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MODES, MODALS, AND BARYCENTRIC ALGEBRAS: A BRIEF SURVEY AND AN ADDITIVITY THEOREM

Abstract. Modes are idempotent and entropic algebras. Modals are both join semilattices and modes, where the mode structure distributes over the join. Barycentric algebras are equipped with binary operations from the open unit interval, satisfying idempotence, skew-commutativity, and skew-associativity. The article aims to give a brief survey of these structures and some of their applications. Special attention is devoted to hierarchical statistical mechanics and the modeling of complex systems. An additivity theorem for the entropy of independent combinations of systems is proved.

Contents

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1. Introduction

Modes are idempotent and entropic algebras, worthy of attention for two reasons. From the standpoint of pure mathematics, the combination of idempotence and entropicity guarantees a number of special properties which lead to a very rich structure, most notably the self-reproducing or fractal property that the set of nonempty submodes of a mode itself forms a mode. In combination with the join of submodes, this set of submodes forms a so-called modal. From the standpoint of applied mathematics, modes and modals are eminently suited to the modeling of a broad range of phenomena, especially in the study of complex systems where traditional algebras such as rings and modules are of little use, or at best are burdened with irrelevant extraneous structure. Barycentric algebras, with binary operations indexed by the open unit interval satisfying idempotence, skew-commutativty, and skew-associativity, represent perhaps the prime example of modes. They model convexity and probability, allowing extensions of these important concepts to complex systems functioning at a number of different levels. In particular, hierarchical statistical mechanics uses barycentric algebras to extend classical, convexity-based statistical mechanics and information theory to the study of complex systems.

The present article aims to give an introduction to these topics, as listed in the table of contents. Furthermore, a new additivity result is obtained for the entropy of independent combinations of systems (Theorem 15.1). It should be stressed that the article is intended as a *brief* survey. No attempt at completeness of attribution or reference is made. Instead, readers are referred to the extensive bibliography and notes in the monograph [16].

2. Algebras

A function $\tau : \Omega \to \mathbb{N}$ is called a *type*, and its domain Ω is known as the *operator domain*. Except in dual contexts, or to follow established notation from an outside area such as statistical mechanics, algebraic notation (with functions to the right of their arguments) will be employed. An *algebra* A or (A, Ω) *of type* τ has an $\omega\tau$-ary *operation*

$$\omega : A^{\omega\tau} \to A; (a_1, \dots, a_{\omega\tau}) \mapsto a_1 \dots a_{\omega\tau}\omega \tag{2.1}$$

for each operator ω. A subset B of A is a *subalgebra* if for each operator ω and elements $a_1, \dots, a_{\omega\tau}$ of A,

$$\big(\forall\ 1 \le i \le \omega\tau,\ a_i \in B\big) \Rightarrow a_1 \dots a_{\omega\tau}\omega \in B.$$

The subset B of A is a *sink* if for each operator ω and elements $a_1, \dots, a_{\omega\tau}$ of A,

$$\big(\exists\ 1 \le i \le \omega\tau.\ a_i \in B\big) \Rightarrow a_1 \dots a_{\omega\tau}\omega \in B.$$

The subset B of A is a *wall* if for each operator ω and elements $a_1, \dots, a_{\omega\tau}$ of A,

$$\bigl(\forall\ 1 \le i \le \omega\tau,\ a_i \in B\bigr) \Leftrightarrow a_1 \dots a_{\omega\tau}\omega \in B.$$

A *reduct* of (A, Ω) is an algebra of the form (A, Ξ), where Ξ is a set of operations on A derived from Ω (compare [21, §IV.1.3]). A *subreduct* of (A, Ω) is a subalgebra of a reduct of (A, Ω).

3. Modes

An algebra (A, Ω) is said to be *idempotent* if the identity

$$x \dots x\omega = x$$

is satisfied for each operation ω, or equivalently, if each singleton subset $\{a\}$ of A is a subalgebra of (A, Ω). The algebra (A, Ω) is said to be *entropic* if the identity

$$\bigl(x_{11} \dots x_{1(\omega'\tau)}\omega'\bigr) \dots \bigl(x_{(\omega\tau)1} \dots x_{(\omega\tau)(\omega'\tau)}\omega'\bigr)\omega$$
$$= \bigl(x_{11} \dots x_{(\omega\tau)1}\omega\bigr) \dots \bigl(x_{1(\omega'\tau)} \dots x_{(\omega\tau)(\omega'\tau)}\omega\bigr)\omega'$$

is satisfied for all ω, ω' in Ω, or equivalently, if each operation (2.1) is a homomorphism. The algebra (A, Ω) is said to be a *mode* if it is both idempotent and entropic, or equivalently, if each polynomial is a homomorphism.

Proposition 3.1. *The class of modes of a given type forms a variety.*

Corollary 3.2. *Products, quotients, limits, and colimits of modes are modes.*

Proposition 3.3. *Subreducts of modes are modes.*

Proposition 3.4. *Given two modes (A, Ω) and (B, Ω) of the same type, the set* $\mathrm{Hom}(A, B)$ *of homomorphisms from (A, Ω) to (B, Ω) forms a subalgebra of the power $(B, \Omega)^A$.*

Proposition 3.5. *Given two modes (A, Ω) and (B, Ω) of the same type, the* tensor product $(A \otimes B, \Omega)$, *defined by the adjointness*

$$\mathrm{Hom}\,(A \otimes B, C) \cong \mathrm{Hom}\,(A, \mathrm{Hom}\,(B, C))$$

for each mode (C, Ω), is again a mode of the given type.

