Abstract. We analyse a finite difference scheme for von Foerster–McKendrick type equations with functional dependence forward in time and backward with respect to one dimensional spatial variable. Some properties of solutions of a scheme are given. Convergence of a finite difference scheme is proved. The presented theory is illustrated by a numerical example.

Introduction

Von Foerster–McKendrick type models are well known models of mathematical biology, describing a population with a structure of its members, given for example by their age [3], size [1] or level of maturation of individuals [7]. Existence, uniqueness and other properties of solutions for above mentioned models are studied in the literature. We are interested in some class of initial problems, originating in [7], which presents erytroid production model, based on a continuous maturation-proliferation mechanism. Far-reaching generalization of this problem is presented in [9]. In this paper we deal with the problem considered in [9] with one dimensional spatial variable.

Let \( T > 0, \tau_0, \tau_1 \in \mathbb{R}_+, \) where \( \mathbb{R}_+ = [0, +\infty) \). Let us introduce

\[
I_0 = [-\tau_0, 0], \quad I = [0, T], \quad B = [-\tau_0, 0] \times [-\tau_1, \tau_1], \\
E_0 = [-\tau_0, 0] \times \mathbb{R}_+, \quad E = [0, T] \times \mathbb{R}_+.
\]

For a given function \( q: I_0 \cup I \to \mathbb{R}, \) \( t \in I \) define the Hale operator \( q_t: I_0 \to \mathbb{R} \) by

\[
q_t(s) = q(t + s), \quad s \in I_0.
\]
see [4]. Denote $\alpha_+ = \max\{0, \alpha\}$. Given a function $w: E_0 \cup E \to \mathbb{R}$ and a point $(t, x) \in E$ define a function $w_{(t,x)}: B \to \mathbb{R}$ by

$$w_{(t,x)}(s, y) = w(t + s, (x + y)_+), \quad (s, y) \in B.$$ 

Our definition of $w_{(t,x)}$ differs from the definition of the Hale operator given in [6] since negative values of the spatial coordinate $x$ do not have biological interpretation.

By $F(X)$ denote the class of all real functions defined on $X$, where $X$ is an arbitrary set. For any metric space $Y$ we denote by $C(Y)$ the class of all continuous real functions on $Y$, whereas $C_+(Y)$ denotes the class of all nonnegative continuous real functions on $Y$. For $q \in C(I_0)$, $w \in C(B)$ we define

$$\|q\|_{C(I_0)} = \max \{|q(t)| : t \in I_0\}, \quad \|w\|_{C(B)} = \max \{|w(t, x)| : (t, x) \in B\}.$$ 

Let $\Omega_0 = E \times C(I_0)$, $\Omega = E \times C(B) \times C(I_0)$. Suppose that $v: E_0 \to \mathbb{R}$ is a given bounded and continuous initial function and $c: \Omega_0 \to \mathbb{R}_+$, $\lambda: \Omega \to \mathbb{R}$.

Consider the differential functional equation

$$\partial_t u(t, x) + c(t, x, z_t) \partial_x u(t, x) = u(t, x) \lambda(t, x, u_{(t,x)}, z_t)$$

with the initial condition

$$u(t, x) = v(t, x), \quad (t, x) \in E_0,$$

where

$$z(t) = \int_0^{+\infty} u(t, y) \, dy, \quad t \in [-\tau_0, T].$$

We assume that $c(t, 0, q) = 0$ for $t \in I$, $q \in C(I_0)$. This condition implies that the boundary condition is not necessary for the problem, cf. [2, 9]. Differential equations, equations with deviated argument, differential integral equations can be derived from (1) by specializing the operators $c, \lambda$, see examples in [9].

This paper extends our previous results concerning approximation of solutions to the above problem without functional dependence by finite difference schemes, see [8].

Solutions to (1)–(2) exist on unbounded domain. However, due to computational constraints solutions of the problem can be approximated only on a finite mesh. The paper is devoted to a difference method for approximation of infinite domain solutions to (1)–(2) by a difference scheme with a finite number of knots. We give conditions on the size of a finite rectangular mesh, which enable to obtain expected error of approximation of (3) for a prescribed initial data and a discretization parameter.
The partial derivative \( \partial_x u \) is approximated by the backward difference quotient, since \( c \geq 0 \). We consider only nonnegative approximation of solutions of the problem. Therefore, the Courant–Friedrich’s–Levy stability condition (cf. [5], p. 274) is replaced by a modified stability condition which implies also nonnegativity of solutions of our scheme. However, the presented theory, under slight modifications, remains valid if solutions of the problem are negative. Values of \( z(t) \) are approximated by a sufficiently large finite rectangular rule. We introduced some class of initial functions to be assured of solutions of the problem. Therefore, the Courant–Friedrich’s–Levy stability condition (cf. [5], p. 274) is replaced by a modified stability condition which implies also nonnegativity of solutions of our scheme. However, the presented theory, under slight modifications, remains valid if solutions of the problem are negative. Values of \( z(t) \) are approximated by a sufficiently large finite rectangular rule. We introduced some class of initial functions to be assured that the rule approximating \( z(t) \) is well defined. Interpolating operators are defined as functions \( c, \lambda \) are dependent on continuous functional argument.

The paper is organized as follows:

(i) a finite difference scheme is introduced and some properties of its solutions are given;
(ii) convergence of the scheme is proved;
(iii) results of numerical experiments illustrating the presented theory are given.

