are defined which are in some analogy to states on orthomodular posets used in axiomatic quantum mechanics. It is shown how certain properties of the set of weak states characterize certain properties of the underlying poset.

Orthomodular posets serve as algebraic models for logics in axiomatic quantum mechanics. States on them are considered which reflect the properties of states of the corresponding physical system. A crucial property of such states is monotonicity. In analogy to these states we define so-called weak states on an arbitrary poset. These weak states are also monotonous and play some role in the characterization of certain algebraic models of quantum systems (cf. [2]). We use properties of the set of weak states in order to characterize certain properties of the underlying poset. In this context semilattices play an important role. For the theory of semilattices we refer the reader to the recent monograph [1].

In the following let $P = (P, \leq)$ be an arbitrary but fixed non-empty poset.

**Definition 1.** We call $P$ trivial if $|P| = 1$. For every positive integer $n$ and $a_1, \ldots, a_n \in P$ put $L(a_1, \ldots, a_n) := \{x \in P \mid x \leq a_1, \ldots, a_n\}$ and $U(a_1, \ldots, a_n) := \{x \in P \mid x \geq a_1, \ldots, a_n\}$. $P$ is called connected if its Hasse diagram is a connected graph. $P$ is called upward directed if $U(a, b) \neq \emptyset$ for any $a, b \in P$.

Now we define weak states on posets.

**Definition 2.** Let $f : P \to [0, 1]$. We call $f$ a 0-weak state on $P$ if both $f$ is monotonous and $(f^{-1}(\{0\}), \leq)$ has a greatest element which we denote...
by \(\alpha(f)\). Let \(W_0(P)\) denote the set of all 0-weak states on \(P\). We call \(f\) a 1-weak state on \(P\) if both \(f\) is monotonous and \((f^{-1}(\{1\}), \leq)\) has a least element which we denote by \(\beta(f)\). Let \(W_1(P)\) denote the set of all 1-weak states on \(P\).

Next we give some examples of weak states.

**Definition 3.** For \((a, s) \in P \times (0, 1]\) let \(f_{a,s}\) denote the mapping from \(P\) to \([0, 1]\) defined by

\[
f_{a,s}(x) := \begin{cases} 0 & \text{if } x \leq a \\ s & \text{otherwise.} \end{cases}
\]

We call the mappings of the form \(f_{a,1}\)—which we will shortly denote by \(f_{a}\)—canonical 0-weak states on \(P\) and denote their set by \(C_0\)\((P)\). For \((a, s) \in P \times [0, 1]\) let \(g_{a,s}\) denote the mapping from \(P\) to \([0, 1]\) defined by

\[
g_{a,s}(x) := \begin{cases} 1 & \text{if } x \geq a \\ s & \text{otherwise.} \end{cases}
\]

We call the mappings of the form \(g_{a,0}\)—which we will shortly denote by \(g_{a}\)—canonical 1-weak states on \(P\) and denote their set by \(C_1\)\((P)\).

**Remark 4.** \(f_{a,s} = f_{b,t}\) if and only if either \((a, s) = (b, t)\) or \(a = b\) is the greatest element of \(P\). \(g_{a,s} = g_{b,t}\) if and only if either \((a, s) = (b, t)\) or \(a = b\) is the least element of \(P\).

**Lemma 5.** If \((a, s) \in P \times (0, 1]\) then \(f_{a,s} \in W_0(P)\) and \(\alpha(f_{a,s}) = a\). If \((a, s) \in P \times [0, 1]\) then \(g_{a,s} \in W_1(P)\) and \(\beta(g_{a,s}) = a\).

**Proof.** Easy.

For weak states we consider pointwise arithmetic mean and pointwise multiplication which not necessarily yields a weak state.

**Definition 6.** For all \(f, g \in [0, 1]^P\) we define \(f \oplus g, fg \in [0, 1]^P\) by \((f \oplus g)(x) := (f(x) + g(x))/2\) and \((f g)(x) := f(x)g(x)\) for all \(x \in P\).

In the subsequent proofs we often use the following formulas:

**Lemma 7.**
\[
(f \oplus g)^{-1}(\{0\}) = L(\alpha(f), \alpha(g)) \text{ for all } f, g \in W_0(P)
\]
\[
(f \oplus g)^{-1}(\{1\}) = U(\beta(f), \beta(g)) \text{ for all } f, g \in W_1(P)
\]
\[
(f g)^{-1}(\{0\}) = L(\alpha(f)) \cup L(\alpha(g)) \text{ for all } f, g \in W_0(P)
\]
\[
(f g)^{-1}(\{1\}) = U(\beta(f), \beta(g)) \text{ for all } f, g \in W_1(P).
\]

**Proof.** Clear.

Now we start our characterization results.
Theorem 8. The following are equivalent:

(i) \( \mathcal{P} \) is connected.