Remark 3.6. Compare [21, §III.3.6] for tensor products of modules. Some universal algebraic details are worked out in [3].

4. Examples of modes

Example 4.1. (Sets and trivial semigroups) Sets form modes (of empty type). Proposition 3.3 then shows that *left trivial semigroups* $(S, \cdot)$, with $x \cdot y = x$ for all x, y in S, form modes.

Example 4.2. (Semilattices) A *semilattice* $(H, \cdot)$ is a set equipped with a single idempotent, commutative, and associative binary operation. A (*meet*) *semilattice* is an ordered set $(H, \leq_\cdot)$ in which any two elements x, y have a greatest lower bound $x \cdot y$, so that

$$x \leq_\cdot y \Leftrightarrow x \cdot y = x.$$

A *join semilattices* $(H, +)$ is defined dually, as an ordered set $(H, \leq_+)$ in which any two elements x, y have a least upper bound $x + y$, so that

$$x \leq_+ y \Leftrightarrow x + y = y.$$

Semilattices are modes.

Example 4.3. (Affine spaces) Suppose that E is a unital module over a commutative, unital ring R. A linear combination

$$\sum_{i=1}^{n} x_i r_i \tag{4.1}$$

is said to be *affine* if $\sum_{i=1}^{n} r_i = 1$. Then E, equipped with the set of all affine linear combinations, is a mode. Such modes are described as *affine spaces* (over the ring R).

Remark 4.4. For $n = 2$, an affine linear combination (4.1) is conveniently written in the form

$$x_1 x_2 \, \underline{r} = x_1(1 - r) + x_2 r \tag{4.2}$$

for r in R. The *Mal'tsev parallelogram* (compare [19]) is the operation

$$(x_1, x_2, x_3)P = x_1 - x_2 + x_3$$

satisfying the identities $(y, y, x)P = x = (x, y, y)P$. General affine linear combinations are derived from the Mal'tsev parallelogram and the binary operations (4.2) for r in R.

Example 4.5. (Convex sets) An affine $\mathbb{R}$-linear combination (4.1) is *convex* if the coefficients r_i are all non-negative. Then convex sets, as subreducts of real affine spaces closed under all the convex linear combinations, form modes.

5. Subalgebra modes and the fractal property

Let A be a set. Given an algebra structure (A, Ω) of type τ on A, the power set 2^A becomes an algebra structure of the same type under the *complex operations*

$$X_1 \ldots X_{\omega\tau} = \{x_1 \ldots x_{\omega\tau}\omega \mid \forall\ 1 \leq i \leq \omega\tau,\ x_i \in X_i\}.$$

Let AS denote the set of nonempty subalgebras of (A, Ω). If (A, Ω) is a mode, then AS forms a subalgebra of $(2^A, \Omega)$ which is again a mode.

Furthermore, there is a mode homomorphism

$$\eta : (A, \Omega) \to (AS, \Omega); a \mapsto \{a\} \tag{5.1}$$

embedding (A, Ω) as an algebra of singletons in (AS, Ω). One may adjoin the empty set to (AS, Ω) as a sink.

In a mode (A, Ω), a *polytope* is defined as a nonempty, finitely generated subalgebra. (The name is taken from the convex set case of Example 4.5.) Let AP denote the set of polytopes of the mode (A, Ω). Then AP forms a subalgebra of (AS, Ω), and (5.1) corestricts to an embedding $\eta : A \to AP$. Again, one may adjoin the empty set to (AP, Ω) as a sink.

Example 5.1. Let $(\mathbb{R}, \Omega)$ be the real line, considered as a convex set mode according to Example 4.5. Then $(\mathbb{R}P, \Omega)$ is the closed northwest halfplane $\{(a, b) \mid a \leq b\}$, again as a convex set mode. The map (5.1) embeds the real line $(\mathbb{R}, \Omega)$ as the diagonal edge $\{(a, a) \mid a \in \mathbb{R}\}$ of $\mathbb{R}P$.

Consider the diagram

$$A \xrightarrow{\eta} AS \xrightarrow{\eta} AS^2 \xrightarrow{\eta} \dots \xrightarrow{\eta} AS^n \xrightarrow{\eta} AS^{n+1} \xrightarrow{\eta} \dots \tag{5.2}$$

of modes and homomorphisms, for natural numbers n. One may take the colimit AS^∞ of this diagram. Replacing S by P, one obtains a colimit AP^∞. Corollary 3.2 then yields the following.

Proposition 5.2. *Let (A, Ω) be a mode of a given type. Then there are colimit modes (AS^∞, Ω) and (AP^∞, Ω) of the same type, with respective endomorphisms $\eta : AS^\infty \to AS^\infty$ and $\eta : AP^\infty \to AP^\infty$.*

Example 5.3. (Supergraphs) The supergraphs of Kisielewicz [7] may be described in the language of colimit modes. Suppose that V is a finite set (of *vertices*). Consider the mode $(V, \varnothing)$ of empty type. Then a *supergraph* on the vertex set V is a finite subset of VP^∞ in which each element that is not at the bottom level V covers exactly two elements under the membership relation.

