Abstract. The main purpose of this paper is to extend our previous construction of $T$-Fock spaces from a given Yang–Baxter operator satisfying the inequalities $-1 \leq T \leq 1$ to the constructions of $T$-symmetric Fock spaces related to the class of Yang–Baxter–Hecke operators meeting a weaker condition that $T \geq -1$. The new representation of the monotone Fock space of N. Muraki will be given. The main idea of this paper is the new class of generalized Gaussian random variables acting on suitable $T$-symmetric Fock spaces. Relations with the row and column operator space will be also given.

Introduction

In this note we will present the following subjects:

2. (a) Hecke operators and (b) positivity of $T$-symmetrizers.
3. Connections of Pusz–Woronowicz operators $T^{\text{CAR}}_\mu$ with monotone Fock space of Muraki–Lu ($\mu = 0$) and with Bose monotone Fock space and also with mixed Bose–Fermi Fock spaces related to $\mu = -1$.
4. Generalized Gaussian random variables as a model for row and column operator spaces and relations to central limit theorems in noncommutative probability, where arcsine law appears. Some other applications of generalized Gaussian random variables were recently found in the framework of the free probability in [BBLS11, BL09].

Let $\mathcal{H}$ be a complex Hilbert space and let $T : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ be a Yang–Baxter operator, i.e., an operator with a property that $T = T^*$, $T \geq -1$ and

$$T_1 T_2 T_1 = T_2 T_1 T_2$$
onumber

on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, where $T_1 = T \otimes 1$, $T_2 = 1 \otimes T$. We define the $T$-symmetrizer operator

$$P_T^{(n)}(T_1, T_2, \ldots, T_{n-1}) = P_T^{(n)} : \mathcal{H}^n \rightarrow \mathcal{H}^n$$
onumber

as follows:

$$P_T^{(n)} = (1 + T_1 + T_2 T_1 + T_3 T_2 T_1 + \cdots + T_{n-1} \cdots T_1) P_T^{(n-1)}(T_2, T_3, \ldots, T_{n-1}),$$

where $P_T^{(1)} = 1$, $P_T^{(2)} = 1 + T_1$ and

$$T_i = \underbrace{1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes T \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i-1 \text{ times}} : \mathcal{H}^n \rightarrow \mathcal{H}^n.$$
2. Hecke operators and positivity of $T$-symmetrizers

2.1. Hecke operators. Now we solve the question posed by L. Accardi, when the $T$-symmetrizer operators $P^{(n)}_T$ are similar to self-adjoint projections, i.e.

$$\left(P^{(n)}_T\right)^2 = \alpha(n) P^{(n)}_T \quad \text{for some } \alpha(n) > 0. \tag{1}$$

First, let us see that if $P^{(2)}_T = 1 + T$ satisfies (1) then

$$(1 + T)^2 = \alpha(1 + T) \quad \text{for } \alpha = \alpha(2),$$

which implies that

$$T^2 = (q - 1)T + q \ 1, \tag{2}$$

where $q = \alpha - 1$. Such an operator satisfying (2) is called Hecke operator with parameter $q$.

We will say that $T = T^*$ is a Yang–Baxter–Hecke operator if it is a Yang–Baxter operator which at the same time is a Hecke operator:

$$T^2 = (q - 1)T + q \ 1, \quad \text{for some } q \geq -1.$$

Typical examples of Hecke operators are the following ones:

$(H_1)$ The flip $T = \sigma : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ given by an exchange of the factors $\sigma(x \otimes y) = y \otimes x$ is a Hecke operator with $q = 1$ and the corresponding "projection"

$$P^{(n)}_T = \sum_{\pi \in S_n} \pi$$

is the classical symmetrizer operator on $\mathcal{H}^{\otimes n}$.

$(H_2)$ For $T = -\sigma$ we obtain the anti-symmetrizer

$$P^{(n)}_T = \sum_{\pi \in S_n} \text{sgn}(\pi) \ \pi,$$

where $\text{sgn}(\pi)$ is the classical sign of a permutation $\pi \in S_n$.