1. Finite difference scheme

We approximate solutions of (1)–(2) on a sufficiently large bounded area since practical computations cannot be performed on unbounded domain. Let \( N_0, N_1 \in \mathbb{N} \), \( h_0 = \frac{\tau_0}{N_0} \), \( h_1 = \frac{\tau_1}{N_1} \) and \( m = \frac{h_0}{h_1} \), \( h = (h_0, h_1) \). There is \( N \in \mathbb{N} \) such that \( Nh_0 < a \leq (N + 1)h_0 \). Define

\[
I_{0,h} = \{ (t^{(i)}): i = -N_0, \ldots, 0 \}, \quad I_h = \{ (t^{(i)}): i = 0, \ldots, N \},
\]

where \( t^{(i)} = ih_0 \). For a given discretization parameter \( h_0 \) define \( N_h \in \mathbb{N} \) such that \( h_0 N_h \to \infty \) as \( h_0 \to 0 \). Define \( M^{(i)} = N_h + N_1 \) for \( i = -N_0, \ldots, 0 \) and \( M^{(i)} = N_h + N_1(N - i) + \) for \( i = 0, \ldots, N + 1 \). Let

\[
E_{0,h} = \{ (t^{(i)}, x^{(j)}): i = -N_0, \ldots, 0, \ j = 0, \ldots, M^{(0)} \},
\]

\[
E_h = \{ (t^{(i)}, x^{(j)}): i = 0, \ldots, N, \ j = 0, \ldots, M^{(i)} \},
\]

where \( x^{(j)} = jh_1 \), be the finite meshes on some bounded parts of \( E_0 \) and \( E \), respectively. Define

\[
E'_h = \{ (t^{(i)}, x^{(j)}): (t^{(i+1)}, x^{(j)}) \in E_h \}, \quad I'_h = \{ t^{(i)} \in I_h: t^{(i+1)} \in I_h \}.
\]

For discrete functions \( u: E_{0,h} \cup E_h \to \mathbb{R} \), \( z: I_{0,h} \cup I_h \to \mathbb{R} \) we write \( u^{(i,j)} = u(t^{(i)}, x^{(j)}), z^{(i)} = z(t^{(i)}) \). Let \( E^* = (\tilde{E}_0 \cup \tilde{E}) \cap (E_0 \cup E) \), where

\[
\tilde{E}_0 = \{ (t, x) \in E_0: x \leq M^{(0)}h_1 \},
\]

\[
\tilde{E} = \{ (t, x) \in E: t \in [t^{(i)}, t^{(i+1)}], \ x \in [0, M^{(i+1)}h_1], \ i = 0, \ldots, N \}.
\]
Introduce the interpolating operator \( T_h \colon F(E_{0,h} \cup E_h) \to C(E^*) \). Define \( S = \{ (\alpha, \beta) : \alpha, \beta \in \{0,1\} \} \). Two cases will be considered.

**I.** Suppose that \((t, x) \in \tilde{E}_0\). There are \((t^{(i)}, x^{(j)}), (t^{(i+1)}, x^{(j+1)}) \in E_{0,h}\) such that \( t^{(i)} \leq t \leq t^{(i+1)} \) and \( x^{(j)} \leq x \leq x^{(j+1)} \). Then

\[
(T_h u)(t, x) = \sum_{(\alpha, \beta) \in S} u^{(\alpha, \beta)} \left( \frac{t - t^{(i)}}{h_0} \right)^\alpha \left( 1 - \frac{t - t^{(i)}}{h_0} \right)^{1-\alpha} \times \left( \frac{x - x^{(j)}}{h_1} \right)^\beta \left( 1 - \frac{x - x^{(j)}}{h_1} \right)^{1-\beta},
\]

provided that \(0^0 = 1\).

**II.** Suppose that \((t, x) \in \tilde{E}\) and \( t \leq h_0N \). There are \(i, j \in \mathbb{N}\) such that \([t^{(i)}, t^{(i+1)}] \times [x^{(j)}, x^{(j+1)}] \subset \tilde{E}\) and \( t^{(i)} \leq t \leq t^{(i+1)}, x^{(j)} \leq x \leq x^{(j+1)} \). Then \((T_h u)(t, x)\) is given by (4). If \((t, x) \in \tilde{E}\) and \( Nh_0 < t \leq T \), then \((T_h u)(t, x) = (T_h u)(Nh_0, x)\).

Note that \( T_h u \) is a continuous function on \( E^* \). The definition of \( T_h \) is based on the definition of the interpolating operator given in [6], page 86.

Define the interpolating operator \( \tilde{T}_{h_0} \colon F(I_{0,h} \cup I_h) \to C(I_0 \cup I) \) by

\[
(\tilde{T}_{h_0} z)(t) = \left( 1 - \frac{t - t^{(i)}}{h_0} \right) z^{(i)} + \frac{t - t^{(i)}}{h_0} z^{(i+1)},
\]

\( t \in [t^{(i)}, t^{(i+1)}], -N_0 \leq i \leq N - 1 \) and \((\tilde{T}_{h_0} z)(t) = (\tilde{T}_{h_0} z)(Nh_0)\) for \( t \in (Nh_0, T]\).