6. Modals

Let $\tau : \Omega \to \mathbb{N}$ be a type. An algebra $(D, +, \Omega)$ is a *modal* if:

(a) $(D, +)$ is a join semilattice;
(b) (D, Ω) is a mode; and
(c) $\forall\ \omega \in \Omega,\ \forall\ 1 \leq i \leq \omega\tau$,

$$x_1 \dots \big(x_i + x_i'\big) \dots x_{\omega\tau}\omega = x_1 \dots x_i \dots x_{\omega\tau}\omega + x_1 \dots x_i' \dots x_{\omega\tau}\omega.$$

The final condition (c) is described as saying that the mode structure (D, Ω) *distributes* over the join semilattice $(D, +)$.

Example 6.1. (Subalgebra modals) *Let* (A, Ω) *be a mode. Use* $+$ *to denote the join of subalgebras of* (A, Ω)*. Then there are modals* $(AS, +, \Omega)$ *and* $(AP, +, \Omega)$*.*

Example 6.2. (Distributive lattices, disemilattices) *A distributive lattice* $(D, +, \cdot)$ *is a modal. Absorption may be expressed as the coincidence of the meet semilattice order* $\leq_{\cdot}$ *with the join semilattice order* $\leq_{+}$*. If one relaxes the absorption requirement, one obtains a* disemilattice*, with potentially distinct orders* $\leq_{\cdot}$ *and* $\leq_{+}$*.*

Example 6.3. (Stammered semilattices) *Let* $(H, \cdot)$ *be a semilattice. Then* $(H, \cdot, \cdot)$ *is a modal: Given the commutativity, the distributivity reduces to* $(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)$*.*

7. Properties of modals

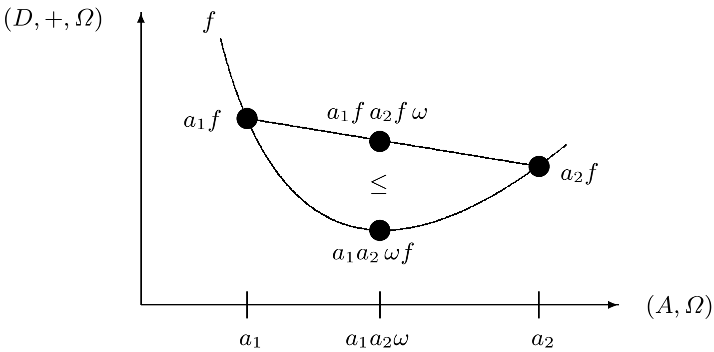

Fig. 1. A convex function.

Modals serve as codomains for generalized convex functions (compare Fig. 1). Indeed, given a mode (A, Ω) and a modal $(D, +, \Omega)$, a function $f : A \to D$ is said to be *convex* if

$$\forall\ \omega \in \Omega,\ \forall\ a_1, \ldots, a_{\omega\tau} \in A,\ a_1 \ldots a_{\omega\tau}\omega f \leq a_1 f \ldots a_{\omega\tau} f \omega.$$

In Figure 1, this condition is illustrated with a binary operation ω. Concave functions $f : A \to D$ are defined dually.

There are three basic results valid in each modal $(D.+, \Omega)$.

Lemma 7.1. (Monotonicity Lemma) *Each operation* $\omega : D^{\omega\tau} \to D$ *is monotone.*

Lemma 7.2. (Convexity Lemma) *For each positive integer* r*,*

$$\Sigma_r : D^r \to D; (x_1, \ldots, x_r) \mapsto x_1 + \cdots + x_r$$

is convex.

Lemma 7.3. (Sum-Superiority Lemma) *For each operation ω, one has $\omega \leq \Sigma_{\omega\tau}$.*

8. Barycentric algebras

Let F be a field. A unary operation of *complementation* is defined by

$$p' = 1 - p$$

for $p \in F$. A binary *dual multiplication* is defined by

$$p \circ q = (p'q')'$$

for $p, q \in F$. A binary *implication* is defined by

$$p \to q = \textbf{if } p = 0 \textbf{ then } 1 \textbf{ else } q/p \tag{8.1}$$

for $p, q \in F$. Note that for $F = \mathrm{GF}(2)$, the definition (8.1) recovers the usual Boolean implication.

Let I° denote the open unit interval $]0, 1[= \{p \in \mathbb{R} \mid 0 < p < 1\}$ in $\mathbb{R}$. Then an algebra (A, I°), with a binary operation $\underline{p}$ for each operator $p \in I^\circ$, is said to be a *barycentric algebra* if it satisfies the identities

$$xx\,\underline{p} = x$$

of *idempotence*,

$$xy\,\underline{p} = yx\,\underline{p'}$$

of *skew-commutativity*, and

$$xy\underline{p}\,z\underline{q} = x\,yz\underline{(p \circ q \to q)}\,\underline{p \circ q}$$

of *skew-associativity* for $p, q \in I^\circ$. The class $\underline{\underline{B}}$ of barycentric algebras forms a variety of modes.

9. Examples of barycentric algebras

Example 9.1. (Semilattices) A semilattice $(H, \cdot)$ may be construed as a barycentric algebra (H, I°) on setting $xy\,\underline{p} = x \cdot y$ for each $p \in I^\circ$. In this case, skew-commutativity and skew-associativity reduce respectively to ordinary commutativity and associativity.

