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THE CONNECTEDNESS OF ARITHMETIC PROGRESSIONS IN FURSTENBERG'S, GOLOMB'S, AND KIRCH'S TOPOLOGIES

Abstract. In this paper we examine the connectedness of arithmetic progressions in the following topologies: Furstenberg's topology on the set of integers, Golomb's topology $\mathcal{D}$ on the set of positive integers, and Kirch's topology $\mathcal{D}'$ on the set of positive integers. Immediate consequences of these studies are theorems concerning the connectedness and the locally connectedness of the topologies $\mathcal{D}$ and $\mathcal{D}'$ proved by S. Golomb in 1959 and A. M. Kirch in 1969.

1. Preliminaries

The letters $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of integers, positive integers, and non-negative integers, respectively. The symbol $\Theta(a)$ denotes the set of all prime factors of $a \in \mathbb{N}$. For all $a, b \in \mathbb{N}$, we use the symbols (a, b) and $\mathrm{lcm}(a, b)$ to denote the greatest common divisor of a and b and the least common multiple of a and b, respectively. Moreover, for all $a, b \in \mathbb{N}$, the symbols $\{an + b\}$ and $\{an\}$ stand for the infinite arithmetic progressions:

$$\{an+b\} \stackrel{\mathrm{df}}{=} a \cdot \mathbb{N}_0 + b \quad \text{and} \quad \{an\} \stackrel{\mathrm{df}}{=} a \cdot \mathbb{N}.$$

For all $a \in \mathbb{N}$ and $b \in \mathbb{Z}$, the symbol $\{az + b\}$ denotes the infinite arithmetic progression:

$$\{az+b\} \stackrel{\mathrm{df}}{=} a \cdot \mathbb{Z} + b.$$

We use standard notation. For the basic results and notions concerning topology and number theory we refer the reader to the monographs of R. Engelking [3] and W. LeVeque [7], respectively.

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2. Introduction

In 1955 H. Furstenberg [4] defined the base of a topology $\mathcal{T}_F$ on $\mathbb{Z}$ by means of *all* arithmetic progressions $\{az+b\}$ and gave an elegant topological proof of the infinitude of primes. Moreover, Furstenberg remarked that the topology $\mathcal{T}_F$ is normal, and hence metrizable. In 2003 K. Broughan [1] defined a metric generating $\mathcal{T}_F$ and proved few interesting theorems concerning its structure. It is known that in Furstenberg's topology $\mathcal{T}_F$ each arithmetic progression is both open and closed [4], i.e. the space $(\mathbb{Z}, \mathcal{T}_F)$ is zero-dimensional [3], whence totally disconnected. In particular, $\mathbb{Z}$ is $\mathcal{T}_F$-disconnected.

In 1959 S. Golomb [5] presented a similar proof of the infinitude of primes using a topology $\mathcal{D}$ on $\mathbb{N}$ with the base

$$\mathcal{B} = \big\{\{an+b\} : (a,b) = 1\big\}, \tag{1}$$

defined in 1953 by M. Brown [2]. In the same paper Golomb proved that $\mathcal{D}$ is Hausdorff, $\mathbb{N}$ is $\mathcal{D}$-connected, and the Dirichlet's theorem (on primes in arithmetic progressions) is equivalent to the $\mathcal{D}$-density of the set of prime numbers in $\mathbb{N}$. For these reasons, $\mathcal{D}$ is often referred to as *Golomb's topology*. Immediately from condition (1) follows that each nonempty open set in Golomb's topology $\mathcal{D}$ is infinite (it contains an arithmetic progression). However, all arithmetic progressions $\{pn\}$, where p is a prime number, are closed but not open in $\mathcal{D}$ [5].

In 1969 A. M. Kirch [6] proved, that the topological space $(\mathbb{N}, \mathcal{D})$ is not locally connected. Moreover, he defined a topology $\mathcal{D}'$ on $\mathbb{N}$ with the base

$$\mathcal{B}' = \big\{\{an+b\} : (a,b) = 1, \ \ b < a, \ \ a - \text{square-free}\big\}, \tag{2}$$

and showed that set $\mathbb{N}$ with topology $\mathcal{D}'$ is Hausdorff, connected and locally connected topological space. When we compare the topologies $\mathcal{D}$ and $\mathcal{D}'$ we have

$$\mathcal{D}' \subsetneq \mathcal{D}.$$

Thus Kirch's topology $\mathcal{D}'$ is weaker then Golomb's topology $\mathcal{D}$. Immediately from condition (2) follows that each nonempty open set in Kirch's topology $\mathcal{D}'$ is infinite. Moreover, all arithmetic progressions $\{pn\}$, where p is a prime number, are closed but not open in $\mathcal{D}'$.