$(H_3)$ If we take $\epsilon = \pm 1$ and we define the operator

$$T = T_\epsilon = \frac{q - 1}{2} + \epsilon \frac{q + 1}{2} \sigma,$$

then we get the Hecke operator with parameter $q$, i.e.

$$T^2 = (q - 1)T + q \ 1.$$

This operator is a Yang–Baxter operator if and only if $q = 1$, which means that $T_\epsilon$ is the symmetrizer ($\epsilon = 1$) or the anti-symmetrizer ($\epsilon = -1$).

$(H_4)$ We get a very interesting example of a Yang–Baxter–Hecke operator for a Hilbert space $\mathcal{H}$ of finite dimension $\dim \mathcal{H} = m$ with an orthonormal
basis \((e_1, e_2, \ldots, e_m)\). We consider the operator \(\tilde{P} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}\) given by

\[
\tilde{P}(e_i \otimes e_j) = -\frac{1}{m} \delta_{ij} \sum_{k=1}^{m} e_k \otimes e_k.
\]

One can see that \(P = (-\tilde{P})\) is the projector operator of the following form:

\[
P(x \otimes y) = \frac{1}{m} \langle x | y \rangle \theta, \quad \text{where} \quad \theta = \sum_{k=1}^{m} e_k \otimes e_k, \quad x, y \in \mathcal{H}
\]

(see [GW98], page 449).

\((H_5)\) The main example of that note is Pusz–Woronowicz twisted CCR (CAR) operators: \(T_{\mu}^{CCR}, T_{\mu}^{CAR}\) defined as [Pus89, PW89]:

\[
T_{\mu}^{CCR}(e_i \otimes e_j) = \begin{cases} 
\mu(e_j \otimes e_i) & \text{if } i < j, \\
\mu^2(e_i \otimes e_i) & \text{if } i = j, \\
-(1-\mu^2)(e_i \otimes e_j) + \mu(e_j \otimes e_i) & \text{if } i > j,
\end{cases}
\]

\[
T_{\mu}^{CAR}(e_i \otimes e_j) = \begin{cases} 
-\mu(e_j \otimes e_i) & \text{if } i < j, \\
-(e_i \otimes e_i) & \text{if } i = j, \\
-(1-\mu^2)(e_i \otimes e_j) - \mu(e_j \otimes e_i) & \text{if } i > j.
\end{cases}
\]

Both the twisted CCR and twisted CAR are Yang–Baxter–Hecke operators with the parameter \(q = \mu^2\), which means that

\[
T^2 = (\mu^2 - 1)T + \mu^2.\]

\((H_6)\) As a special case we consider \(T_{0}^{CAR} = T^{M}\), where \(T^{M}\) was discovered by N. Muraki [Mur97]:

\[
T^{M}(e_i \otimes e_j) = \begin{cases} 
0 & \text{if } i < j, \\
-(e_i \otimes e_j) & \text{if } i \geq j.
\end{cases}
\]

It is connected with Muraki–Lu monotone Fock space, as we will see later.

\((H_7)\) Also it will be interesting to see the corresponding \(T\)-Fock space in the case when the twisted CCR operator has parameter \(\mu = 0\) and then we get the following operator:

\[
T_{0}^{CCR}(e_i \otimes e_j) = \begin{cases} 
0 & \text{if } i \leq j, \\
-(e_i \otimes e_j) & \text{if } i > j.
\end{cases}
\]

Later we will use this operator to construct the Bose monotone Fock space.
In the paper with Lytvynov and Wysoczanski [BLW12] we introduced another type (called anyonic) of the Yang–Baxter–Hecke operator $T_z$ on $L^2(\mathbb{R}, \sigma)$, where $\sigma$ is a non-atomic Radon measure on $\mathbb{R}$ defined for $f \in L^2(\mathbb{R}^2, \sigma \otimes \sigma)$ as follows:

$$T_z f(x, y) = Q(x, y) f(y, x),$$

where $|z| = 1$ and

$$Q(x, y) = \begin{cases} 
    z & \text{if } x < y, \\
    \bar{z} & \text{if } x > y.
\end{cases}$$

Then $T_z$ is a Yang–Baxter–Hecke operator with parameter $q = 1$.

2.2. Positivity of $P^{(n)}_T$ for Yang–Baxter–Hecke operators. We are in position to prove the main result of this paper.