Example 9.2. (Convex sets) Using (4.2) to define binary operations for each element p of I°, a convex set C becomes a barycentric algebra (C, I°). Indeed, the class $\underline{\underline{C}}$ of all convex sets forms the quasivariety of barycentric algebras defined by the quasi-identity

$$xy\,\underline{p} = xz\,\underline{p} \;\Rightarrow\; y = z$$

of *cancellation* for any given element p of I°.

Example 9.3. (The extended real line) The disjoint union $\mathbb{R}^\infty$ of the convex set $(\mathbb{R}, I^\circ)$ with a singleton sink $\{\infty\}$ forms a barycentric algebra

$(\mathbb{R}^\infty, I^\circ)$, the *extended real line*. Note that the two-element semilattice, a barycentric algebra according to Example 9.1, appears as the subalgebra $\{0, \infty\}$ of $\mathbb{R}^\infty$. Note further that for each real number s, the function

$$s : \mathbb{R}^\infty \to \mathbb{R}^\infty; x \mapsto \textbf{if } x < \infty \textbf{ then } xs \textbf{ else } \infty$$

is an endomorphism of $\mathbb{R}^*$.

Example 9.4. (Dual barycentric algebras) For a barycentric algebra A, the *dual* A^* is defined as $\mathrm{Hom}(A, \mathbb{R}^\infty)$ (compare Proposition 3.4). With a natural identification, one has $A \leq A^{**}$.

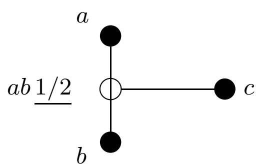

Fig. 2. The barycentric algebra T.

Example 9.5. (The "T") Let T denote the union of two copies of the closed unit interval (I, I°), one generated by endpoints a, b and the other by endpoints $ab\,\underline{2^{-1}}, c$. Suppose that $\langle a, b\rangle \cap \langle ab\,\underline{2^{-1}}, c\rangle = \{ab\,\underline{2^{-1}}\}$ (Figure 2). Define $xy\,\underline{p} = \big(ab\,\underline{2^{-1}}\big)\,(1-p) + yp$ for $x \in \langle a, b\rangle$ and $y \notin \langle a, b\rangle$. As shown by Ignatov [4], (T, I°) forms a barycentric algebra generating a quasivariety that covers the quasivariety of convex sets.

10. Free barycentric algebras and probability

For a set X, the free barycentric algebra XB and the free convex set XC over X coincide. Elements or words w in XB may be interpreted as finitely-supported probability distributions $\sum_{x \in X} xp_x$ over X: The probability of a generator x is its coefficient p_x in the convex linear combination $w = \sum_{x \in X} xp_x$. For a nonempty finite set X of cardinality n, the free convex set XC is a simplex of geometric dimension $n - 1$ (compare Figure 3).

Consider the closed unit interval barycentric algebra $A = (I, I^\circ)$. As the free barycentric algebra $\{0, 1\}B$, it represents one random bit. According to the definition given in Proposition 3.5, the tensor product $A \otimes A$ satisfies the same universality property as the free barycentric algebra on the direct product $\{0, 1\} \times \{0, 1\} = \{00, 01, 10, 11\}$ that is illustrated in Figure 3 (compare [21, III, Ex. 3.6.3] for the comparable case of modules). This picture may be contrasted with the direct product $A \times A$ illustrated in Figure 4.

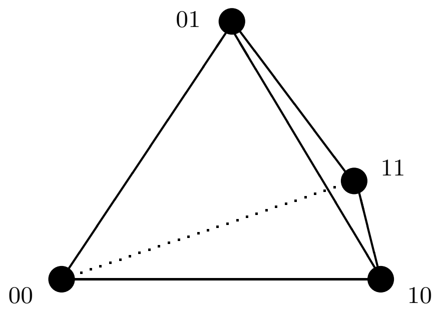

Fig. 3. The free barycentric algebra over $\{00, 01, 10, 11\}$.

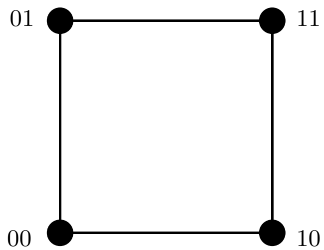

Fig. 4. The direct product $\{0, 1\}B \times \{0, 1\}B$.

In terms of probability theory, the direct product in Figure 4 represents two independent bits. By contrast, the tensor product in Figure 3 represents two entangled bits.

11. Hierarchical statistical mechanics

Classical statistical mechanics may be founded on convex sets [5]. One of the motivating applications of modes, modals, and barycentric algebras is a hierarchical statistical mechanics that extends the classical theory to complex systems functioning on a number of different levels.

Consider a finite set X known as the *state space*. The different elements of X represent the possible states of a system. Consider a function $f : X \to A$ from X to a barycentric algebra A that is generated by the image Xf of f. The function f is known as the *valuation function*. For example, it might be the energy function $E : X \to [\min_{x \in X} E(x), \max_{x \in X} E(x)]$ (measured in joules) of a real physical system X, in which case A is a closed interval in the real line.

Elements α of A may be expressed as various barycentric algebra words $\sum_{x \in X} xfp_x$ in the values xf of the states x under the valuation function.