In this paper we study the connectedness of arithmetic progressions in Furstenberg's, Golomb's, and Kirch's topologies (Theorems 3.1, 3.3 and 3.5), and the connectedness of the set of primes in each of three given topologies (Theorems 5.1, 5.2 and 5.3). The characterizations we obtained for $(\mathbb{N}, \mathcal{D})$ and $(\mathbb{N}, \mathcal{D}')$ are generalizations of Theorem 3 proved by Golomb in [5] and Theorems 1, 2 and 5 proved by Kirch in [6].

The paper is organized as follows. In the next section we present our main results, and their proofs are given in Section 4. In the last section we examine the connectedness of the set of primes.

3. Main results

First we present the theorem concerning the connectedness of arithmetic progressions in Furstenberg's topology $\mathcal{T}_F$ on $\mathbb{Z}$.

Theorem 3.1. *Every arithmetic progression in $\mathbb{Z}$ is $\mathcal{T}_F$-disconnected.*

Clearly, all bases of the topology $\mathcal{T}_F$ contain some arithmetic progression, and $\mathbb{Z}$ is equal to the arithmetic progression $\{z+1\}$. So, using Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *The topology $\mathcal{T}_F$ is not connected and not locally connected.*

In the next theorem we give full characterization of the connectedness of arithmetic progressions in Golomb's topology $\mathcal{D}$ on $\mathbb{N}$.

Theorem 3.3. *Let $a, b \in \mathbb{N}$. The arithmetic progression $\{an + b\}$ is connected in $(\mathbb{N}, \mathcal{D})$ if and only if $\Theta(a) \subseteq \Theta(b)$. In particular,*

i) *the progression $\{an\}$ is $\mathcal{D}$-connected, and*
ii) *if the progression $\{an + b\}$ is an element of the basis $\mathcal{B}$, then it is $\mathcal{D}$-connected only for $a = 1$.*

We can easily see that every base of the topology $\mathcal{D}$ contains some disconnected arithmetic progression. Moreover, we have $\mathbb{N} = \{n + 1\}$. So, using Theorem 3.3, we obtain (independently of Golomb's [5, Theorem 3] and Kirch's [6, Theorem 1] results) the following corollary.

Corollary 3.4. *The topology $\mathcal{D}$ is connected and not locally connected.*

In the theorem below we present the connectedness of arithmetic progressions in Kirch's topology $\mathcal{D}'$.

Theorem 3.5. *Every arithmetic progression in $\mathbb{N}$ is $\mathcal{D}'$-connected.*

Clearly, immediate consequences of Theorem 3.5 are the following results proved by Kirch [6, Theorems 2 and 5].

Corollary 3.6. *The topology $\mathcal{D}'$ is connected and locally connected.*

4. Proofs

Proof of Theorem 3.1. Since the space $(\mathbb{Z}, \mathcal{T}_F)$ is totally disconnected and all arithmetic progressions are infinite, every arithmetic progression in $\mathbb{Z}$ is $\mathcal{T}_F$-disconnected. ■

Proof of Theorem 3.3. Let $\mathcal{B}$ be the base of the topology $\mathcal{D}$ (see (1)). Let us fix $a, b \in \mathbb{N}$.