Given discrete functions \( u : E_{0,h} \cup E_h \to \mathbb{R} \), \( z : I_{0,h} \cup I_h \to \mathbb{R} \) denote

\[
c^{(i,j)}[z] = c(t^{(i)}, x^{(j)}, (\tilde{T}_{h_0} z)_{t^{(i)}}),
\]

\[
\lambda^{(i,j)}[u, z] = \lambda(t^{(i)}, x^{(j)}, (T_h u)_{(t^{(i)}, x^{(j)})}, (\tilde{T}_{h_0} z)_{t^{(i)}}).
\]

Introduce the difference operators \( \delta_0, \delta_1 \):

\[
\delta_0 u^{(i,j)} = (u^{(i+1,j)} - u^{(i,j)})/h_0, \quad \delta_1 u^{(i,j)} = (u^{(i,j)} - u^{(i,j-1)})/h_1.
\]

Consider the finite difference scheme corresponding to (1)–(2)

\[
(5) \quad \delta_0 u^{(i,j)} + c^{(i,j)}[z] \delta_1 u^{(i,j)} = u^{(i,j)} \lambda^{(i,j)}[u, z] \quad \text{on} \quad E_h', \quad j > 0,
\]

where

\[
(6) \quad z^{(i)} = h_1 \sum_{j=0}^{M^{(i)}-1} u^{(i,j)}, \quad i = -N_0, \ldots, N,
\]

with the initial condition

\[
(7) \quad u^{(i,j)} = v^{(i,j)} \quad \text{on} \quad E_{0,h}.
\]
It follows from \( c(t, 0, q) = 0, t \in [0, a], q \in C(I_0), \) that
\[
\delta_0 u^{(i,0)} = u^{(i,0)} \lambda^{(i,0)}[u, z], \quad i = 0, \ldots, N - 1.
\]
There exists exactly one solution of (5)–(8).

Suppose that \( \varphi: \{x^{(j)}: j \in \mathbb{N}\} \to \mathbb{R} \) is a bounded and summable function. Define
\[
\|\varphi\| = \sup\{|\varphi^{(j)}|: j \in \mathbb{N}\}, \quad \|\varphi\|_1 = h_1 \sum_{j=0}^\infty |\varphi^{(j)}|.
\]
By \( L^1 \) we denote a class of Lebesgue integrable functions defined on \( \mathbb{R}_+ \) with the standard norm denoted by \( \| \cdot \|_{L^1} \). Define the following class of functions.

**Definition 1.1.** Let \( f: \mathbb{R}_+ \to \mathbb{R}, f \in L^1 \). The function \( f \in L^1_M \) if there is a decreasing function \( g: \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( g \in L^1 \) and \( |f(x)| \leq g(x) \) for \( x \in \mathbb{R}_+ \).

Given \( h_1 > 0 \) and \( f: \mathbb{R}_+ \to \mathbb{R} \) let us denote \( f_{h_1} = f|_{\{x^{(j)}: j \in \mathbb{N}\}} \).

**Lemma 1.2.** If \( f: \mathbb{R}_+ \to \mathbb{R} \) and \( f \in L^1_M \), then \( \|f_{h_1}\|_1 < \infty \).

**Proof.** One can assume that \( h_1 \leq 1 \). There is a decreasing function \( F: \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( F \in L^1 \) and \( |f(x)| \leq F(x), x \in \mathbb{R}_+ \). We have
\[
\|F_{h_1}\|_1 = h_1 \sum_{j=0}^\infty F(x^{(j)}) \leq h_1 F(0) + \sum_{j=1}^\infty \int_{x^{(j-1)}}^{x^{(j)}} F(x)dx \leq F(0) + \|F\|_{L^1}.
\]
The assertion follows from the inequality \( |f(x)| \leq F(x), x \in \mathbb{R}_+ \). ■

We make the following assumptions:

**Assumption [V].** \( v: E_0 \to \mathbb{R}_+ \), there exists a decreasing function \( V: \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( V \in L^1 \) and \( v(t, x) \leq V(x) \) for \( (t, x) \in E_0 \).

**Assumption [C].** \( c: \Omega_0 \to \mathbb{R}_+ \) is continuous, there exist constants \( L_c, L_c^* \), \( M_c > 0 \) such that \( c(t, x, q) \leq M_c \) and
\[
|c(t, x, q) - c(t, x, \bar{q})| \leq L_c|x - \bar{x}| + L_c^*\|q - \bar{q}\|_{C(I_0)}
\]
for \( (t, x), (t, \bar{x}) \in E, q, \bar{q} \in C_+(I_0) \).

**Assumption [A].** \( \lambda: \Omega \to \mathbb{R} \) is continuous, there exist constants \( M_\lambda, L_\lambda, L_z > 0 \) such that \( \lambda(t, x, w, q) \leq M_\lambda \) and
\[
|\lambda(t, x, w, q) - \lambda(t, x, \bar{w}, \bar{q})| \leq L_\lambda\|w - \bar{w}\|_{C(B)} + L_z\|q - \bar{q}\|_{C(I_0)}
\]
for \( (t, x) \in E, w, \bar{w} \in C_+(B), q, \bar{q} \in C_+(I_0) \).

**Assumption [SN].** \( c: \Omega_0 \to \mathbb{R}_+, \lambda: \Omega \to \mathbb{R} \) and a discretization parameter \( h \) satisfy
\[
1 - \frac{h_0}{h_1} c(t, x, q) + h_0 \lambda(t, x, w, q) \geq 0
\]
for \( (t, x) \in E, w \in C_+(B), q \in C_+(I_0) \).
We give some properties of solutions of (5)–(8).