(ii) For all \( a, b \in \mathcal{P} \) there exist a positive integer \( n \) and \( a_0, \ldots, a_n \in \mathcal{P} \) with \( a_0 = a \) and \( a_n = b \) such that for all \( i = 1, \ldots, n \) either \( (f_{a_{i-1}} \oplus f_{a_i})^{-1}(\{0\}) \neq \emptyset \) or \( (g_{a_{i-1}}g_{a_i})^{-1}(\{1\}) \neq \emptyset \).

Proof. According to Lemma 7, \((f_{a_{i-1}} \oplus f_{a_i})^{-1}(\{0\}) = L(a_{i-1}, a_i) \) and \((g_{a_{i-1}}g_{a_i})^{-1}(\{1\}) = U(a_{i-1}, a_i) \) for all \( i = 1, \ldots, n \).

Theorem 9. Let \( a, b \in \mathcal{P} \). Then the following are equivalent:

(i) \( \mathcal{P} \) is the disjoint union of \( \{a\} \) and \( \{b\} \).

(ii) \( f_a = g_b \).

Proof. Let \( c \in \mathcal{P} \).

(i)\( \Rightarrow \) (ii): If \( c \leq a \) then \( c \not\geq b \) and hence \( f_a(c) = 0 = g_b(c) \) and if \( c \not\geq a \) then \( c \geq b \) and hence \( f_a(c) = 1 = g_b(c) \).

(ii)\( \Rightarrow \) (i): If \( c \not\geq a \) then \( g_b(c) = f_a(c) = 1 \) and hence \( c \geq b \). Moreover, \( c \leq a \) and \( c \geq b \) together would imply \( f_a(c) = 0 \neq 1 = g_b(c) \), a contradiction.

Theorem 10. The following are equivalent:

(i) \( \mathcal{P} \) is upward directed.

(ii) \((fg)^{-1}(\{1\}) \neq \emptyset \) for all \( f, g \in W_1(\mathcal{P}) \).

Proof. (i)\( \Rightarrow \) (ii): If \( f, g \in W_1(\mathcal{P}) \) then according to Lemma 7 \((fg)^{-1}(\{1\}) = U(\beta(f), \beta(g)) \neq \emptyset \).

(ii)\( \Rightarrow \) (i): If \( a, b \in \mathcal{P} \) then again according to Lemma 7
\[ U(a, b) = U(\beta(g_a), \beta(g_b)) = (g_ag_b)^{-1}(\{1\}) \neq \emptyset. \]

Theorem 11. The following are equivalent:

(i) \( \mathcal{P} \) is a join-semilattice.

(ii) \( W_1(\mathcal{P}) \) is a subsemigroup of \( ([0, 1], \cdot)^{P} \).

Proof. Let \( a, b \in \mathcal{P} \).

(i)\( \Rightarrow \) (ii): If \( f, g \in W_1(\mathcal{P}) \) then \( fg \) is a monotonous mapping from \( \mathcal{P} \) to \([0, 1]\) and according to Lemma 7 \((fg)^{-1}(\{1\}) = U(\beta(f), \beta(g)) = U(\beta(f) \lor \beta(g)) \) and hence \( fg \in W_1(\mathcal{P}) \) and \( \beta(fg) = \beta(f) \lor \beta(g) \).

(ii)\( \Rightarrow \) (i): \( g_a(\beta(g_ag_b))g_b(\beta(g_ag_b)) = (g_ag_b)(\beta(g_ag_b)) = 1 \) and hence \( g_a(\beta(g_ag_b)) = g_b(\beta(g_ag_b)) = 1 \) which implies \( \beta(g_ag_b) \geq a, b \). If \( c \geq a, b \) then \( (g_ag_b)(c) = g_a(c)g_b(c) = 1 \cdot 1 = 1 \) and hence \( c \geq \beta(g_ag_b) \). This shows \( a \lor b = \beta(g_ag_b) \).