**Proposition 1.** Let $T = T^*$ be a Yang–Baxter–Hecke operator. Then for each $n \geq 1$

$$(P_T^{(n)})^2 = n! P_T^{(n)} \geq 0,$$

where $n = 1 + q + \ldots + q^{n-1}$ and $n! = 1 \cdot 2 \cdot \ldots \cdot n$.

Moreover, for $q \geq -1$

$$\|P_T^{(n)}\| = n! = \prod_{k=1}^{n} \frac{(1-q^k)}{(1-q)^n}.$$

**Remark 2.** Proposition 1 solves the problem of L. Accardi: $P_T^{(n)}$ is similar to a projection if and only if $T$ is a Hecke operator.

**Remark 3.** Since the spectrum of the Hecke operator $\sigma(T) = \{-1, q\}$, therefore $\|T\| = \max\{1, |q|\}$, so the norm $\|T\|$ can be arbitrarily large.

**Proof of Proposition 1.** Since $T^2 = (q-1)T + q1$, hence

$$(**): T_i(1 + T_i) = (1 + T_i)T_i = q(1 + T_i).$$

Let us first prove the formula $(*)$ for $n = 3$:

$$P_T^{(3)} = (1 + T_2 + T_1 T_2)(1 + T_1) = (1 + T_1 + T_2 T_1)(1 + T_2) = (P_T^{(3)})^*.$$  

Here we used the Yang–Baxter equation $T_1 T_2 T_1 = T_2 T_1 T_2$. Now let us calculate

$$(P_T^{(3)})^2 = (1 + T_2 + T_1 T_2)(1 + T_1)(1 + T_1)(1 + T_2 + T_2 T_1) = P_T^{(3)} (P_T^{(3)})^*$$

$$= (1 + q)(1 + T_2 + T_1 T_2)(1 + T_1)(1 + T_2 + T_2 T_1)$$

$$= (1 + q)(1 + T_1 + T_2 T_1)(1 + T_2)(1 + T_2 + T_2 T_1).$$

But then the formula $(**)$ implies that we have

$$(1 + T_2)(1 + T_2 + T_2 T_1) = (1 + T_2)((1 + q)1 + q T_1).$$
Therefore since
\[
(P_T^{(3)})^2 = (1 + q)(1 + T_1 + T_2 T_1)(1 + T_2)((1 + q)1 + qT_1)
\]
\[
= (1 + q)(1 + T_2 + T_1 T_2)(1 + T_1)((1 + q)1 + qT_1)
\]
\[
= (1 + q)(1 + T_2 + T_1 T_2)(1 + T_1)((1 + q + q^2) = 1 \cdot 2 \cdot 3 P_T^{(3)}.
\]

For general case, we recall that using the Yang–Baxter property of $T$ we have
\[
P_T^{(n)} = S_i(1 + T_i) = (1 + T_i)S_i^*,
\]
where
\[
S_i = \sum T_{i_1} \cdots T_{i_k} \quad \text{and} \quad T_{i_k} \neq T_i.
\]

Therefore since
\[
P_T^{(n)} = P_T^{(n-1)}(T_1, \ldots, T_{n-2})(1 + T_{n-1} + T_{n-1}T_{n-2} + \cdots + T_{n-1} \cdots T_2 T_1)
\]
we get
\[
(P_T^{(n)})^2 = P_T^{(n-1)}(T_1, \ldots, T_{n-2})(1 + T_{n-1} + T_{n-1}T_{n-2} + \cdots + T_{n-1} \cdots T_2 T_1)
\]
\[
\quad \cdot (1 + T_1)S_i^*
\]
\[
= P_T^{(n-1)}(T_1, \ldots, T_{n-2})(1 + T_{n-1} + T_{n-1}T_{n-2} + \cdots + T_{n-1} \cdots T_2 \cdot q)
\]
\[
\quad \cdot (1 + T_2)S_2^*
\]
\[
= P_T^{(n-1)}(T_1, \ldots, T_{n-2})(1 + T_{n-1} + T_{n-1} \cdots T_3 \cdot q + T_{n-1} \cdots T_3 \cdot q^2)
\]
\[
\quad \cdot (1 + T_3)S_3^*
\]
\[
\quad \cdots
\]
\[
= P_T^{(n-1)}(T_1, \ldots, T_{n-2})(1 + q + q^2 + \cdots + q^{n-1})P_T^{(n)}
\]
\[
= n P_T^{(n-1)}(T_1, \ldots, T_{n-2})P_T^{(n-1)}(T_1, \ldots, T_{n-2})
\]
\[
\quad \cdot (1 + T_{n-1} + T_{n-1}T_{n-2} + \cdots + T_{n-1} \cdots T_2 T_1).
\]