**Lemma 1.3.** If Assumptions [V], [SN] are satisfied, then any solution of (5)–(8) is nonnegative.

**Proof.** The proof is by induction on \( i \). For \( i = 0 \) the assertion holds since \( v \geq 0 \). Formulas (5) and (8) can be rewritten in the explicit form:

\[
(9) \quad u^{(i+1,j)} = u^{(i,j)} \left( 1 - \frac{h_0}{h_1} c^{(i,j)}[z] + h_0 \lambda^{(i,j)}[u, z] \right) + \frac{h_0}{h_1} c^{(i,j)}[z] u^{(i,j-1)}
\]
on \( E_h' \) for \( j > 0 \) and

\[
(10) \quad u^{(i+1,0)} = u^{(i,0)} \left( 1 + h_0 \lambda^{(i,0)}[u, z] \right), \quad i = 0, \ldots, N - 1.
\]

Suppose that for some \( 0 < i \leq N - 1 \) the functions \( u^{(k,\cdot)} \), \( i - N_0 \leq k \leq i \), are nonnegative. As Assumption [SN] is satisfied it follows from (9), (10) that the assertion holds for \( i + 1 \).

Define an auxiliary function \( U : E_{0,h} \cup E_h \to \mathbb{R}_+ \) by

\[
U^{(i,j)} = \begin{cases} V^{(j)}, & (t^{(i)}, x^{(j)}) \in E_{0,h}, \\
(1 + h_0 M_A)^i V^{((j-i) \lambda)}, & (t^{(i)}, x^{(j)}) \in E_h. \end{cases}
\]

**Lemma 1.4.** If Assumptions [V], [\( A \)], [SN] are satisfied, then \( u^{(i,j)} \leq U^{(i,j)} \) on \( E_{0,h} \cup E_h \).

**Proof.** The proof is by induction on \( i \). For \( i = -N_0, \ldots, 0 \) the assertion follows from Assumption [V]. Suppose that \( u^{(i,j)} \leq U^{(i,j)} \) for some \( 0 < i \leq N - 1 \) and \( j = 0, \ldots, M^{(i)} \). For \( j > 0 \) from Assumptions [V], [SN] and (9) we get

\[
u^{(i+1,j)} \leq U^{(i,j)} \left( 1 - \frac{h_0}{h_1} c^{(i,j)}[z] + h_0 \lambda^{(i,j)}[u, z] \right) + \frac{h_0}{h_1} c^{(i,j)}[z] U^{(i,j-1)} \\
\leq U^{(i,j-1)} \left( 1 + h_0 M_A \right) = (1 + h_0 M_A)^i V^{((j-i+1) \lambda)} = U^{(i+1,j)}.
\]

For \( j = 0 \) we proceed similarly. The proof is completed.

**Corollary 1.5.** Under assumptions of Lemma 1.4 we have the estimates

\[
\|u^{(i,\cdot)}\| \leq e^{aM_\lambda} V(0), \quad \|u^{(i,\cdot)}\|_1 \leq e^{aM_\lambda} (V(0)(1 + a/m) + \|V\|_{L^1})
\]

for \( i = 0, \ldots, N \).

2. **Convergence**

Suppose that \( \tilde{u} : E_0 \cup E \to \mathbb{R}_+ \) is a solution of (1)–(2) and \( \tilde{z} : I_0 \cup I \to \mathbb{R}_+ \) is given by (3). Denote \( \tilde{u}_h = \tilde{u}_{E_{0,h} \cup E_h} \) and \( \tilde{u}_{h}^{(i,j)} = \tilde{u}_h(t^{(i)}, x^{(j)}) \). The local discretization error \( \xi : E_h \to \mathbb{R} \) is defined as follows:

\[
(11) \quad \xi^{(i,j)} = \delta_0 u^{(i,j)} + c^{(i,j)}[\hat{z}_{h_0}] \delta_1 u^{(i,j)} - \tilde{u}_h^{(i,j)} \lambda^{(i,j)}[\tilde{u}_h, \hat{z}_{h_0}] \text{ on } E_h', \ j > 0,
\]
Convergence of a finite difference scheme  

\[ \xi^{(i,0)} = \delta_0 u_h^{(i,0)} - u_h^{(i,0)} \lambda^{(i,0)} [\tilde{u}_h, \tilde{z}_h], \quad i = 0, \ldots, N - 1, \]

where \[ \tilde{z}^{(i)}_{h_0} = h_1 \sum_{j=0}^{M^{(i)}-1} \tilde{u}^{(i,j)}_h, \quad i = -N_0, \ldots, N. \]

Let \( V: \mathbb{R}_+ \to \mathbb{R}_+ \) be the function given in Assumption [V].

**Assumption [\( \tilde{U} \)]**. \( \tilde{u}: E_0 \cup E \to \mathbb{R}_+ \) is a solution of (1)–(2) of class \( C^2 \) and

1) \( \tilde{u}(t, \cdot) \in L^1_M, \quad t \in I_0 \cup I \), and there exists a constant \( D_0 > 0 \) such that \( \tilde{u}(t, x) \leq D_0 V(x) \) on \( E_0 \cup E \);

2) \( \partial_t \tilde{u}(t, \cdot), \partial_x \tilde{u}(t, \cdot) \in L^1_M, \quad t \in I_0 \cup I \), and there exists \( D_1 > 0 \) such that \( |\partial_t \tilde{u}(t, x)| \leq D_1 V(x), \quad |\partial_x \tilde{u}(t, x)| \leq D_1 V(x) \) on \( E_0 \cup E \);

3) there exists a constant \( C > 0 \) such that

\[ |\partial_t \tilde{u}(t, x)|, |\partial_x \tilde{u}(t, x)|, |\partial_{xx} \tilde{u}(t, x)| \leq C \quad \text{on} \quad E_0 \cup E; \]