Part "only if". Assume that $\Theta(a) \nsubseteq \Theta(b)$. Hence $a > 1$. Then there is a prime number p such that $p \mid a$ and $(p, b) = 1$. We shall show that in this case *the arithmetic progression* $\{an + b\}$ *is* $\mathcal{D}$*-disconnected.*

Since $p \mid a$, we obtain

$$\{an + b\} \subseteq \{pn + b\}. \tag{3}$$

Moreover, the assumption $(p, b) = 1$ implies $\{pn+b\} \in \mathcal{B}$ and $(pn+b, p^s) = 1$ for all $n, s \in \mathbb{N}_0$. Let us choose $t \in \mathbb{N} \setminus \{1\}$ such that $p^{t-1} \mid a$ and $p^t \nmid a$. Then for $k \in \{0, \dots, p^{t-1} - 1\}$ the progressions $\{p^t n + (pk + b)\}$ are pairwise disjoint and $\mathcal{D}$-open (as elements of the basis $\mathcal{B}$) and it is easy to check that

$$\{pn + b\} = \bigcup_{k=0}^{p^{t-1}-1} \{p^t n + (pk + b)\}. \tag{4}$$

From (3) and (4), we obtain

$$\{an + b\} = \{an + b\} \cap \bigcup_{k=0}^{p^{t-1}-1} \{p^t n + (pk + b)\} = X \cup Y,$$

where

$$X = \{an + b\} \cap \{p^t n + b\},$$
$$Y = \bigcup_{k=1}^{p^{t-1}-1} \Big(\{an + b\} \cap \{p^t n + (pk + b)\}\Big).$$

Consequently, the arithmetic progression $\{an + b\}$ splits into two disjoint sets X and Y, which are $\mathcal{D}$-open in $\{an + b\}$.

Now we will show both the sets, X and Y, are nonempty. Obviously the number $b \in \{an + b\} \cap \{p^t n + b\} = X$, whence X is nonempty. Further, by (3) we have $a + b \in \{an + b\} \subseteq \{pn + b\}$, whence

$$a + b \in \{pn + b\} \cap \{an + b\}. \tag{5}$$

Since $p^t \nmid a$, we have $a + b \notin \{p^t n + b\}$. Hence

$$a + b \notin \{p^t n + b\} \cap \{an + b\} = X. \tag{6}$$

From conditions (5) and (6) we obtain $a+b \in Y$, and so, Y is nonempty, too. We thus have proved that if $\Theta(a) \nsubseteq \Theta(b)$, then the arithmetic progression $\{an + b\}$ is $\mathcal{D}$-disconnected, as claimed.

Part "if". Now suppose the condition

$$\Theta(a) \subseteq \Theta(b) \tag{7}$$

is satisfied. We shall prove that *the $\mathcal{D}$-disconnectedness of the set $\{an+b\}$ is impossible.*

Assume the contrary: there are two disjoint nonempty sets O_1 and O_2, which are $\mathcal{D}$-open in $\{an+b\}$, such that $\{an+b\} = O_1 \cup O_2$. Hence there exist two $\mathcal{D}$-open sets U_1, U_2, such that

$$O_1 = U_1 \cap \{an+b\} \quad \text{and} \quad O_2 = U_2 \cap \{an+b\}. \tag{8}$$

Since O_1 and O_2 are nonempty, there are positive integers b_1 and b_2, such that $b_1 \in O_1 \subset U_1$ and $b_2 \in O_2 \subset U_2$. So, there are arithmetic progressions $\{a_1n+b_1\}, \{a_2n+b_2\} \in \mathcal{B}$, such that

$$\{a_1n+b_1\} \subseteq U_1 \quad \text{and} \quad \{a_2n+b_2\} \subseteq U_2. \tag{9}$$

Moreover, by (1), we have $(a_1, b_1) = 1$ and $(a_2, b_2) = 1$.

If there was a prime number p with $p \mid a$ and $p \mid a_1$, we would have, by (7), that $p \mid b$. But since $b_1 \in \{an+b\}$, then $p \mid b_1$, which contradicts the condition $(a_1, b_1) = 1$. Hence, we must have

$$(a, a_1) = 1. \tag{10}$$

Similarly, we can show that

$$(a, a_2) = 1. \tag{11}$$

Now let us define the set $P_1 \overset{\text{df}}{=} \{an+b\} \cap \{a_1n\}$. We claim that $P_1 \neq \emptyset$ and $P_1 \subset O_1$. Since $(a, a_1) = 1$, by the Chinese Remainder Theorem (CRT), there is $\alpha \in P_1$. So, the set P_1 is nonempty indeed. Let β be an arbitrary fixed element of P_1. Since

