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NEIGHBOURHOODS OF CERTAIN p-VALENTLY ANALYTIC FUNCTIONS DEFINED BY USING SALAGEAN OPERATOR

Abstract. By making use of the familiar concept of neighbourhood of analytic and p-valent functions, the author prove coefficient bounds and distortion inequalities and associated inclusion relations for the \((j, \theta)\)-neighbourhoods of a family of p-valent functions with negative coefficients and defined by using Salagean operator which is defined by means of a certain non-homogenous Cauchy–Euler differential equation.

1. Introduction

Let \(T(j, p)\) denote the class of functions of the form :

\[
f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \quad (a_k \geq 0; \ p, j \in N = \{1, 2, \ldots \}),
\]

which are analytic and p-valent in the open unit disc \(U = \{z : |z| < 1\}\).

A function \(f(z) \in T(j, p)\), is said to be p-valently starlike of order \(\alpha\) if it satisfies the inequality :

\[
\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U; \ 0 \leq \alpha < p; \ p \in N).
\]

We denote by \(T^*_j(p, \alpha)\) the class of all p-valently starlike functions of order \(\alpha\). Also a function \(f(z) \in T(j, p)\) is said to be p-valently convex of order \(\alpha\) if it satisfies the inequality :

\[
\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U; \ 0 \leq \alpha < p; \ p \in N).
\]

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We denote by $C_j(p, \alpha)$ the class of all $p$-valently convex functions of order $\alpha$. We note that (see for example Duren [10])

$$f(z) \in C_j(p, \alpha) \iff \frac{zf'(z)}{p} \in T_j^*(p, \alpha) \quad (0 \leq \alpha < p; \ p \in \mathbb{N}).$$

The classes $T_j^*(p, \alpha)$ and $C_j(p, \alpha)$ are studied by Owa [15].

For a function $f(z)$ in $T(j, p)$, we have

$$D_0^f(z) = f(z),$$
$$D_1^f(z) = Df(z) = \frac{z}{p} f'(z) = z^p - \sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right) a_k z^k,$$
$$D_2^f(z) = D(D_1^f(z)) = \frac{z}{p} \left( \frac{z}{p} f'(z) \right)' = z^p - \sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right)^2 a_k z^k,$$
and

$$D_n^f(z) = D(D_{n-1}^f(z)) \quad (n \in \mathbb{N}).$$

It is easy to see that

$$D_n^f(z) = z^p - \sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right)^n a_k z^k \quad (n \in N_0 = N \cup \{0\}).$$

For $j = p = 1$, the differential operator $D^n$ was introduced by Salagean [19].

Now, making use of the differential operator $D_n^f(z)$ given by (1.5), we introduce a new class $S_n(j, p, \lambda, b, \beta)$ of the $p$-valently analytic functions $f(z) \in T(j, p)$ satisfying the following inequality:

$$\left| \frac{1}{b} \left( \frac{zF_{n,p,\lambda}(z)}{F_{n,p,\lambda}(z)} - p \right) \right| < \beta$$

(1.6) $(z \in U; \ p, j \in \mathbb{N}; \ n \in N_0; 0 \leq \lambda \leq 1; \ b \in C\backslash\{0\}; \ 0 < \beta \leq 1)$,

where

$$F_{n,p,\lambda}(z) = (1 - \lambda)D_p^n f(z) + \lambda z(D_p^n f(z))'.$$

We note that:

(i) $S_0(j, p, \lambda, p - \alpha, 1) = T_j(p, \alpha, \lambda)$ $(0 \leq \alpha < p)$ (Altintas et al. [3] and [7]);

(ii) $S_0(j, 1, \lambda, 1 - \alpha, 1) = P(j, \lambda, \alpha)$ $(j \in \mathbb{N}; 0 \leq \alpha < 1; 0 \leq \lambda \leq 1)$ (Altintas [1]);

(iii) $S_n(j, 1, 0, 1 - \alpha, 1) = P(j, \alpha, n)$ $(j \in \mathbb{N}; n \in N_0; 0 \leq \alpha < 1)$ (Aouf and Srivastava [9]);