One may associate a numerical quantity with each such word, its *entropy* $-\sum_{x\in X} p_x \log p_x$ (setting $0 \log 0 = 0$). The entropy is zero if $p_x = 1$ for some state x (for example when $\{\alpha\}$ is a wall of A and $|f^{-1}\{\alpha\}| = 1$). The entropy attains a maximum of $\log |X|$ if $p_x = |X|^{-1}$ for each state x. One thus defines the *entropy function*

$$H : A \to [0, \log |X|]; \quad \alpha \mapsto \sup\Big\{ -\sum_{x\in X} p_x \log p_x \;\Big|\; \alpha = \sum_{x\in X} x f p_x \Big\}. \tag{11.1}$$

For a classical physical system, the thermodynamic entropy S is kH with Boltzmann's constant $k \approx 1.38 \times 10^{-23}$ joules/°K.

Elements β of the dual barycentric algebra $A^* = \mathrm{Hom}(A, \mathbb{R}^\infty)$ are known as *potentials.* Dual to the entropy (11.1), one then has the *partition function*

$$Z : A^* \to [0, \infty]; \beta \mapsto \sum_{x\in X} \exp(-x f \beta) \tag{11.2}$$

with $\exp(-\infty) = 0$. For a classical physical system in equilibrium at a temperature of T°K, the potential $\beta : \big[\min_{x\in X} E(x), \max_{x\in X} E(x)\big] \to \mathbb{R}^*$ is multiplication by the scalar $1/kT$, and the *Helmholtz free energy* F (in joules) is given by $-kT \log Z(\beta)$.

Physicists understand partition functions as generating functions. For mathematicians, an example from number theory may be helpful. For a positive integer N, the state space is $\{1, 2, \dots, N\}$. The valuation is the logarithm function $\log : \{1, 2, \dots, N\} \to [0, \log N]$. For a real number s, consider the potential $s : [0, \log N] \to \mathbb{R}^*; r \mapsto rs$. Then the partition function is the partial sum $Z(s) = \sum_{n=1}^{N} n^{-s}$ of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

12. Independent systems

For the study of independent systems, such as the independent bits of Figure 4, an elementary lemma of universal algebra is fundamental.

Lemma 12.1. *Suppose that idempotent algebras M_i of a given type are generated by respective subsets Y_i, for $i = 1, 2$. Then the product algebra $M_1 \times M_2$ is generated by $Y_1 \times Y_2$.*

Proof. Consider respective elements m_i of M_i, with $m_i = y_{1i} \dots y_{n_i i} u_i$ for suitable n_i-ary derived operations u_i and elements y_{ji} of Y_i. Then

$$\begin{aligned}
(m_1, m_2) &= (y_{11} \dots y_{n_1 1} u_1, y_{12} \dots y_{n_2 2} u_2) \\
&= \big((y_{11} \dots y_{n_1 1} u_1) \dots (y_{11} \dots y_{n_1 1} u_1) u_2, y_{12} \dots y_{n_2 2} u_2\big) \\
&= (y_{11} \dots y_{n_1 1} u_1, y_{12}) \dots (y_{11} \dots y_{n_1 1} u_1, y_{n_2 2}) u_2
\end{aligned}$$

$$= (y_{11} \dots y_{n_1 1} u_1, y_{12} \dots y_{12} u_1) \dots (y_{11} \dots y_{n_1 1} u_1, y_{n_2 2} \dots y_{n_2 2} u_1) u_2$$
$$= \big((y_{11}, y_{12}) \dots (y_{n_1 1}, y_{12}) u_1\big) \dots \big((y_{11}, y_{n_2 2}) \dots (y_{n_1 1}, y_{n_2 2}) u_1\big) u_2.$$

Here, the second and fourth equalities are instances of idempotence, while the third and fifth equalities reflect the componentwise structure of $M_1 \times M_2$. Thus an arbitrary element (m_1, m_2) of $M_1 \times M_2$ is exhibited as an element of the subalgebra of $M_1 \times M_2$ generated by $Y_1 \times Y_2$. ∎

Remark 12.2. It is essential to make the assumption of idempotence in Lemma 12.1: While the additive group $M_1 = M_2 = (\mathbb{Z}/2, +, 0)$ of integers modulo 2 is generated by the subset $Y_1 = Y_2 = \{1\}$, the set $Y_1 \times Y_2$ only generates the diagonal subgroup $\{(0,0),(1,1)\}$ of $M_1 \times M_2$.

Now consider two valuation functions $f_i : X_i \to A_i$ with $A_i = \langle X_i f_i \rangle$, for $i = 1, 2$.

Corollary 12.3. *Suppose that the codomains A_i of the valuation functions f_i are generated by their respective images $X_i f_i$. Then the product algebra $A_1 \times A_2$ is generated by $(X_1 \times X_2)(f_1 \times f_2)$.*

Proof. Barycentric algebras are idempotent. ∎

In the context of Corollary 12.3, the system represented by the state space $X_1 \times X_2$ and the valuation function $f_1 \times f_2 : X_1 \times X_2 \to A_1 \times A_2$ is considered as an *independent* combination of the constituent systems represented by the respective valuation functions $f_i : X_i \to A_i$. Define the (homomorphic) projections $\pi_i : A_1 \times A_2 \to A_i; (\alpha_1, \alpha_2) \mapsto \alpha_i$. Then for constituent potentials $\beta_i \in A_i^*$, define

$$\beta_1 \oplus \beta_2 : A_1 \times A_2 \to \mathbb{R}^\infty; (\alpha_1, \alpha_2) \mapsto \alpha_1 \beta_1 + \alpha_2 \beta_2. \tag{12.1}$$

Since $\beta_1 \oplus \beta_2 = (\pi_1 \beta_1)(\pi_2 \beta_2)\, \underline{2^{-1}}\, 2$, it follows from Proposition 3.4 and Example 9.3 that (12.1) is a potential of $A_1 \times A_2$. One then has the following multiplicative property of partition functions.