$$\beta \in \{an+b\} = O_1 \cup O_2 \tag{12}$$

and $O_1 \cap O_2 = \emptyset$, we must have

$$\beta \in O_1 \quad \text{or} \quad \beta \in O_2. \tag{13}$$

We shall show that the second case in (13) is impossible. Otherwise, the inclusion $O_2 \subset U_2$ would imply an existence of an arithmetic progression $\{An+\beta\} \in \mathcal{B}$, such that

$$\{An+\beta\} \subseteq U_2 \quad \text{and} \quad (A, \beta) = 1 \tag{14}$$

(recall that U_2 is $\mathcal{D}$-open). Since $\beta \in \{a_1n\}$, we would have

$$(a_1, A) = 1 \quad \text{and} \quad (a, A) = 1 \tag{15}$$

(in the second case in (15), if $(a, A) > 1$, then from (7) and (12) we would obtain $(A, \beta) > 1$, which, by (14), is impossible). By CRT, applied to (10) and (15), we would get $(a_1A, a) = 1$, and hence

$$\{a_1n+b_1\} \cap \{An+\beta\} \cap \{an+b\} \neq \emptyset.$$

By (8), (9) and (14), we would have

$$O_1 \cap O_2 = U_1 \cap U_2 \cap \{an + b\} \neq \emptyset,$$

which contradicts our assumption $O_1 \cap O_2 = \emptyset$. We thus have proved that the second case in (13) is impossible. Therefore $\beta \in O_1$ for arbitrary $\beta \in P_1$, as claimed.

In a similar way we can prove that the set $P_2 \stackrel{\text{df}}{=} \{an + b\} \cap \{a_2 n\}$ is nonempty and $P_2 \subset O_2$. Let $c = \operatorname{lcm}(a_1, a_2)$. Now we define the set

$$P \stackrel{\text{df}}{=} \{an + b\} \cap \{cn\}.$$

From the definitions of P_1, P_2 and c it follows that

$$P \subset P_1 \cap P_2.$$

Since $(a, c) = 1$ (see (10) and (11)), from CRT again, we obtain $P \neq \emptyset$. Finally,

$$P \subset P_1 \cap P_2 \subset O_1 \cap O_2,$$

whence $O_1 \cap O_2 \neq \emptyset$, a contradiction. So, the assumption, that the progression $\{an + b\}$ may be $\mathcal{D}$-disconnected, was false.

Part (i). Observe that, if $b = a$, then

$$\{an + a\} = a \cdot \mathbb{N}_0 + a = a \cdot \mathbb{N} = \{an\},$$

and obviously $\Theta(a) = \Theta(b)$. Hence $\{an\}$ is $\mathcal{D}$-connected.

Part (ii). Obvious.

The proof of Theorem 3.3 is complete. ■

In the proof of Theorem 3.5 we will need the technical lemma below.

Lemma 4.1. *Assume that $a, b, c, d \in \mathbb{N}$ and $b < a$. If $\{an+b\} \cap \{cn+d\} \neq \emptyset$ and $a \mid c$, then $\{cn + d\} \subseteq \{an + b\}$.*

Proof. Let us fix $a, b, c, d \in \mathbb{N}$ and let $b < a$. Since $\{an + b\} \cap \{cn + d\} \neq \emptyset$ and $a \mid c$, then there is an element $x \in \{an + b\} \cap \{cn + d\}$ such that

$$x \equiv b (\operatorname{mod} a) \quad \text{and} \quad x \equiv d (\operatorname{mod} a).$$

Hence we have

$$b \equiv d (\operatorname{mod} a). \tag{16}$$

Now let $y \in \{cn + d\}$. Therefore $y \equiv d(\operatorname{mod} c)$. Since $a \mid c$, then $y \equiv d(\operatorname{mod} a)$ and, by (16), we obtain $y \equiv b(\operatorname{mod} a)$. Finally, using assumption $b < a$, we have $y \in \{an + b\}$. ■

Proof of Theorem 3.5. Let $\mathcal{B}'$ be the base of the topology $\mathcal{D}'$ (see (2)). Let us fix $a, b \in \mathbb{N}$. We shall prove that *the arithmetic progression $\{an + b\}$ is $\mathcal{D}'$-connected in* $\mathbb{N}$.