(iv) $S_0(j, p, 0, p - \alpha, 1) = T_j^*(p, \alpha)$ $(p, j \in \mathbb{N}; 0 \leq \alpha < p)$ (Owa [15] and Yamakawa [22]);
Certain \( p \)-valently analytic functions  

\[ S_0(j, p, 1, p - \alpha, 1) = C_j(p, \alpha) \ (p, j \in N; 0 \leq \alpha < p) \ \text{(Owa [15] and Yamakawa [22])}. \]

Now, following the earlier investigation by Goodman [11], Ruscheweyh [18], and others including Altintas and Owa [5], Altintas et al. ([6] and [7]), Murgusundaramoorthy and Srivastava [12], Raina and Srivastava [17], Aouf [8], Prajapat et al. [16] and Srivastava and Orhan [20] (see also [13], [14] and [21]), we define the \((j, \delta)\)-neighbourhood of a function \( f(z) \in T_p(n) \) by (see, for example, [7, p. 1668])

\[
N_{j, \theta}(f; g) = \left\{ g : g \in T(j, p), \ g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k \right. \\
\left. \quad \text{and} \quad \sum_{k=j+p}^{\infty} k|a_k - b_k| \leq \theta \right\}.
\]

In particular, if

\[
h(z) = z^p \quad (p \in N),
\]

we immediately have

\[
N_{j, \theta}(h; g) = \left\{ g : g \in T(j, p), \ g(z) = z^p - \sum_{k=j+1}^{\infty} b_k z^k \right. \\
\left. \quad \text{and} \quad \sum_{k=j+p}^{\infty} k|b_k| \leq \theta \right\}.
\]

The main object of this paper is to derive several coefficient bounds, distortion inequalities and associated inclusion relations for the \((j, \theta)\)-neighbourhood of function in the class \( H_n(j, p, \lambda, b, \beta; \delta) \) which consists of functions \( f(z) \in T(j, p) \) satisfying the following non-homogenous Cauchy-Euler differential equation :

\[
\frac{d^2 w}{dz^2} + 2(\delta + 1) \frac{dw}{dz} + \delta(\delta + 1)w = (p + \delta)(p + \delta + 1)g(z) \\
(w = f(z) \in T(j, p); g \in S_n(j, p, \lambda, b, \beta); \delta > -p \ (\delta \in R)).
\]

2. Coefficient bounds and distortion inequalities

In our present investigation of the class \( S_n(j, p, \lambda, b, \beta) \) we shall require Lemmas 1 and 2 below.

**Lemma 1.** Let the function \( f(z) \in T(j, p) \) be defined by (1.1). Then \( f(z) \) is in the class \( S_n(j, p, \lambda, b, \beta) \) if and only if

\[
\sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right)^n (k + \beta|b| - p)[1 + \lambda(k - 1)]a_k \leq \beta|b|[1 + \lambda(p - 1)].
\]
Proof. Let a function \( f(z) \) of the form (1.1) belong to the class \( S_n(j, p, \lambda, b, \beta) \). Then, in view of (1.5) and (1.6), we obtain the following inequality:

\[
(2.2) \quad \text{Re} \left\{ \frac{zF'_{n,p,\lambda}(z)}{F_{n,p,\lambda}(z)} - p \right\} > -\beta|b| \quad (z \in U),
\]

or, equivalently,

\[
(2.3) \quad \text{Re} \left\{ \frac{-\sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right)^n (k-p)[1 + \lambda(k-1)]a_k z^{k-p}}{[1 + \lambda(p-1)] - \sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right)^n [1 + \lambda(k-1)]a_k z^{k-p}} \right\} > -\beta|b|,
\]

\( (z \in U) \).

Setting \( z = r \) \((0 \leq r < 1)\) in (2.3), we observe that the expression in the denominator of the left-hand side of (2.3) is positive for \( r = 0 \) and also for \( 0 < r < 1 \). Thus, by letting \( r \to 1^- \) through real values, (2.3) leads us to the desired assertion of Lemma 1.