By induction the latter expression is equal to
\[
\frac{n}{n-1}! P_T^{(n-1)}(T_1, \ldots, T_{n-2})(1 + T_{n-1} + T_{n-1}T_{n-2} + \cdots + T_{n-1} \cdots T_2 T_1)
\]
\[
= n! P_T^{(n)}.
\]

Since
\[
(P_T^{(n)})^2 = n! P_T^{(n)} = n!(P_T^{(n)})^*
\]
we obtain at once that
\[
\|P_T^{(n)}\| = n!
\]
which finishes the proof. ■
Now we can construct the $T$-symmetric Fock Hilbert space

$$\mathcal{F}_T(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} n! = \mathbb{C}\Omega \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots ,$$

where

$$\mathcal{H}^{\otimes n} = P_T^{(n)}(\mathcal{H}^{\otimes n})$$

is the space of $T$-symmetric tensors. By Proposition 1 we have that for $f \in \mathcal{H}^{\otimes n}$, $P_T^{(n)}(f) = n! f$. Therefore the Hilbert norm $\|f\|_T^2$ for $f = (f_0, f_1, f_2, \ldots) \in \mathcal{F}_T(\mathcal{H})$ is defined as:

$$\|f\|_T^2 = \langle P_T(f)|f\rangle = \sum_{n=0}^{\infty} \langle P_T^{(n)}(f_n)|f_n\rangle = \sum_{n=0}^{\infty} n! \|f_n\|^2 \leq \infty.$$

One can see that we have the following description of $T$-symmetric tensors:

**Lemma 4.** For $n > 1$ we have

$$\mathcal{H}^{\otimes n} = \left\{ f \in \mathcal{H}^{\otimes n} : T_j(f) = q f \text{ for } j \in \{1, 2, \ldots, n-1\} \right\} = \left\{ f \in \mathcal{H}^{\otimes n} : \tilde{P}_T^{(n)}(f) = f \right\},$$

where $\tilde{P}_T^{(n)} = \frac{1}{n!} P_T^{(n)}$.

**Proof.** For the special case $q = 1$ of this result, see [BLW12, Proposition 2.3].

If $T_j(f) = q f$ for $j \in \{1, 2, \ldots, n-1\}$ then by the induction we have

$$P_T^{(n)}(f) = (1 + T_1 + T_2 T_1 + \cdots + T_1 \cdots T_{n-1}) P_T^{(n-1)}(f) = (n-1)! (1 + T_1 + T_2 T_1 + \cdots + T_1 \cdots T_{n-1})(f) = n! f.$$

On the other hand, $P_T^{(n)}(f) = n! f = (1 + T_i)S_i(f)$, therefore

$$n! T_i(f) = T_i(1 + T_i)S_i(f) = q (1 + T_i)S_i(f) = q n! f.$$

So $T_i(f) = q f$ for $i < n$. ■

**Remark 5.** Let us observe that the $T$-creation and $T$-annihilation operators on the $T$-Fock space can be defined as follows: for $f \in \mathcal{H}$

$$a_T^+(f) = \tilde{P} l^+(f) \tilde{P} = \tilde{P} l^+(f),$$

$$a_T(f) = \tilde{P} l(f) \tilde{P} = l(f) \tilde{P},$$

where $\tilde{P} = \tilde{P}_T = \sum_{n=0}^{\infty} \frac{1}{n!} P_T^{(n)}$ is the orthogonal projection onto $T$-symmetric tensors and $l^+(f), l(f)$ are the free creation and free annihilation operators.
2.3. **Boolean Fock spaces.** The simplest among deformed $T$-symmetric Fock spaces is the Boolean Fock space $\mathcal{F}_{-1}(\mathcal{H}) = \mathbb{C}\Omega \oplus \mathcal{H}$ which are obtained for the Yang–Baxter–Hecke operator $T = -1$. In that case $P_T^{(n)} = 0$ for $n > 1$.