Next we show that if \( W_1(\mathcal{P}) \) is a subsemigroup of \( ([0, 1], \cdot)^{P} \) then it contains a subsemigroup which is both isomorphic to \( (P, \lor) \) and a homomorphic image of \( W_1(\mathcal{P}) := (W_1(\mathcal{P}), \cdot) \).
Theorem 12. If $\mathcal{P}$ is a join-semilattice $(P, \lor)$ then $C_1(\mathcal{P})$ is the set of all idempotents of $\mathcal{W}_1(\mathcal{P})$ and a subsemilattice of $\mathcal{W}_1(\mathcal{P})$ which is both isomorphic to $(P, \lor)$ and a homomorphic image of $\mathcal{W}_1(\mathcal{P})$.

Proof. Since $f \in W_1(\mathcal{P})$ is idempotent if and only if $f(P) \subseteq \{0, 1\}$, $C_1(\mathcal{P})$ is the set of all idempotents of $\mathcal{W}_1(\mathcal{P})$. It is easy to see that $x \mapsto g_x$ is an embedding of $(P, \lor)$ into $\mathcal{W}_1(\mathcal{P})$. Moreover, it is easy to see that $\beta$ is a homomorphism from $\mathcal{W}_1(\mathcal{P})$ onto $(P, \lor)$. Hence $f \mapsto g_\beta(f)$ is a homomorphism from $\mathcal{W}_1(\mathcal{P})$ onto $(C_1(\mathcal{P}), \cdot)$. □

Theorem 13. Assume $\mathcal{P}$ to be a join-semilattice. Then the following are equivalent:

(i) $\mathcal{P}$ has a least element.
(ii) $W_1(\mathcal{P})$ has a unit element.

Proof. (i)⇒(ii): If 0 denotes the least element of $\mathcal{P}$ then $g_0$ is the constant function from $P$ to $[0, 1]$ with value 1 and therefore a unit element of $\mathcal{W}_1(\mathcal{P})$.
(ii)⇒(i): If $f$ denotes the unit element of $\mathcal{W}_1(\mathcal{P})$ and $a \in P$ then
$$f(a) = f(a) \cdot 1 = f(a)g_a(a) = (fg_a)(a) = g_a(a) = 1$$
and hence $a \geq \beta(f)$ showing that $\beta(f)$ is the least element of $\mathcal{P}$. ■

Theorem 14. Assume $\mathcal{P}$ to be a join-semilattice. Then the following are equivalent:

(i) $\mathcal{P}$ has a greatest element.
(ii) $W_1(\mathcal{P})$ has a zero element.

Proof. (i)⇒(ii): If 1 denotes the greatest element of $\mathcal{P}$ then $g_1(1) = 1$ and $g_1(P \setminus \{1\}) = \{0\}$ and therefore $g_1$ is a zero element of $\mathcal{W}_1(\mathcal{P})$.
(ii)⇒(i): If $f$ denotes the zero element of $\mathcal{W}_1(\mathcal{P})$ then
$$g_a(\beta(f)) = 1 \cdot g_a(\beta(f)) = f(\beta(f))g_a(\beta(f)) = (fg_a)(\beta(f)) = f(\beta(f)) = 1$$
and hence $\beta(f) \geq a$ showing that $\beta(f)$ is the greatest element of $\mathcal{P}$. ■

For the following theorem we need some definitions from semigroup theory.

Definition 15. Let $S = (S, \cdot)$ be a semigroup. $S$ is called regular if for every $a \in S$ there exists an element $b$ of $S$ with $aba = a$. $S$ is called cancellative if whenever $a, b, c \in S$ and either $ac = bc$ or $ca = cb$ then $a = b$.

Theorem 16. The following are equivalent:

(i) $\mathcal{P}$ is a chain.
(ii) $\mathcal{P}$ is a meet-semilattice $(P, \land)$ and every canonical 0-weak state on $\mathcal{P}$ is a homomorphism from $(P, \land)$ to $(\{0, 1\}, \cdot)$.
(iii) $W_0(\mathcal{P})$ is a subsemigroup of $([0, 1], \cdot)^P$. 
**Proof.** Let \(a, b, c \in P\).

(i) \(\Rightarrow\) (ii): \(f_a(b \land c) = f_a(b)f_a(c)\).

(ii) \(\Rightarrow\) (i): \(f_{a \land b}(a)f_{a \land b}(b) = f_{a \land b}(a \land b) = 0\) and hence \(f_{a \land b}(a) = 0\) or \(f_{a \land b}(b) = 0\) which implies \(a \leq a \land b\) or \(b \leq a \land b\) showing \(a \leq b\) or \(b \leq a\).