Conversely, by applying the hypothesis (2.1) and letting \( |z| = 1 \), we find from (1.6) that

\[
(2.4) \quad \left| \frac{zF'_{n,p,\lambda}(z)}{F_{n,p,\lambda}(z)} - p \right| = \left| \frac{\sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right)^n (k-p)[1 + \lambda(k-1)]a_k z^{k-p}}{[1 + \lambda(p-1)] - \sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right)^n [1 + \lambda(k-1)]a_k z^{k-p}} \right| \leq \frac{\beta|b| \{[1 + \lambda(p-1)] - \sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right)^n [1 + \lambda(k-1)]a_k \}}{[1 + \lambda(p-1)] - \sum_{k=j+p}^{\infty} \left( \frac{k}{p} \right)^n [1 + \lambda(k-1)]a_k} = \beta|b|.
\]

Hence, by the maximum modulus theorem, we have \( f(z) \in S_n(j, p, \lambda, b, \beta) \), which evidently completes the proof of Lemma 1.

Lemma 2. Let the function \( f(z) \) given by (1.1) be in the class \( S_n(j, p, \lambda, b, \beta) \). Then

\[
(2.4) \quad \sum_{k=j+p}^{\infty} a_k \leq \frac{\beta|b|[1 + \lambda(p-1)]}{(j+p)^n (j + \beta|b|)[1 + \lambda(j + p - 1)]}
\]

and

\[
(2.5) \quad \sum_{k=j+p}^{\infty} k a_k \leq \frac{(j + p)\beta|b|[1 + \lambda(p-1)]}{(j+p)^n (j + \beta|b|)[1 + \lambda(j + p - 1)]} \quad (p > |b|).
\]
Proof. By using Lemma 1, we find from (2.1) that
\[
\left(\frac{j + p}{p}\right)^n (j + \beta |b|)[1 + \lambda(j + p - 1)] \sum_{k=j+p}^\infty a_k 
\leq \sum_{k=j+p}^\infty \left(\frac{k}{p}\right)^n [k + \beta |b| - p][1 + \lambda(k - 1)]a_k 
\leq \beta |b|[1 + \lambda(p - 1)],
\]
which immediately yields the first assertion (2.4).

For the proof of the second assertion, by appealing to (2.1), we have
\[
\left(\frac{j + p}{p}\right)^n [1 + \lambda(j + p - 1)] \sum_{k=j+p}^\infty ka_k 
\leq \beta |b|[1 + \lambda(p - 1)] + \left(\frac{j + p}{p}\right)^n (p - \beta |b|)[1 + \lambda(j + p - 1)] \sum_{k=j+p}^\infty a_k 
\leq \beta |b|[1 + \lambda(p - 1)] + (p - \beta |b|)\frac{\beta |b|[1 + \lambda(p - 1)]}{(j + \beta |b|)} 
= \frac{(j + p)\beta |b|[1 + \lambda(p - 1)]}{(j + \beta |b|)}.
\]
Hence
\[
\sum_{k=j+p}^\infty ka_k \leq \frac{(j + p)\beta |b|[1 + \lambda(p - 1)]}{(j + \beta |b|)[1 + \lambda(j + p - 1)]} \quad (p > |b|),
\]
which implies the second assertion (2.5). □

Our main distortion inequalities for functions in the class $H_n(j, p, \lambda, b, \beta, \delta)$ are given by Theorem 1 below.

**Theorem 1.** Let a function $f(z) \in T(j, p)$ be in the class $H_n(j, p, \lambda, b, \beta; \delta)$. then for $z \in U$ we have
\[
|f(z)| \leq |z|^p + \frac{\beta |b|[1 + \lambda(p - 1)](p + \delta)(p + \delta + 1)}{(j + \beta |b|)[1 + \lambda(j + p - 1)](j + p + \delta)}|z|^{j+p},
\]
\[
|f(z)| \geq |z|^p - \frac{\beta |b|[1 + \lambda(p - 1)](p + \delta)(p + \delta + 1)}{(j + \beta |b|)[1 + \lambda(j + p - 1)](j + p + \delta)}|z|^{j+p},
\]
\[
|f^{(m)}(z)| \leq \begin{cases} 
\frac{p!}{(p-m)!} + \\
\frac{\beta |b|[1 + \lambda(p - 1)](p + \delta)(p + \delta + 1)(j + p)!}{(j + \beta |b|)[1 + \lambda(j + p - 1)](j + p + \delta)(j + p - m)!}|z|^j 
\end{cases} |z|^{p-m}
\]
and

\[ |f^{(m)}(z)| \geq \left\{ \frac{p!}{(p-m)!} - \frac{\beta|b|[1 + \lambda(p-1)](p+\delta)(p+\delta+1)(j+p)!}{(\frac{j+p}{p})^n(j + \beta|b|)(1 + \lambda(j+p-1))(j+p+\delta)(j+p-m)!} |z|^j \right\} |z|^{p-m}. \]