The Boolean creation and annihilation operators have a very simple form:

$$b^+(f)\xi = \begin{cases} 0 & \text{if } \xi \in \mathcal{H}, \\ f & \text{if } \xi = \Omega, \end{cases}$$

$$b(f)\xi = \begin{cases} \langle f|\xi \rangle & \text{if } \xi \in \mathcal{H}, \\ 0 & \text{if } \xi \in \mathbb{C}\Omega. \end{cases}$$

They satisfy the following relations: if $(e_1, e_2, \ldots, e_N)$ is an orthonormal basis of $\mathcal{H}$ and $b^\pm_i := b^\pm(e_i)$ then

$$b_i b^+_j = \delta_{i,j} \left( 1 - \sum_{k=0}^{N} b^+_k b_k \right) = \delta_{i,j} P_\Omega,$$

where $P_\Omega$ is the projection on the vacuum vector $\Omega$.

For the Boolean Gaussian random variables $G^B(f) = b(f) + b^+(f)$, the following Proposition is known to be true (see [BKW06, Theorem 3]):

**Proposition 6.** For arbitrary operators $\alpha_i \in B(\mathcal{H})$ and $f_i \in \mathcal{H}_R$, $\|f_i\| = 1$, we have

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes G^B(f_i) \right\| = \max \left\{ \left\| \left( \sum_{i=1}^{N} \alpha_i^* \right)^{1/2} \right\|, \left\| \left( \sum_{i=1}^{N} \alpha_i \right)^{1/2} \right\| \right\}.$$

That means that Boolean Gaussian random variables span the operator space completely isometrically isomorphic to row and column operator space. Similar results was obtained for the free and $q$-Gaussian random variables and free generators (see [HP93, BS92]). In the next chapter we prove the similar result for the monotone Gaussian random variables.

2.4. **Monotone Fock spaces.** Now we recall the definition of the monotone Fock space following the fundamental paper of Muraki [Mur97] and we show that it is equal to special case of the $T$-symmetric Fock space for the Pusz–Woronowicz operator considered in the example $(H_6) T_0^{CAR} = T^M$.

Let $\mathbb{N}$ be the set of all natural numbers. For $r \geq 1$ we define $I_r = \{(i_1, i_2, \ldots, i_r) : i_1 < i_2 < \cdots < i_r, i_j \in \mathbb{N}\}$ and for $r = 0$ we set $I_0 = \{\emptyset\}$, where $\emptyset$ denotes the null sequence. We define $\text{Inc}(\mathbb{N}) = \bigcup_r I_r$. Let $\mathcal{H}_r = \ell^2(I_r)$ be the $r$-particle Hilbert space and $\Phi = \bigoplus_{r=0}^{\infty} \mathcal{H}_r$ the monotone Fock space. For an increasing sequence $\sigma = (i_1, i_2, \ldots, i_r) \in \text{Inc}(\mathbb{N})$, denote by
\[ \sigma = \{ i_1, i_2, \ldots, i_r \} \] the associated set and by \( \{ e_\sigma \} \) the cannonical basis vector in the monotone Fock space \( \Phi \).

We will write \( [\sigma] < [\tau] \) if for each \( i \in [\sigma] \) and \( j \in [\tau] \) we have \( i < j \). The monotone creation operator \( \delta^+_i \) and the annihilation operator \( \delta^-_i \) are defined for each \( i \in \mathbb{N} \) by:

\[
\delta^+_i e_{(i_1, \ldots, i_r)} = \begin{cases} e_{(i, i_1, \ldots, i_r)} & \text{if } \{ i \} < \{ i_1, \ldots, i_r \}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\delta^-_i e_{(i_1, \ldots, i_r)} = \begin{cases} e_{(i_2, \ldots, i_r)} & \text{if } r \geq 1, i = i_1, \\ 0 & \text{otherwise}. \end{cases}
\]

Let us observe that if \( P = \bigoplus P^{(n)} \) is the orthogonal projection from the full Fock space onto the monotone Fock space, then \( \delta^+_i = P l^+_i P \), where \( l^+_i \) are the free creation and the free annihilation operators. Moreover, the following relations hold:

\[
\delta^+_i \delta^+_j = \delta^-_j \delta^-_i = 0 \quad \text{for } i \geq j,
\]

\[
\delta^-_i \delta^+_j = 0 \quad \text{for } i \neq j,
\]

\[
\delta^-_i \delta^+_i = 1 - \sum_{j \leq i} \delta^+_j \delta^-_j \quad \text{for } i = j.
\]