Lemma 12.4. *Suppose that the respective partition functions of the systems X_1, X_2, and $X_1 \times X_2$ are Z_1, Z_2, and Z. Then for respective potentials β_i of A_i,*

$$Z(\beta_1 \oplus \beta_2) = Z_1(\beta_1) Z_2(\beta_2).$$

Proof. Using (11.2), one computes

$$Z(\beta_1 \oplus \beta_2) = \sum_{(x_1, x_2) \in X_1 \times X_2} \exp\big(-(x_1 f_1, x_2 f_2)(\beta_1 \oplus \beta_2)\big)$$
$$= \sum_{(x_1, x_2) \in X_1 \times X_2} \exp(-x_1 f_1 \beta_1 - x_2 f_2 \beta_2)$$

$$= \sum_{(x_1,x_2)\in X_1\times X_2} \exp(-x_1 f_1\beta_1)\exp(-x_2 f_2\beta_2)$$
$$= \sum_{x_1\in X_1} \exp(-x_1 f_1\beta_1) \sum_{(x_2\in X_2} \exp(-x_2 f_2\beta_2)$$
$$= Z_1(\beta_1)Z_2(\beta_2).$$

Note that the third equality remains valid if $x_i f_i \beta_i = \infty$ for some i. ■

13. Superadditivity

A basic property of the entropy function (11.1) is its superadditivity for independent systems.

Proposition 13.1. *For $i = 1, 2$, consider constituent systems represented by valuation functions $f_i : X_i \to A_i$, with respective entropy functions H_i. Suppose that the valuation function $f_1 \times f_2 : X_1 \times X_2 \to A_1 \times A_2$ of the independent combination of the constituent systems has entropy function H. Then*

$$H\big((\alpha_1, \alpha_2)\big) \geq H_1(\alpha_1) + H_2(\alpha_2) \tag{13.1}$$

for elements α_i of A_i.

Proof. Given $\varepsilon > 0$, suppose

$$\alpha_i = \sum_{x_i\in X_i} x_i f_i p_{x_i}$$

with

$$-\sum_{x_i\in X_i} p_{x_i}\log p_{x_i} > H(\alpha_i) - \frac{\varepsilon}{2}$$

for $i = 1, 2$. Then

$$(\alpha_1, \alpha_2) = \Big(\sum_{x_1\in X_1} x_1 f_1 p_{x_1}, \sum_{x_2\in X_2} x_2 f_2 p_{x_2}\Big)$$
$$= \sum_{(x_1,x_2)\in X_1\times X_2} (x_1 f_1, x_2 f_2) p_{x_1} p_{x_2}.$$

By (11.1),

$$H\big((\alpha_1, \alpha_2)\big) \geq -\sum_{(x_1,x_2)\in X_1\times X_2} p_{x_1} p_{x_2} \log\left(p_{x_1} p_{x_2}\right)$$
$$= -\sum_{(x_1,x_2)\in X_1\times X_2} p_{x_1} p_{x_2} \left(\log p_{x_1} + \log p_{x_2}\right)$$
$$= -\sum_{x_1\in X_1} \left(p_{x_1}\log p_{x_1}\right) \sum_{x_2\in X_2} p_{x_2} - \sum_{x_2\in X_2} \left(p_{x_2}\log p_{x_2}\right) \sum_{x_1\in X_1} p_{x_1}$$

$$= - \sum_{x_1 \in X_1} p_{x_1} \log p_{x_1} - \sum_{x_2 \in X_2} p_{x_2} \log p_{x_2}$$
$$> H(\alpha_1) + H(\alpha_2) - \varepsilon.$$

Since $H\big((\alpha_1, \alpha_2)\big) > H(\alpha_1) + H(\alpha_2) - \varepsilon$ holds for all positive ε, the desired inequality (13.1) follows. ∎

14. Legendre transforms

The set $\mathbb{R}^\infty$ of extended reals carries a modal structure $(\mathbb{R}^\infty, \max, I^\circ)$ which may be used as the codomain for convex and concave functions whose domain is a barycentric algebra. One then has the following Legendre transform theorem for hierarchical statistical mechanics [16, Th. 9.8.2].

Theorem 14.1. *Let X be a finite state space. Let $f : X \to A$ be a valuation function, taking values in a barycentric algebra A that is generated by the image Xf. Then:*

(a) $\forall\ \alpha \in A,\ \forall\ \beta \in A^*,\ \alpha\beta \geq H(\alpha) - \log Z(\beta)$.
(b) $-\log Z : A^* \to \mathbb{R}^\infty$ *is concave, with*
$$\forall\ \beta \in A^*,\ -\log Z(\beta) = \inf\{\alpha\beta - H(\alpha) \mid \alpha \in A\}.$$
(c) $H : A \to [0, |X|]$ *is concave, with*
$$\forall\ \alpha \in A,\ H(\alpha) = \inf\{\alpha\beta + \log Z(\beta) \mid \beta \in A^*\}.$$