**Proof.** Suppose that \( f(z) \in T(j, p) \) is given by (1.11). Also let the function \( g(z) \in S_n(j, p, \lambda, b, \beta) \), occurring in the non-homogenous Cauchy-Euler differential equation (1.11), be given as in the definitions (1.8) and (1.10) with

\[ b_k \geq 0 \quad (k = j + p, j + p + 1, \ldots). \]

Then we easily see from (1.11) that

\[ a_k = \frac{(p+\delta)(p+\delta+1)}{(k+\delta)(k+\delta+1)} b_k \quad (k = j + p, j + p + 1, \ldots), \]

so that

\[ f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k = z^p - \sum_{k=j+p}^{\infty} \frac{(p+\delta)(p+\delta+1)}{(k+\delta)(k+\delta+1)} b_k z^k, \]

\[ |f(z)| \leq |z|^p + |z|^{j+p} \sum_{k=j+p}^{\infty} \frac{(p+\delta)(p+\delta+1)}{(k+\delta)(k+\delta+1)} b_k \]

and

\[ |f(z)| \geq |z|^p - |z|^{j+p} \sum_{k=j+p}^{\infty} \frac{(p+\delta)(p+\delta+1)}{(k+\delta)(k+\delta+1)} b_k. \]

Since \( g(z) \in S_n(j, p, \lambda, b, \beta) \), the first assertion (2.4) of Lemma 2 yields the following inequality:

\[ b_k \leq \frac{\beta|b|[1 + \lambda(p-1)]}{(\frac{j+p}{p})^n(j + \beta|b|)(1 + \lambda(j+p-1))} \frac{(p+\delta)(p+\delta+1)}{(k+\delta)(k+\delta+1)} \quad (k = j + p, j + p + 1, j + p + 2, \ldots). \]

This, in conjunction with (2.12) gives

\[ |f(z)| \leq |z|^p + \frac{\beta|b|[1 + \lambda(p-1)](p+\delta)(p+\delta+1)}{(\frac{j+p}{p})^n(j + \beta|b|)(1 + \lambda(j+p-1))} |z|^{j+p} \times \sum_{k=j+p}^{\infty} \frac{1}{(k+\delta)(k+\delta+1)} \quad (z \in U). \]
Observe that also the following identity holds:

\[
(2.16) \quad \sum_{k=j+\delta}^{\infty} \frac{1}{(k+\delta)(k+\delta+1)} = \sum_{k=j+\delta}^{\infty} \left( \frac{1}{(k+\delta)} - \frac{1}{(k+\delta+1)} \right) = \frac{1}{j+p+\delta} \quad (\delta \in R\{-j-p, -j-p-1, -j-p-2, \ldots\}).
\]

Now the assertion (2.6) of Theorem 1 follows at once from (2.15) together with (2.16). The second assertion (2.7) of Theorem 1 can be proven by, similarly applying (2.13), (2.14) and (2.16).

**Remark 1.** (i) Putting \( n = 0, \beta = 1 \) and \( b = p - \alpha, 0 \leq \alpha < p \), in Theorem 1, we obtain the result obtained by Altintas et al. [7, Theorem 1];

(ii) Putting \( n = 0, \beta = 1 \) and \( b = p - \alpha, 0 \leq \alpha < p \), in Theorem 1, we obtain the result obtained by Altintas [2, Theorem 1 with \( q = 0 \)];

(iii) Putting \( n = 0 \), in Theorem 1, we obtain the result obtained by Altintas et al. [4, Theorem 1 with \( q = 0 \)].

### 3. Neighborhoods for the classes \( S_n(j, p, \lambda, b, \beta) \) and \( H_n(j, p, \lambda, b, \beta; \delta) \)

In this section, we determine inclusion relations for the classes \( S_n(j, p, \lambda, b, \beta) \) and \( H_n(j, p, \lambda, b, \beta; \delta) \) involving the \((j, \delta)\)-neighbourhoods defined by (1.8) and (1.10).