4) \( \partial_t \tilde{u}(t, \cdot), \partial_x \tilde{u}(t, \cdot) \in L^1_M, \quad t \in I_0 \cup I \), and there exists \( D_2 > 0 \) such that

\[ |\partial_t \tilde{u}(t, x)| \leq D_2 V(x), \quad |\partial_{xx} \tilde{u}(t, x)| \leq D_2 V(x) \quad \text{on} \quad E_0 \cup E; \]

5) there exists a constant \( D_3 > 0 \) such that

\[ |\tilde{z}(\tilde{t}) - \tilde{z}(t)| \leq D_3 |\tilde{t} - t|, \quad \tilde{t}, t \in I_0 \cup I. \]

**Lemma 2.1.** If the function \( \tilde{u}: E_0 \cup E \to \mathbb{R} \) satisfies Assumption [\( \tilde{U}3 \)], then

\[ \|(T_{h} \tilde{u}_h)_{(t^{(i)}, x^{(j)})} - \tilde{u}_{(t^{(i)}, x^{(j)})})\|_{C(B)} \leq C_{T_{h}} h_0^2 \]

for \( (t^{(i)}, x^{(j)}) \in E_h \) such that \( (t^{(i)}, x^{(j+N_1)}) \in E_h \), where \( C_{T_{h}} = C(1 + 2m + m^2) \).

The proof of the Lemma is similar to the proof of Theorem 3.18 in [6].

**Lemma 2.2.** Suppose that Assumption [\( \tilde{U}1, 2, 5 \)] is satisfied. Then there is \( \alpha: (0, +\infty) \to \mathbb{R}_+ \) such that \( \lim_{h_0 \to 0} \alpha(h_0) = 0 \) and

\[ \|(\tilde{T}_{h_0} \tilde{z}_h)_{(t^{(i)})} - \tilde{z}_{(t^{(i)})})\|_{C(I_0)} \leq h_0 C_{\tilde{T}_{h_0}} + \alpha(h_0), \quad 0 \leq i \leq N, \]

where \( C_{\tilde{T}_{h_0}} = (1 + \frac{1}{2m}) D_1 \Gamma_V + D_3, \quad \Gamma_V = V(0) + \|V\|_{L^1} \).

**Proof.** Let \( s \in [t^{(i-N_0)}, t^{(i)}] \). There is \( i - N_0 \leq k \leq i - 1 \) such that \( s \in [t^{(k)}, t^{(k+1)}] \) and

\[ |\tilde{T}_{h_0} \tilde{z}_h(s) - \tilde{z}(s)| \leq |\tilde{z}(t^{(k)}) - \tilde{z}(s)| + |\tilde{z}^{(k)}_{h_0} - \tilde{z}(t^{(k)})| + \frac{s - t^{(k)}}{h_0} |\tilde{z}^{(k+1)}_{h_0} - \tilde{z}^{(k)}_{h_0}|. \]
Let
\[ P(t^{(k)}, h_0) = D_0 \int_{x^{(M(k))}}^\infty V(x) dx, \quad R(t^{(k)}, h_0) = D_0 h_1 \sum_{j=0}^{N_1-1} V^{(M(k+1)+j)}. \]

The function \( M^{(\cdot)} \) is nonincreasing, hence \( x^{(M(N))} \leq x^{(M(k))} \) for \(-N_0 \leq k \leq N\) and \( x^{(M(N))} \to \infty \) as \( h_0 \to 0 \). Therefore
\[ P(t^{(k)}, h_0) \leq \int_{x^{(M(N))}}^\infty V(x) dx =: \bar{P}(h_0) \]
and \( \bar{P}(h_0) \to 0 \) as \( h_0 \to 0 \). Since \( V^{(M(k+1)+j)} \leq V^{(M(N)+j)} \leq V^{(M(N))} \) we have
\[ R(t^{(k)}, h_0) \leq D_0 h_1 \sum_{j=0}^{N_1-1} V^{(M(N))} \leq D_0 \tau_1 V^{(M(N))} =: \bar{R}(h_0). \]

The function \( V \) is decreasing, therefore \( \bar{R}(h_0) \to 0 \) as \( h_0 \to 0 \). From Assumption \([\bar{U}2]\) and Lemma 1.2 we obtain the estimates
\[ |\hat{z}_{h_0}^{(k)} - \hat{z}(t^{(k)})| \leq D_1 h_1^2 \sum_{j=0}^{M(k)-1} V^{(j)} + P(t^{(k)}, h_0) \leq D_1 h_1 \Gamma_V + \bar{P}(h_0), \]
\[ |\hat{z}_{h_0}^{(k+1)} - \hat{z}_{h_0}^{(k)}| \leq D_1 h_0 h_1 \sum_{j=0}^{M(k+1)-1} V^{(j)} + R(t^{(k)}, h_0) \leq D_1 h_0 \Gamma_V + \bar{R}(h_0). \]

The remaining part of the proof follows from Assumption \([\bar{U}5]\) and the both above estimates with \( \alpha(h_0) = \bar{P}(h_0) + \bar{R}(h_0) \). □

**Theorem 2.3.** If Assumptions \([V], [C], [\Lambda], [\bar{U}]\) are satisfied, then
\[ \|\xi^{(i,\cdot)}\| \leq V^{(0)} \beta(h_0), \quad \|\xi^{(i,\cdot)}\|_1 \leq \Gamma_V \beta(h_0), \]
where \( \Gamma_V = V^{(0)} + \|V\|_{L^1}, D = D_0 L_z + D_1 L_c^*, \)
\[ \beta(h_0) = h_0 [C_{\bar{T}h_0} D + D_2 (1 + M_c/m)] + \alpha(h_0) D + D_0 L_\lambda C T h_0^2. \]