Assume the contrary: there are two disjoint nonempty sets O_1 and O_2, which are $\mathcal{D}'$-open in $\{an+b\}$, such that $\{an+b\} = O_1 \cup O_2$. Hence there exist two $\mathcal{D}'$-open sets U_1, U_2, such that

$$O_1 = U_1 \cap \{an+b\} \quad \text{and} \quad O_2 = U_2 \cap \{an+b\}. \tag{17}$$

Since O_1 and O_2 are nonempty, there are positive integers a_1 and a_2, such that $a_1 \in O_1 \subset U_1$ and $a_2 \in O_2 \subset U_2$. So, there are arithmetic progressions $\{qn+b_1\}, \{rn+b_2\} \in \mathcal{B}'$, such that

$$a_1 \in \{qn+b_1\} \subseteq U_1 \quad \text{and} \quad a_2 \in \{rn+b_2\} \subseteq U_2. \tag{18}$$

By (2), the numbers q and r are square-free, $b_1 < q$, $b_2 < r$, $(q, b_1) = 1$, and $(r, b_2) = 1$. Now we consider two cases.

Case 1. $\mathrm{lcm}(a, q) = a$ or $\mathrm{lcm}(a, r) = a$.

Assume that $\mathrm{lcm}(a, q) = a$. Since

$$a_1 \in \{an+b\} \cap \{qn+b_1\}, \quad b_1 < q \quad \text{and} \quad q \mid a,$$

then Lemma 4.1 implies $\{an+b\} \subseteq \{qn+b_1\}$. By conditions (17) and (18) we immediately obtain $O_1 = \{an+b\}$. Therefore $O_2 = \emptyset$. If $\mathrm{lcm}(a, r) = a$, then similarly we show that $O_1 = \emptyset$. So, in this case the assumption, that the progression $\{an+b\}$ may be $\mathcal{D}'$-disconnected, was false.

Case 2. $\mathrm{lcm}(a, q) \neq a$ and $\mathrm{lcm}(a, r) \neq a$.

Since q and r are square-free, there are square-free numbers $q_1, r_1 \geq 2$, such that

$$\mathrm{lcm}(a, q) = aq_1 \quad \text{and} \quad \mathrm{lcm}(a, r) = ar_1. \tag{19}$$

Observe that $q_1 \mid q$, $r_1 \mid r$, $(a, q_1) = 1$, and $(a, r_1) = 1$. Hence $(a, q_1 r_1) = 1$ and, by CRT, we obtain $\{an+b\} \cap \{q_1 r_1 n\} \neq \emptyset$. Let us choose

$$b' \in \{an+b\} \cap \{q_1 r_1 n\}. \tag{20}$$

Without loss of generality we can assume that $b' \in O_1$. Then $b' \neq a_2$. From (17) there is an arithmetic progression $\{sn+b_3\} \in \mathcal{B}'$, such that

$$b' \in \{sn+b_3\} \subseteq U_1. \tag{21}$$

By (2), the number s is square-free, $b_3 < s$ and $(s, b_3) = 1$. Moreover, we have

$$(s, q_1 r_1) = 1. \tag{22}$$

Indeed, if $d = (s, q_1 r_1) > 1$, then by (20) and (21) we would have $d \mid b_3$, which contradicts the condition $(s, b_3) = 1$. Now observe that $a_2, b' \in \{an+b\}$. Hence

$$a_2 - b' = ka \quad \text{for some } k \in \mathbb{Z} \setminus \{0\}. \tag{23}$$

By (22) and (23), using Euclid's algorithm, we obtain that

$$\alpha as - \beta aq_1r_1 = a_2 - b' \quad \text{for some } \alpha, \beta \in \mathbb{N}. \tag{24}$$

Put $\xi = b' + \alpha as$. Then by (20) we have $\xi \in \{an+b\}$, and by (21) we obtain $\xi \in \{sn + b_3\} \subseteq U_1$. Hence

$$\xi \in \{an + b\} \cap U_1 = O_1.$$

Now observe that from (24) we also have

$$\xi = a_2 + \beta aq_1r_1.$$

By (18) and (19) we obtain $\xi \in \{rn + b_2\} \subseteq U_2$, whence

$$\xi \in \{an + b\} \cap U_2 = O_2.$$

Finally $O_1 \cap O_2 \neq \emptyset$, a contradiction. So, the progression $\{an + b\}$ is $\mathcal{D}'$-connected. The proof of Theorem 3.5 is complete. ■