**Theorem 2.** If \( f(z) \in T(j, p) \) is in the class \( S_n(j, p, \lambda, b, \beta) \), then

\[
S_n(j, p, \lambda, b, \beta) \subset N_{j,\theta}(h; f),
\]

where \( h(z) \) is given by (1.9) and

\[
\theta = \frac{(j+p)\beta|b|[1+\lambda(p-1)]}{(j+\delta)^n(j+\beta|b|)[1+\lambda(j+p-1)]}. \tag{3.2}
\]

**Proof.** Assertion (3.1) follows easily from the definition of \( N_{j,\theta}(h; f) \), which is given by (1.10) with \( g(z) \) replaced by \( f(z) \), and the second assertion (2.5) of Lemma 2.

**Theorem 3.** Let the function \( f(z) \in T(j, p) \) be in the class \( H_n(j, p, \lambda, b, \beta; \delta) \). Then

\[
H_n(j, p, \lambda, b, \beta; \delta) \subset N_{j,\theta}(g; f),
\]

where \( g(z) \) is given by (1.11) and

\[
\theta = \frac{(j+p)\beta|b|[1+\lambda(p-1)][j+(p+\delta)(p+\delta+2)]}{(j+\delta)^n(j+\beta|b|)[1+\lambda(j+p-1)](j+p+\delta)} \quad (p > |b|). \tag{3.4}
\]
**Proof.** Suppose that \( f(z) \in H_n(j, p, \lambda, b, \beta; \delta) \). Then we obtain

\[
(3.6) \quad \sum_{k=j+p}^{\infty} k|b_k - a_k| \leq \sum_{k=j+p}^{\infty} k b_k + \sum_{k=j+p}^{\infty} \frac{(p+\delta)(p+\delta+1)}{(k+\delta)(k+\delta+1)} k b_k.
\]

Next, since \( g(z) \in S_n(j, p, \lambda, b, \beta) \), the second assertion (2.5) of Lemma 2 yields

\[
(3.7) \quad k b_k \leq \frac{(j+p)\beta|b|[1+\lambda(p-1)]}{(\frac{j+p}{p})^n(j+\beta|b|)[1+\lambda(j+p-1)]} \times \left(1 + \sum_{k=j+p}^{\infty} \frac{(p+\delta)(p+\delta+1)}{(k+\delta)(k+\delta+1)}\right),
\]

which, by virtue of the identity (2.16), immediately yields that

\[
(3.8) \quad \sum_{k=j+p}^{\infty} k|b_k - a_k| \leq \frac{(j+p)\beta|b|[1+\lambda(p-1)]}{(\frac{j+p}{p})^n(j+\beta|b|)[1+\lambda(j+p-1)]} \times \left(1 + \sum_{k=j+p}^{\infty} \frac{(p+\delta)(p+\delta+1)}{(k+\delta)(k+\delta+1)}\right),
\]

Finally, by making use of (2.5) as well as (3.7) on the right-hand side of (3.6), we find that

\[
(3.9) \quad \sum_{k=j+p}^{\infty} k|b_k - a_k| \leq \frac{(j+p)\beta|b|[1+\lambda(p-1)]}{(\frac{j+p}{p})^n(j+\beta|b|)[1+\lambda(j+p-1)]} \times \left(1 + \sum_{k=j+p}^{\infty} \frac{(p+\delta)(p+\delta+1)}{(k+\delta)(k+\delta+1)}\right) = \theta \quad (p > |b|).
\]

Thus, by definition (1.8) with \( g(z) \) interchanged by \( f(z) \), \( f(z) \in N_{j,\theta}(g; f) \). This evidently completes the proof of Theorem 3.

**Remark 2.** (i) Putting \( n = 0 \), \( \beta = 1 \) and \( b = p - \alpha, 0 \leq \alpha < p \), in Theorem 3, we obtain the result obtained by Altintas et al. [7, Theorem 3];

(ii) Putting \( n = 0 \), \( \beta = 1 \) and \( b = p - \alpha, 0 \leq \alpha < p \), in Theorem 3, we obtain the result obtained by Altintas [2, Theorem 3 with \( q = 0 \)];

(iii) Putting \( n = 0 \), in Theorem 3, we obtain the result obtained by Altintas et al. [4, Theorem 3 with \( q = 0 \)].

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