**Proof.** Let us subtract (1) at the point \((t^{(i)}, x^{(j)}) \in E_{h}' ; j > 0, \) from (11). Then, by the mean value theorem and Assumptions \([\bar{U}1, 2, 4], [C], [\Lambda]\), we have
\[ |\xi^{(i,j)}| \leq h_0 D_2 V^{(j)} + D_0 V^{(j)} (L_\lambda \Delta_h^{(i,j)} + L_z \tilde{\Delta}_h^{(i)}) + h_1 D_2 M_c V^{(j-1)} + D_1 L_c^* V^{(j)} \tilde{\Delta}_h^{(i)}, \quad j > 0, \]
where
\[ \Delta_h^{(i,j)} = \|(T_h \tilde{u}_h)(t^{(i)}, x^{(j)}) - \tilde{u}_{(t^{(i)}, x^{(j)})}\|_{C(B)}, \quad \tilde{\Delta}_h^{(i)} = \|\tilde{T}_h \hat{z}_{h_0}^{(i)} - \tilde{z}_{(i)}\|_{C(I_0)}. \]
Similarly, subtracting (1) at the point \((t^{(i)}, 0), t^{(i)} \in I'_h\), from (12) we get
\[ |\xi^{(i,0)}| \leq h_0D_2V^{(0)} + D_0V^{(0)}(L\lambda(\Delta_{h}^{(i,0)} + L_z\bar{\Delta}_{h_0}^{(i)}).\]
In force of Lemmas 2.1, 2.2 we get inequalities
\[ \Delta_{h}^{(i,j)} \leq C_{T_h}h_0^2, \quad \bar{\Delta}_{h_0} \leq h_0C_{T_h} + \alpha(h_0),\]
which applied to the above estimates for \(|\xi^{(i,j)}|, |\xi^{(i,0)}|\) lead to the assertion. ■

**Lemma 2.4.** Let \(u_1, u_2 : E_{0,h} \cup E_h \to \mathbb{R}_+, z_1, z_2 : I_{0,h} \cup I_h \to \mathbb{R}_+\) be arbitrary bounded discrete functions. Then for \(t^{(i)} \in I_h\) and \((t^{(i)}, x^{(j)}) \in E_h\) such that \((t^{(i)}, x^{(j+N_1)}) \in E_h\) we have
\[ \|(T_hu_1)_{(i)}(t^{(i)}, x^{(j)}) - (T_hu_2)_{(i)}(t^{(i)}, x^{(j)})\|_{C(B)} \leq \max_{i-N_0 \leq k \leq i} |u_1^{(k,l)} - u_2^{(k,l)}|, \]
for all \(i \geq i_0\), \(0 < k \leq i\), \(0 < l \leq i\) where \(T_h\) is a finite difference operator.

**Theorem 2.5.** Suppose that Assumptions \([V], [C], [\Lambda], [SN], [\bar{U}]\) are satisfied, there is \(\gamma_0 : (0, +\infty) \to \mathbb{R}_+\) such that \(\lim_{h_0 \to 0} \gamma_0(h_0) = 0\) and
\[ \gamma_0(h_0)V^{(j)} \quad \text{on} \quad E_{0,h}.\]
Then \(\|\tilde{u}_h^{(i,i)} - u^{(i,i)}\|, \|\tilde{u}_h^{(i,i)} - u^{(i,i)}\|_1 \to 0\) as \(h_0 \to 0\), uniformly with respect to \(i\).

**Proof.** Denote \(\varepsilon^{(i,j)} = \tilde{u}_h^{(i,j)} - u^{(i,j)}\). Subtraction of the both sides of (11), (5), and (12), (8) lead to the recurrence error equations
\[ \varepsilon^{(i+1,j)} = \varepsilon^{(i,j)}(1 - \frac{h_0}{h_1}c^{(i,j)}[z] + h_0\lambda(\varepsilon^{(i,j)})[u, z] + \frac{h_0}{h_1}\varepsilon^{(i,j-1)}c^{(i,j)}[z]) \]
\[ + \frac{h_0}{h_1}C^{(i,j)}(\tilde{u}_h^{(i,j)} - u^{(i,j-1)}) + h_0\tilde{u}_h^{(i,j)}\Lambda^{(i,j)} + h_0\varepsilon^{(i,j)}, \quad j \geq 1, \]
and
\[ \varepsilon^{(i+1,0)} = \varepsilon^{(i,0)}(1 + h_0\lambda(\varepsilon^{(i,0)})[u, z]) + h_0\tilde{u}_h^{(i,0)}\Lambda^{(i,0)} + h_0\varepsilon^{(i,0)}, \]
respectively, where
\[ \Lambda^{(i,j)} = \lambda^{(i,j)}[\tilde{u}_h, \tilde{\zeta}_{h_0}] - \lambda^{(i,j)}[u, z], \quad C^{(i,j)} = c^{(i,j)}[z] - c^{(i,j)}[\tilde{\zeta}_{h_0}].\]
It follows from Assumption \([\bar{U}2]\) that \(\|\tilde{u}_h^{(i,j)} - \bar{u}_h^{(i,j-1)}\| \leq h_1D_1V^{(j-1)}\). By Assumptions \([SN], [C], [\Lambda], [\bar{U}]\) we conclude that
\[ |\varepsilon^{(i+1,j)}| \leq |\varepsilon^{(i,j)}|(1 - \frac{h_0}{h_1}c^{(i,j)}[z] + h_0\lambda(\varepsilon^{(i,j)})[u, z]) + \frac{h_0}{h_1}c^{(i,j)}[z]|\varepsilon^{(i,j-1)}| \]
\[ + h_0D_1V^{(j-1)}|C^{(i,j)}| + h_0\tilde{u}_h^{(i,j)}|\Lambda^{(i,j)}| + h_0|\varepsilon^{(i,j)}|, \quad j \geq 1, \]
and
\[ |\varepsilon^{(i+1,0)}| \leq (1 + h_0\lambda(\varepsilon^{(i,0)})[u, z])|\varepsilon^{(i,0)}| + h_0\tilde{u}_h^{(i,0)}|\Lambda^{(i,0)}| + h_0|\varepsilon^{(i,0)}|. \]
Since $|z^{(k)}_{h_0} - z^{(k)}| \leq \|\varepsilon^{(k,\cdot)}\|_1$, in force of Lemma 2.4 and Assumptions [C], [A], we obtain