5. Prime numbers

As we mentioned earlier, using Furstenberg's and Golomb's topologies we can prove the infinitude of primes. Obviously in Kirch's topology Golomb's proof of the infinitude of primes is true, too [5, Theorem 1]. Since these proofs are very elegant, the following question can rise: Might the same methods be used to show the infinitude of some special subset of primes (e.g. twin primes or Mersenne primes)? It turns out that this is not possible. Consider, for example, Furstenberg's proof. In Furstenberg's topology $\mathcal{T}_F$ each arithmetic progression is both open and closed. As the result the union of any finite number of arithmetic progressions is closed. Note that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in P} \{pz\},$$

where P denotes the set of all primes. Since $\mathcal{T}_F$ is Hausdorff, the set $\{-1, 1\}$ is closed but not open. Hence $\bigcup_{p\in P}\{pz\}$ is not finite union of closed sets which proves that there are an infinity of primes. This proof used the obvious fact that the complement of all multiples of all primes is finite. Now let P' be some infinite subset of P. Then the complement of all multiples of all primes which belongs to P' is infinite, and it is very hard to say whether such infinite set is closed (or possible not open) in any one of the three given topologies.

In [5, Theorems 6 and 7] Golomb showed that the set of primes is $\mathcal{D}$-dense and its interior is empty (in particular, the set of primes is not $\mathcal{D}$-open). In the same way we can prove that the set of primes is $\mathcal{D}'$-dense and its interior is empty in $\mathcal{D}'$. So, the set of primes is not $\mathcal{D}'$-open. But in Furstenberg's

topology $\mathcal{T}_F$ on $\mathbb{Z}$ the set of primes P is not dense. Indeed,

$$\mathrm{cl}_{\mathcal{T}_F} P = P \cup \{-1, 1\},$$

which was proved by Broughan [1, Theorem 4.2]. Now we will show that the interior of P in $(\mathbb{Z}, \mathcal{T}_F)$ is empty. If $\mathrm{int}\, P \neq \emptyset$, then there was an arithmetic progression $\{az + b\} \subseteq \mathrm{int}\, P$. Recall, that the base of the topology $\mathcal{T}_F$ is the family of all arithmetic progressions $\{az + b\}$, where $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ are fixed. Without loss of generality we can assume that $b > a$. Then, for $z_0 = a + b + 1$, we have

$$az_0 + b = a(a + b + 1) + b = a^2 + ab + a + b = (a + b)(a + 1),$$

whence $az_0 + b$ is composite (see also [5, Theorem 7]). Moreover, since the space $(\mathbb{Z}, \mathcal{T}_F)$ is totally disconnected, the set of primes P is totally disconnected in $(\mathbb{Z}, \mathcal{T}_F)$, also. In particular, P is $\mathcal{T}_F$-disconnected in $\mathbb{Z}$.

Now we will prove another properties of primes.

Theorem 5.1. *The set of all prime numbers is disconnected in Golomb's and Kirch's topologies.*

Proof. First we will show that the set of primes P is $\mathcal{D}'$-disconnected. We must find two sets A and B which are disjoint, nonempty, $\mathcal{D}'$-open in P, and such that $P = A \cup B$. Define

$$A \stackrel{\mathrm{df}}{=} P \cap A_1 \quad \text{and} \quad B \stackrel{\mathrm{df}}{=} P \cap B_1,$$

where

$$A_1 = \{3n + 2\} \cup \{5n + 1\} \cup \{5n + 2\} \cup \{5n + 3\},$$
$$B_1 = \{15n + 4\}.$$

The sets A and B are $\mathcal{D}'$-open in P. Moreover, A and B are nonempty (by Dirichlet's theorem) and disjoint. We will show that $P = A \cup B$. Observe that

$$\begin{aligned} P = P \cap \mathbb{N} = P \cap \bigcup_{k=1}^{15} \{15n + k\} &= \\ = P \cap \big(A_1 \cup B_1 \cup \{15n + 9\} \cup \{15n + 10\} \cup \{15n\}\big) &= \\ = A \cup B \cup \big(P \cap \{15n + 9\}\big) \cup \big(P \cap \{15n + 10\}\big) \cup \big(P \cap \{15n\}\big)&. \end{aligned}$$

Since

$$P \cap \{15n + 9\} = P \cap \{15n + 10\} = P \cap \{15n\} = \emptyset,$$

then $P = A \cup B$. This proves that P is $\mathcal{D}'$-disconnected.