**Proposition 7.** If \( T^M = T_{0 CAR} \) is the Pusz–Woronowicz Yang–Baxter–Hecke operator defined as

\[
T^M(e_i \otimes e_j) = \begin{cases} 0 & \text{if } i < j, \\ -(e_i \otimes e_j) & \text{if } i \geq j \end{cases}
\]

then the \( T \)-symmetric Fock space is exactly Muraki monotone Fock space and the corresponding creation and annihilation operators are the following:

\[ a^+_i = \delta^+_i, \quad a_i = \delta^-_i. \]

**Proof.** Since \( T^2 = -T \), therefore \( T \) is Yang–Baxter–Hecke operator with parameter \( q = 0 \) and \( n = 1 \), for all natural \( n \).

One can see that the operator \( P^{(2)}_T = 1 + T \) is the orthogonal projection on the linear span of the vectors: \( e_{i_1} \otimes e_{i_2} \), where \( i_1 < i_2 \), and more generally \( P^{(n)}_T \) is the projection of \( \mathcal{H}^\otimes n \), onto the linear space generated by the monotone sequences \( \{ e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} : i_1 < i_2 < \cdots < i_n \} \).

The other parts of Proposition follow at once from the above observation. 

Now we are in position to present the following proposition:
Proposition 8. Let $\alpha_i \in B(H)$ and $G_i = \delta_i^- + \delta_i^+$ be the monotone Gaussian operators. Then

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^+ \right\| = \left\| \sum_{i=1}^{N} \alpha_i^* \alpha_i \right\|^{1/2},$$

(3)

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^- \right\| = \left\| \sum_{i=1}^{N} \alpha_i^* \alpha_i \right\|^{1/2},$$

(4)

$$\left\| \delta_i^- \right\| = \left\| \delta_i^+ \right\| = 1,$$

(5)

$$1 \leq \|G_i\| \leq 2,$$

(6)

$$\max\left\{ \left\| \left( \sum_{i=1}^{N} \alpha_i \alpha_i^* \right)^{1/2} \right\|, \left\| \left( \sum_{i=1}^{N} \alpha_i^* \alpha_i \right)^{1/2} \right\| \right\} \leq \left\| \sum_{i=1}^{N} \alpha_i \otimes G_i \right\| \leq 2 \max\left\{ \left\| \left( \sum_{i=1}^{N} \alpha_i \alpha_i^* \right)^{1/2} \right\|, \left\| \left( \sum_{i=1}^{N} \alpha_i^* \alpha_i \right)^{1/2} \right\| \right\}. \tag{7}$$

Proof. Equalities (3) and (4) follow at once if we observe that

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^- \right\|^2 = \left\| \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j^* \otimes \delta_i^- \delta_j^+ \right\| = \left\| \sum_{i=1}^{N} \alpha_i \alpha_i^* \otimes \delta_i^- \delta_i^+ \right\|$$

$$= \left\| \sum_{i=1}^{N} \alpha_i \alpha_i^* \otimes (1 - \sum_{j \leq i} \delta_j^+ \delta_j^-) \right\| \leq \left\| \sum_{i=1}^{N} \alpha_i \alpha_i^* \right\|.$$

By the same argument we have

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^+ \right\| \leq \left\| \sum_{i=1}^{N} \alpha_i^* \alpha_i \right\|.$$

But

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^- \right\|^2 \geq \sup\left\{ \left\| \left( \sum_{i=1}^{N} \alpha_i \alpha_i^* \otimes \delta_i^- \delta_i^+ \right)(\xi \otimes \Omega) \right\| : \|\xi\| = 1 \right\}$$

$$= \left\| \sum_{i=1}^{N} \alpha_i \alpha_i^* \right\|.$$

Hence

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^- \right\| = \left\| \sum_{i=1}^{N} \alpha_i \alpha_i^* \right\|^{1/2}.$$
and
\[ \left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^+ \right\| = \left\| \sum_{i=1}^{N} \alpha_i^* \alpha_i \right\|^{1/2}. \]

Equality (5) is a trivial consequence of (3) and (4).