$$|C^{(i,j)}| \leq L^*_c \max_{i-N_0 \leq k \leq i} \|\varepsilon^{(k,\cdot)}\|_1,$$

$$|\Lambda^{(i,j)}| \leq L_\lambda \max_{i-N_0 \leq k \leq i} \|\varepsilon^{(k,\cdot)}\| + L_z \max_{i-N_0 \leq k \leq i} \|\varepsilon^{(k,\cdot)}\|_1.$$

Note that

$$\sum_{j=1}^{M^{(i)-1}} c^{(i,j)}[z](|\varepsilon^{(i,j-1)}| - |\varepsilon^{(i,j)}|) \leq h_1 L_c \sum_{j=1}^{M^{(i)-2}} |\varepsilon^{(i,j)}| \leq L_c \|\varepsilon^{(i,\cdot)}\|_1.$$

Summation of (15) and (14) over $j \geq 1$ yields

$$\|\varepsilon^{(i+1,\cdot)}\|_1 \leq (1 + h_0 M_\lambda + h_0 L_c)\|\varepsilon^{(i,\cdot)}\|_1 + h_0 L^*_c D_1 \Gamma_V \max_{i-N_0 \leq k \leq i} \|\varepsilon^{(k,\cdot)}\|_1 + h_0 D_0 \Gamma_V (L_\lambda \max_{i-N_0 \leq k \leq i} \|\varepsilon^{(k,\cdot)}\| + L_z \max_{i-N_0 \leq k \leq i} \|\varepsilon^{(k,\cdot)}\|_1) + h_0 \|\xi^{(i,\cdot)}\|_1.$$

From (14), (15) we obtain

$$\|\varepsilon^{(i+1,\cdot)}\| \leq (1 + h_0 M_\lambda) \|\varepsilon^{(i,\cdot)}\| + h_0 L^*_c D_1 V^{(0)} \max_{i-N_0 \leq k \leq i} \|\varepsilon^{(k,\cdot)}\|_1 + h_0 D_0 V^{(0)} (L_\lambda \max_{i-N_0 \leq k \leq i} \|\varepsilon^{(k,\cdot)}\| + L_z \max_{i-N_0 \leq k \leq i} \|\varepsilon^{(k,\cdot)}\|_1) + h_0 \|\xi^{(i,\cdot)}\|.$$

By Theorem 2.3 we have $\|\xi^{(i,\cdot)}\|, \|\xi^{(i,\cdot)}\|_1 \leq \Gamma_V \beta(h_0)$.

Define an auxiliary comparison function $\Psi: I_{0,h} \cup I_h \to \mathbb{R}_+$,

$$\Psi^{(i)} = \gamma_0(h_0) \Gamma_V, \quad -N_0 \leq i \leq 0,$$

$$\Psi^{(i+1)} = (1 + h_0 \Gamma) \Psi^{(i)} + h_0 \Gamma_V \beta(h_0), \quad 0 \leq i \leq N - 1,$$

where $\Gamma = M_\lambda + L_c + \Gamma_V [D_1 L^*_c + D_0 (L_\lambda + L_z)]$. It is easy to verify that

$$\Psi^{(i)} \leq \gamma_0(h_0) e^{t^{(i)} \Gamma} + \beta(h_0) \Gamma_V t^{(i)} e^{t^{(i)} \Gamma} \quad \text{for} \quad 0 \leq i \leq N.$$

We show by induction on $i$ that $\|\varepsilon^{(i,\cdot)}\|, \|\varepsilon^{(i,\cdot)}\|_1 \leq \Psi^{(i)}$ for $-N_0 \leq i \leq N$. The assertion for $-N_0 \leq i \leq 0$ follows from (13). Suppose that the assertion holds for some $0 \leq i \leq N - 1$. Then applying the inductive assumption to (16), (17) we obtain the assertion for $i + 1$. The proof is completed. ■

Remark 2.6. Suppose that $H > 0$ is a sufficiently small real number, $h_1 \in (0, H)$. Given a decreasing Lebesgue integrable function $V: \mathbb{R}_+ \to \mathbb{R}_+$ and
\( \phi: (0, H) \to \mathbb{R}_+ \) such that \( \lim_{h_1 \to 0} \phi(h_1) = 0 \), we determine \( N_h \) satisfying

\[
\int_{h_1N_h}^{\infty} V(x) dx = \phi(h_1).
\]

**I.** If \( V(x) = e^{-ax}, a > 0 \), then \( N_h = \left[ \frac{1}{ah_1} \ln \frac{1}{a\phi(h_1)} \right] \), where \( [x] \) denotes the integral part of \( x \in \mathbb{R} \).