(i) \(\Rightarrow\) (iii): If \(f, g \in W_0(P)\) then \(fg\) is a monotonous mapping from \(P\) to \([0, 1]\) and according to Lemma 7 \((fg)^{-1}(\{0\}) = L(af) \cup L(ag) = L(af) \cup (ag)\) and hence \(fg \in W_0(P)\) and \(af = af\) \(= \alpha(af)\). Now \(f_a(af_a)f_b(af_b) = (f_a(af_a))(af_a) = 0\) and hence \(f_a(af_a) = 0\) or \(f_b(af_b) = 0\) whence \(\alpha(af) \leq a\) or \(\alpha(af) \leq b\). Together we obtain \(b \leq a\) or \(a \leq b\).

Similarly as before we can show that if \(W_0(P)\) is a subsemigroup of \((\mathbb{R}, \cdot)^\mathbb{R}\) then it contains a subsemigroup which is both isomorphic to \((P, \lor)\) and a homomorphic image of \(W_0(P) := (W_0(P), \cdot)\).

**Theorem 17.** If \(P\) is a chain then \(C_0(P)\) is the set of all idempotents of \(W_0(P)\) and a subsemilattice of \(W_0(P)\) which is both isomorphic to \((P, \lor)\) and a homomorphic image of \(W_0(P)\).

**Proof.** Since \(f \in W_0(P)\) is idempotent if and only if \(f(P) \subseteq \{0, 1\}\), \(C_0(P)\) is the set of all idempotents of \(W_0(P)\). It is easy to see that \(x \mapsto f_{x}\) is an embedding of \((P, \lor)\) into \(W_0(P)\). Moreover, it is easy to see that \(\alpha\) is a homomorphism from \(W_0(P)\) onto \((P, \lor)\). Hence \(f \mapsto f_{\alpha(f)}\) is a homomorphism from \(W_0(P)\) onto \((C_0(P), \cdot)\).

**Theorem 18.** The following are equivalent:

(i) \(P\) is trivial.

(ii) \(W_0(P)\) is a regular subsemigroup of \((\mathbb{R}, \cdot)^\mathbb{R}\).

(iii) \(W_1(P)\) is a regular subsemigroup of \((\mathbb{R}, \cdot)^\mathbb{R}\).

(iv) \(W_1(P)\) is a cancellative subsemigroup of \((\mathbb{R}, \cdot)^\mathbb{R}\).

(v) \(W_0(P)\) is a subgroup of \((\mathbb{R}, \cdot)^\mathbb{R}\).

(vi) \(W_1(P)\) is a subgroup of \((\mathbb{R}, \cdot)^\mathbb{R}\).

**Proof.** The implications (i) \(\Rightarrow\) (ii)–(vi), (v) \(\Rightarrow\) (ii) and (vi) \(\Rightarrow\) (iii), (iv) are trivial.

(ii) \(\Rightarrow\) (i): Suppose \(|P| > 1\). Then there exist two different elements \(a\) and \(b\) of \(P\). Without loss of generality, \(a \leq b\). Then for every \(f \in W_0(P)\) \((f_{a_{\frac{1}{2}}}(f_{a_{\frac{1}{2}}}))(a) \leq 1/4 < 1/2 = f_{a_{\frac{1}{2}}}(a)\) and hence \(f_{a_{\frac{1}{2}}}(f_{a_{\frac{1}{2}}} \neq f_{a_{\frac{1}{2}}}\) showing that \(W_0(P)\) is not regular in this case.

(iii) \(\Rightarrow\) (i): Suppose \(|P| > 1\). Then there exist two different elements \(a\) and \(b\) of \(P\). Without loss of generality, \(a \leq b\). Then for every \(f \in W_1(P)\) \((g_{a_{\frac{1}{2}}}(g_{a_{\frac{1}{2}}}))(b) \leq 1/4 < 1/2 = g_{a_{\frac{1}{2}}}(b)\) and hence \(g_{a_{\frac{1}{2}}}(g_{a_{\frac{1}{2}}} \neq g_{a_{\frac{1}{2}}}\) showing that \(W_1(P)\) is not regular in this case.
(iv)⇒(i): Suppose $|P| > 1$. Then there exist two distinct elements $a$ and $b$ of $P$. Without loss of generality, $a < b$. Then $g_ag_a = g_{a,1/2}$, but $g_a \neq g_{a,1/2}$ since $g_a(b) = 0 \neq 1/2 = g_{a,1/2}(b)$ showing that $W_1(P)$ is not cancellative in this case.

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