The Legendre transform theorem may be illustrated on an example from semilattice character theory. Suppose that A is a finite (join) semilattice, considered as a state space. As the valuation, take the identity function $\mathrm{id}_A : A \to A$. Consider the two-element semilattice $S = \{0, \infty\}$ inside $\mathbb{R}^\infty$ (compare Example 9.3). Recall that a *character* of the semilattice A is a semilattice homomorphism $\chi : A \to S$. For an element a of A, define the *principal wall* $[a] = \{x \in A \mid x \leq a\}$. Then define the character
$$\widehat{a} : A \to S; x \mapsto \textbf{if } x \in [a] \textbf{ then } 0 \textbf{ else } \infty.$$
The concavity of $-\log Z : A^* \to \mathbb{R}^\infty$ and $H : A \to [0, \log|A|]$ give the respective inequalities
$$\big|[a] \cap [b]\big| \leq \big|[a]\big|^{1-p}\big|[b]\big|^p \leq \big|[a] \cup [b]\big|$$
for $p \in I^\circ$ and $a, b \in A$.

15. Additivity

Proposition 13.1 gave a superadditivity inequality for the entropy of an independent combination of constituent systems. The Legendre transform enables that result to be refined to an additivity, a dual of the recast version
$$-\log Z(\beta_1 \oplus \beta_2) = -\log Z_1(\beta_1) - \log Z_2(\beta_2)$$
of Lemma 12.4.

Theorem 15.1. *For $i = 1, 2$, consider constituent systems represented by valuation functions $f_i : X_i \to A_i$, with respective entropy functions H_i. Let $f : X \to A$ be the valuation function $f_1 \times f_2 : X_1 \times X_2 \to A_1 \times A_2$ of the independent combination of the constituent systems, with entropy function H. Then*

$$H\big((\alpha_1, \alpha_2)\big) = H_1(\alpha_1) + H_2(\alpha_2)$$

for elements α_i of A_i.

Proof. Proposition 13.1 already gives the superadditivity inequality

$$H\big((\alpha_1, \alpha_2)\big) \geq H_1(\alpha_1) + H_2(\alpha_2).$$

It remains to establish the reverse inequality (subadditivity). Given $\varepsilon > 0$, Theorem 14.1(c) shows that there are potentials β_i such that

$$\alpha_i\beta_i + \log Z_i(\beta_i) < H_i(\alpha_i) + \frac{\varepsilon}{2}.$$

Then by Theorem 14.1(a), (12.1), and Lemma 12.4,

$$\begin{aligned} H\big((\alpha_1, \alpha_2)\big) &\leq (\alpha_1, \alpha_2)(\beta_1 \oplus \beta_2) + \log Z(\beta_1 \oplus \beta_2) \\ &= \alpha_1\beta_1 + \log Z_1(\beta_1) + \alpha_2\beta_2 + \log Z_2(\beta_2) \\ &< H_1(\alpha_1) + H_2(\alpha_2) + \varepsilon. \end{aligned}$$

Since the inequality $H\big((\alpha_1, \alpha_2)\big) < H_1(\alpha_1) + H_2(\alpha_2) + \varepsilon$ holds for all positive constants ε, the desired subadditivity follows. ■

16. A toy model for complex systems

Complex systems are characterized by their function on many different levels. While there are already mathematical techniques (such as multiscale methods in numerical analysis) to deal with systems whose levels are comparable, truly complex systems involve levels that differ by more than a mere rescaling. This phenomenon is exhibited in biology by the levels of demography and ecology. Demography deals with the internal (age or stage) structure of a single species. Ecology is concerned with competition between different species. This section presents a toy model to show how barycentric algebras are able to handle such complexity.

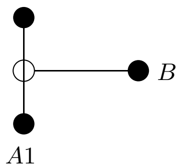

Fig. 5. Demographic and ecological levels

Consider two species, A and B. Species B is unstructured, while species A exists in two stages, $A1$ and $A2$. Ecologically, the two species are in competition for a limited supply of food. This competition is expressed by the specification that the total number of individuals in species B and species A (regardless of stage) is held constant. The system is modeled by the barycentric algebra T (compare Example 9.5) that is illustrated in Figure 5. Demographic states (mixes of the stages $A1$ and $A2$) appear as elements of the subalgebra $\langle A1, A2\rangle$, while ecological states are elements of the subalgebra $\langle A1\, A2\, \underline{2^{-1}}, B\rangle$. The biological point is that, when it comes to competition between the species A and B, it does not matter to which stage a particular individual of species A happens to belong. Thus at the ecological level, each particular individual of species A is represented by a uniform mix $A1\, A2\, \underline{2^{-1}}$ of the two stages.

It is curious to observe how the barycentric algebra T, arising in Ignatov's abstract universal-algebraic classification of the quasivarieties of barycentric algebras [4], is now able to capture the key features of this toy model of a complex system.

17. Other applications, developments, and problems

The topics chosen for this survey represent no more than a sample of the range of work being done in the theory and practice of modes, modals, and barycentric algebras. Readers may find a more comprehensive treatment, current at the turn of the millennium, in [16]. To complete the present survey, a supplementary list of applications, developments, and problems is appended:

(1) Classical real-valued support functions describe compact convex sets [8, pp. 106, 144, 231]. For arbitrary convex sets, modal theory provides support functions taking values in more general modals than the real line modal $(\mathbb{R}, \max, I^{\circ})$ used for Minkowski support functions [2, 13].