**II.** If \( V(x) = a/(1 + x^2), a > 0 \), then \( N_h = \left[ \frac{1}{h_1} \tan \left( \frac{\pi}{2} - \frac{\phi(h_1)}{a} \right) \right] \).

3. Numerical experiment

Let \( E = [0, 1] \times \mathbb{R}_+, I = [0, 1], E_0 = [-\frac{1}{10}, 0] \times \mathbb{R}_+, I_0 = [-\frac{1}{10}, 0] \). Consider the differential integral equation with delay

\[
\partial_t u(t, x) + \frac{t \sin^2 x}{1 + x} \sin^2(z(t - 0.1)) \partial_x u(t, x) = u(t, x) \left\{ \frac{1}{1 + t} + \frac{f(t) \sin(2x)}{(1 + \int_{t/2}^{t} z(s)ds) (1 + x)} \right\} \]

with the initial condition

\[
u(t, x) = (t + 1) \sin^2 x/(1 + x^2) \quad \text{for } (t, x) \in E_0,
\]

where \( z \) is given by (3), \( A = \pi(1 - e^{-2})/4 \) and

\[
f(t) = \sin^2(A(t + 0.9)) [1 + 0.5At (0.75t + 1)], \quad g(t) = \frac{20t \sin^2(A(t + 0.9))}{t + 0.95}.
\]

The function \( \bar{u}(t, x) = (t + 1) \sin^2 x/(1 + x^2) \) is the solution of (18)–(19) and \( \bar{z}(t) = A(t + 1) \).

Note that there is no deviation with respect to the spatial variable in (18)–(19). Therefore, \( M^{(i)} = N_h, -N_0 \leq i \leq N \). We applied the following difference method for (18)–(19):

\[
\delta_0 u^{(i,j)} + t^{(i)} \sin^2(x^{(j)}) \sin^2(z^{(i-N_0)}) \delta_1 u^{(i,j)} = u^{(i,j)} \lambda^{(i,j)}[u, z],
\]

on \( E'_h \) for \( j > 0 \), with the initial condition

\[
u^{(i,j)} = (1 + t^{(i)}) \sin^2(x^{(j)})/(1 + (x^{(j)})^2) \quad \text{on } E_{0,h},
\]
where \( u^{(i,0)} \), \( z^{(i)} \) are given by (8), (6), respectively, and

\[
\lambda^{(i,j)}[u, z] = \frac{1}{1 + t^{(i)}} + f(t^{(i)}) \frac{\sin(2x^{(j)})}{1 + z_h^{(i)}} - g(t^{(i)}) \frac{x^{(j)}}{1 + x^{(j)}} U_h^{(i,j)},
\]

\[
U_h^{(i,j)} = \frac{h_0}{2} (u^{(i-N_0,j)} + u^{(i,j)}) + h_0 \sum_{k=i-N_0+1}^{i-1} u^{(k,j)},
\]

\[
Z_{h_0}^{(i)} = \left\{ \begin{array}{ll}
\frac{h_0}{2} \left( z^{(i)} + \sum_{k=\frac{i+1}{2}+1}^{i-1} z^{(k)} + \sum_{k=\frac{i}{2}+1}^{i-1} \frac{h_0}{2} (z^{(i)} + z^{(k)}) \right), & \text{if } i \text{ is odd}, \\
\frac{h_0}{2} \left( z^{(i)} + \sum_{k=\frac{i}{2}+1}^{i-1} z^{(k)} + \sum_{k=\frac{i+1}{2}+1}^{i-1} \frac{h_0}{2} (z^{(i)} + z^{(k)}) \right), & \text{if } i \text{ is even}.
\end{array} \right.
\]

Suppose that \( u: E_{0,h} \cup E_h \to \mathbb{R}_+ \) is the solution of (20)–(21) and \( z: I_{0,h} \cup I_h \to \mathbb{R}_+ \) is given by (6). Let \( \bar{u}: E_0 \cup E \to \mathbb{R}_+ \) be the solution of (18)–(19) with \( \bar{z}: I_0 \cup I \to \mathbb{R}_+ \) given by (3) and denote \( \bar{u}_h = \bar{u}|_{E_{0,h} \cup E_h}, \bar{z}_h = \bar{z}|_{I_{0,h} \cup I_h} \). Let \( \varepsilon^{(i,j)} = \bar{u}_h^{(i,j)} - u^{(i,j)} \). We define error of the approximation:

\[
\Delta u = \max_{0 \leq i \leq N} \{ \| \varepsilon^{(i,\cdot)} \| \}, \quad \Delta_1 u = \max_{0 \leq i \leq N} \{ \| \varepsilon^{(i,\cdot)} \|_1 \}.
\]

Additionally, we define

\[
\Delta z = \max_{0 \leq i \leq N} \{ |z^{(i)} - \bar{z}_h^{(i)}| \}.
\]

The results of computations with \( N_h \) defined in Remark 2.6 for \( \phi(h) = \sqrt{h}/2 \) and \( V(x) = 1/(1 + x^2) \) are presented in the tables. Estimates of the functions \( c, \lambda \) for the above data are given. During computations we checked that Assumption [SN] was satisfied. The computations were performed by PC.

\[
h_1 = h_0, \ 0 \leq c \leq 0.38, \ 0.06 \leq \lambda \leq 1
\]

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Convergence of a finite difference scheme

\[ h_1 = 2h_0, \ 0 \leq c \leq 0.36, \ 0.06 \leq \lambda \leq 1 \]

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References


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Received September 14, 2009; revised version December 28, 2011.