The proof of (7) follows by the inequality (3), the triangle inequality and the fact that for all positive real numbers \( x, y \) we have
\[ x + y \leq 2 \max(x, y). \]

Therefore we have
\[
\left\| \sum_{i=1}^{N} \alpha_i \otimes G_i \right\| \leq \left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^+ \right\| + \left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^- \right\|
\leq \left\| \sum_{i=1}^{N} \alpha_i \alpha_i^* \right\|^{1/2} + \left\| \sum_{i=1}^{N} \alpha_i^* \alpha_i \right\|^{1/2}
\leq 2 \max \left\{ \left\| \left( \sum_{i=1}^{N} \alpha_i \alpha_i^* \right)^{1/2} \right\|, \left\| \left( \sum_{i=1}^{N} \alpha_i^* \alpha_i \right)^{1/2} \right\| \right\}.
\]

On the other hand since
\[ \varepsilon(G_i G_j) = \langle \delta_i^- \delta_j^+ | \Omega \rangle = \delta(i, j) \]
so we have
\[
\left\| \sum_{i=1}^{N} \alpha_i \otimes G_i \right\|^2 \geq \sup (\varphi \otimes \varepsilon) \left\{ \left( \sum_{i=1}^{N} \alpha_i \otimes G_i \right) \left( \sum_{i=1}^{N} \alpha_i \otimes G_i \right)^* : \varphi \text{ is a state on } B(H) \right\}
\geq \left\| \sum_{i=1}^{N} \alpha_i^* \alpha_i \right\|,
\]
and similarly, we get
\[
\left\| \sum_{i=1}^{N} \alpha_i \otimes G_i \right\|^2 \geq \left\| \sum_{i=1}^{N} \alpha_i \alpha_i^* \right\|,
\]
which finishes the proof. ■

2.5. Remarks on Bose Monotone Fock spaces. If we consider bosonic type of the operator Pusz–Woronowicz defined as
\[
T^B(e_i \otimes e_j) = T^{CCR}_0(e_i \otimes e_j) = \begin{cases} 
0 & \text{if } i \leq j, \\
-(e_i \otimes e_j) & \text{if } i > j,
\end{cases}
\]
then one can see that in that case the $n$-th particle space of the corresponding $T$-symmetric Fock space $\mathcal{F}_{TB}(\mathcal{H})$ is of the following form:

$$P_T^{(n)}(\mathcal{H}^\otimes n) = \mathcal{H}_T^\otimes n = \text{Lin}\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} : i_1 \leq i_2 \leq \cdots \leq i_n\}.$$  

The action of the creation and annihilation operators is following:

$$\Delta_j^+(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = \begin{cases} e_j \otimes e_{i_j} \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} & \text{if } j \leq i_1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\Delta_j(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = \begin{cases} e_{i_2} \otimes \cdots \otimes e_{i_n} & \text{if } j = i_1, \\ 0 & \text{otherwise}, \end{cases}$$

and they satisfy the following commutation relations:

$$\Delta_i \Delta_j^+ = 0 \quad \text{for } i \neq j,$$

$$\Delta_i \Delta_j^+ = 1 - \sum_{k<i} \Delta_k^+ \Delta_k \quad \text{for } i = j.$$  

From the last formulas we get $\Delta_1 \Delta_1^+ = 1$ and $\|\Delta_1\| = 1$. Moreover, $\|\Delta_j^+\| \leq 1$ and since $\|\Delta_j^+ \Omega\| = 1$ we have $\|\Delta_j^\pm\| = 1$.