(2) The transition from affine geometry to projective geometry is usually made in an *ad hoc* fashion by adjoining a hyperplane at infinity. Modal theory provides a natural, fully invariant way to make the transition, based on the universal-algebraic technique of replication [12].

(3) Modes known as *differential groupoids* provide a purely algebraic basis for elementary differential calculus [14].

(4) Cancellative modes (of arbitrary type) embed as subreducts of affine spaces over commutative, unital rings [15]. Binary modes embed as subreducts of semiaffine spaces over commutative, unital semi-rings [6]. The problem of embedding general modes as subreducts of semiaffine

spaces remained open for some time, until examples of non-embeddable modes were provided by Stanovský [22] and Stronkowski [23].

(5) Barycentric algebras provide an efficient way to determine all the embeddings of a finite poset into linear orders [1].

(6) The existence of a full duality for barycentric algebras remains as one of the major open problems of the theory. Certainly the Legendre transform presented in §14 offers one kind of duality (especially as illustrated by the example of finite semilattices). Other fragments of a duality have appeared in [1, 9, 10, 11].

(7) The free barycentric algebra functor is used to define the type of coalgebras that yield permutation representations of quasigroups, models of approximate symmetry that generalize exact symmetry as given by permutation representations of groups [20].

(8) A more abstract axiomatization of barycentric algebras is given in [18], allowing interpretations of Boolean algebras and B-sets. In [17], this formulation is used to translate Boolean logic automatically into a logic based on barycentric algebras, providing accurate models of the kind of "fuzzy" logic used by cells in gene expression.

References

[1] G. Bińczak, A. B. Romanowska, J. D. H. Smith, *Poset extensions, convex sets, and semilattice presentations*, Discrete Math. 307 (2007), 1–11.

[2] D.-H. Choi, J. D. H. Smith, *Support functions of general convex sets*, Algebra Universalis 49 (2003), 305–319.

[3] B. A. Davey, G. Davis, *Tensor products and entropic varieties*, Algebra Universalis 21 (1985), 68–88.

[4] V. V. Ignatov, *Quasivarieties of convexors* (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. 29 (1985), 12–14.

[5] R. B. Israel, *Convexity in the Theory of Lattice Gases*, Princeton University Press, Princeton, NJ, 1979.

[6] J. Ježek, T. Kepka, *Medial Groupoids*, Academia, Praha, 1983.

[7] A. Kisielewicz, *Supergraphs and combinatorial complexity of permutation groups*, Ars Combin., to appear.

[8] E. Minkowski, *Gesammelte Abhandlungen* Bd. 2, Teubner, Leipzig, 1911.

[9] K. J. Pszczoła, A. B. Romanowska, J. D. H. Smith, *Duality for some free modes*, Discuss. Math. Gen. Algebra Appl. 23 (2003), 45–61.

[10] K. J. Pszczoła, A. B. Romanowska, J. D. H. Smith, *Duality for quadrilaterals*, Contributions to General Algebra 13 (2004), 127–134.

[11] A. B. Romanowska, P. Ślusarski, J. D. H. Smith, *Duality for convex polytopes*, J. Aust. Math. Soc. 86 (2009), 399–412.

[12] A. B. Romanowska, J. D. H. Smith, *From affine to projective geometry via convexity*, pp. 255-269 in "*Universal Algebra and Lattice Theory*" (ed. S. D. Comer), Springer Lecture Notes in Mathematics, Berlin, 1985.

[13] A. B. Romanowska, J. D. H. Smith, *Support functions and ordinal products*, Geom. Dedicata 30 (1989), 281–296.

[14] A. B. Romanowska, J. D. H. Smith, *Differential groupoids*, Contributions to General Algebra 7 (1991), 283–290.

[15] A. B. Romanowska, J. D. H. Smith, *Embedding sums of cancellative modes into functorial sums of affine spaces*, pp. 127-139 in "*Unsolved Problems on Mathematics for the 21st Century - A Tribute to Kiyoshi Iséki's 80th Birthday*" (eds. J. M. Abe and S. Tanaka), IOS Press, Amsterdam, 2001.

[16] A. B. Romanowska, J. D. H. Smith, *Modes*, World Scientific, River Edge, NJ, 2002.

[17] A. B. Romanowska, J. D. H. Smith, *Barycentric algebras and gene expression*, pp. 20–27 in "WILF 2009" (eds. V. di Gesù, S. K. Pal and A. Petrosino), Springer Lecture Notes in Artificial Intelligence, Berlin, 2009

[18] A. B. Romanowska, J. D. H. Smith, E. Orłowska, *Abstract barycentric algebras*, Fund. Informaticae 81 (2007), 257–273.

[19] J. D. H. Smith, *Mal'cev Varieties*, Springer Lecture Notes in Mathematics, Berlin, 1976.

[20] J. D. H. Smith, *An Introduction to Quasigroups and their Representations*, Chapman & Hall, Boca Raton, FL, 2007.

[21] J. D. H. Smith, A. B. Romanowska, *Post-Modern Algebra*, Wiley, New York, NY, 1999.

[22] D. Stanovský, *Idempotent subreducts of semimodules over commutative semirings*, Rend. Semin. Mat. Univ. Padova 121 (2009), 33–43.

[23] M. M. Stronkowski, *On free modes*, Comment. Math. Univ. Carolin. 47 (2006), 561–568.

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