Since $\mathcal{D}' \subset \mathcal{D}$, set P is $\mathcal{D}$-disconnected also. ∎

Theorem 5.2. *The set of all prime numbers is locally connected in Furstenberg's topology.*

Proof. We will show that the set of primes P is locally connected at a point $p \in P$. Let G be $\mathcal{T}_F$-open in P and $p \in G$. We must find a $\mathcal{T}_F$-connected in P neighborhood H of p which is contained in G. Since G is $\mathcal{T}_F$-open in P and $p \in G$, there is an arithmetic progression $\{az+b\}$ such that

$$p \in \{az+b\} \cap P \subset G.$$

Let $H \stackrel{\text{df}}{=} \{p\}$. Then $p \in \{p\} \subset \{az+b\} \cap P \subset G$. Clearly, $\{p\}$ is $\mathcal{T}_F$-connected in P and $\{p\} = \{pz\} \cap P$, whence $\{p\}$ is $\mathcal{T}_F$-open in P. (Recall that $\{pz\}$ is $\mathcal{T}_F$-open in $\mathbb{Z}$.) So, the set of primes P is locally connected at a point p, which proves that P is locally connected in $(\mathbb{Z}, \mathcal{T}_F)$. ∎

Theorem 5.3. *The set of all prime numbers is not locally connected in Golomb's and Kirch's topologies.*

Proof. First we will examine the locally connectedness of the set of primes P in Kirch's topology. Suppose that P is locally connected in $(\mathbb{N}, \mathcal{D}')$. Since $\{3n+2\} \cap P$ is $\mathcal{D}'$-open in P and $2 \in \{3n+2\} \cap P$, there are $\mathcal{D}'$-open set H_0 and $\mathcal{D}'$-connected set H, such that

$$2 \in H_0 \subset H \subset \{3n+2\} \cap P.$$

Since H_0 is $\mathcal{D}'$-open in P, there is an arithmetic progression $\{an+b\} \in \mathcal{B}'$, such that

$$2 \in \{an+b\} \cap P \subset H_0.$$

Recall that $(a,b) = 1$. By Dirichlet's theorem there is a prime number $p_1 \in \{an+b\} \setminus \{2\}$. Choose $p \in \{3n+1\} \cap P$ such that $p > p_1$. Then obviously $p \notin \{3n+2\}$. Note that

$$P = P \cap \mathbb{N} = P \cap \bigcup_{k=1}^{p} \{pn+k\},$$

whence, since $P \cap \{pn\} \cap \{3n+2\} = \emptyset$, we obtain

$$P \cap \{3n+2\} = P \cap \bigcup_{k=1}^{p-1} \{pn+k\} \cap \{3n+2\}.$$

Moreover we have $2, p_1 \in H$,

$$2 \in \{pn+2\} \subset \bigcup_{k=1}^{p-1} \{pn+k\}, \quad p_1 \in \{pn+p_1\} \subset \bigcup_{k=1}^{p-1} \{pn+k\},$$

and $\{pn+2\} \cap \{pn+p_1\} = \emptyset$. Define

$$A \stackrel{\text{df}}{=} P \cap \{pn+p_1\} \cap \{3n+2\} \quad \text{and} \quad B \stackrel{\text{df}}{=} P \cap \{3n+2\} \setminus A.$$

Then $A \cap B = \emptyset$ and $P \cap \{3n+2\} = A \cup B$. Since the set $\{pn+k\}$ is open in $(\mathbb{N}, \mathcal{D}')$ for all $k \in \{1, 2, \ldots, p-1\}$, the sets A and B are $\mathcal{D}'$-open

in P. Finally, since $p_1 \in A$ and $2 \in B$, we obtain that $A \cap H$ and $B \cap H$ separate H, a contradiction. So, the set of primes P is not locally connected in $(\mathbb{N}, \mathcal{D}')$.

Since $\mathcal{D}' \subset \mathcal{D}$, set P is not locally connected in $(\mathbb{N}, \mathcal{D})$ also. ∎

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