By the similar considerations like in the Fermi-monotone Muraki–Fock space we have the following:

**Proposition 9.** Let $\alpha_i \in B(\mathcal{H})$ and $g_i = \Delta_i^- + \Delta_i^+$ be the monotone Bose Gaussian operators, then

$$\max\left\{ \left\| \left( \sum_{i=1}^{N} \alpha_i \alpha_i^* \right)^{1/2} \right\|, \left\| \left( \sum_{i=1}^{N} \alpha_i^* \alpha_i \right)^{1/2} \right\| \right\} \leq \left\| \sum_{i=1}^{N} \alpha_i \otimes g_i \right\| \leq 2 \max\left\{ \left\| \left( \sum_{i=1}^{N} \alpha_i \alpha_i^* \right)^{1/2} \right\|, \left\| \left( \sum_{i=1}^{N} \alpha_i^* \alpha_i \right)^{1/2} \right\| \right\}.$$  

**Remark 10.** If we take the vacuum state $\varepsilon(T) = \langle T \Omega, \Omega \rangle$, then one can show the following central limit theorem for the Bose-monotone Gaussian random variables $g_i = \Delta_i^- + \Delta_i^+$.

**Proposition 11.** If $S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} g_i$ then

$$\lim_{N \to \infty} \varepsilon(S_N^{2n}) = \binom{2n}{n},$$

i.e., $S_N$ weakly tends to arcsine law $\frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}$.

In the case of the Fermi monotone case this same law was obtained by N. Muraki [Mur97]. See also the paper of J. Wysoczanski [Wys10] for related generalization of the central limit theorems for the Boolean-monotonic case.
Remark 12. The model of Fermi monotone Fock space as was presented above was inspired N. Muraki to form monotone independence and monotone convolution for probability measures on the real line (see [Mur00]).

We think that our Bose monotone Fock space also will give us a new concept of noncommutative independence. This will be done in forthcoming paper.

See also last paper of J. Wysoczanski [Wys10], about (bm) independence, which is also very close to our considerations.

Remark 13. When we will consider the Pusz–Woronowicz Hecke operator $T^{CCR}_{\mu}$ for $\mu = -1$, defined as

$$T^B(e_i \otimes e_j) = T^{CCR}_{-1}(e_i \otimes e_j) = \begin{cases} (e_i \otimes e_j) & \text{if } i = j, \\ -(e_i \otimes e_j) & \text{if } i \neq j, \end{cases}$$

so we obtain the model of mixed Bose–Fermi commutation relations on the corresponding T-symmetric Fock space. The commutations relations in that case are following:

$$b_ib_j^+ + b_j^+b_i = 0 \quad \text{if } i \neq j,$$

$$b_ib_j^+ - b_j^+b_i = 1 \quad \text{if } i = j.$$

That models correspond to so called $q_{ij}$-CCR commutations relations of the form

$$A_i A_j^+ - q_{ij} A_j^+ A_i = \delta_{ij} 1,$$

where $q_{ij} = \bar{q}_{ji}$ and $|q_{ij}| \leq 1$. Such models were considered in many papers done by [Spe93, BS92, LP99, JSW95, Kró00] and others.

In our last case we have case of “anicommuting bosons” i.e.: $q_{ii} = 1$ and $q_{ij} = -1$ for $i \neq j$. Similarly, if we consider the Pusz–Woronowicz–Hecke operator $T^{CAR}_{\mu}$, for $\mu = -1$, we obtain again $q_{ij}$-CCR commutation relations of the type “commuting fermions", when $q_{ii} = -1$ and $q_{ij} = 1$ for $i \neq j$.

We finish that note with the natural problem for $q_{ij}$- Gaussian random variables $G_i = A_i + A_i^+$ considered above.

Problem 14. Is it true that the following inequality holds for some positive scalar $C$?

$$\max \left\{ \left\| \left( \sum_{i=1}^{N} \alpha_i \alpha_i^* \right)^{1/2} \right\|, \left\| \left( \sum_{i=1}^{N} \alpha_i^* \alpha_i \right)^{1/2} \right\| \right\} \leq \left\| \sum_{i=1}^{N} \alpha_i \otimes G_i \right\|$$

$$\leq C \max \left\{ \left\| \left( \sum_{i=1}^{N} \alpha_i \alpha_i^* \right)^{1/2} \right\|, \left\| \left( \sum_{i=1}^{N} \alpha_i^* \alpha_i \right)^{1/2} \right\| \right\}. $$
In the case $|q_{ij}| < 1$ this inequality was proved by [BS92], so is there more general problem for all $|q_{ij}| = 1$?

References


Deformed Fock spaces, Hecke operators and monotone Fock space of Muraki


INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
Pl. Grunwaldzki 2/4
50-384 WROCŁAW, POLAND
E-mail: bozejko@gmail.com

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