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FOUR-DIMENSIONAL MATRIX TRANSFORMATION
AND $A$-STATISTICAL FUZZY KOROVKIN
TYPE APPROXIMATION

Abstract. In this paper, we prove a fuzzy Korovkin-type approximation theorem for fuzzy positive linear operators by using $A$-statistical convergence for four-dimensional summability matrices. Also, we obtain rates of $A$-statistical convergence of a double sequence of fuzzy positive linear operators for four-dimensional summability matrices.

1. Introduction

Anastassiou [3] first introduced the fuzzy analogue of the classical Korovkin theory (see also [1], [2], [4], [10]). Recently, some statistical fuzzy approximation theorems have been obtain by using the concept of statistical convergence (see, [5], [8]). In this paper, we prove a fuzzy Korovkin-type approximation theorem for fuzzy positive linear operators by using $A$-statistical convergence for four-dimensional summability matrices. Then, we construct an example such that our new approximation result works but its classical case does not work. Also we obtain rates of $A$-statistical convergence of a double sequence of fuzzy positive linear operators for four-dimensional summability matrices.

We now recall some basic definitions and notations used in the paper.

A fuzzy number is a function $\mu : \mathbb{R} \to [0, 1]$, which is normal, convex, upper semi-continuous and the closure of the set $\text{supp}(\mu)$ is compact, where $\text{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$. The set of all fuzzy numbers are denoted by $\mathbb{R}_F$. Let

$$[\mu]^0 = \{x \in \mathbb{R} : \mu(x) > 0\} \text{ and } [\mu]^r = \{x \in \mathbb{R} : \mu(x) \geq r\}, \text{ } (0 < r \leq 1).$$

Then, it is well-known [11] that, for each $r \in [0, 1]$, the set $[\mu]^r$ is a closed

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Now denote the interval \( [0, 1] \) necessarily bounded. The definition and the characterization of regularity briefly, RH for four dimensional matrices is known as Robison–Hamilton conditions, or RH. This assumption was made because a double \( P \)-convergent sequence into a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations is known as Silverman–Toeplitz conditions (see, for instance, [13]). In 1926, Robison [17] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double \( P \)-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison–Hamilton conditions, or briefly, RH-regularity (see, [12], [17]).

Recall that a four dimensional matrix \( A = [a_{j,k,m,n}] \) is said to be RH-regular if it maps every bounded \( P \)-convergent sequence into a \( P \)-convergent sequence.
sequence with the same $P$-limit. The Robison–Hamilton conditions state that a four dimensional matrix $A = [a_{j,k,m,n}]$ is RH-regular if and only if

1. $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each $(m, n) \in \mathbb{N}^2$,
2. $P - \lim_{j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n} = 1$,
3. $P - \lim_{j,k} \sum_{m \in \mathbb{N}} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,
4. $P - \lim_{j,k} \sum_{n \in \mathbb{N}} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,
5. $\sum_{(m,n) \in \mathbb{N}^2} |a_{j,k,m,n}|$ is $P$-convergent for each $(j, k) \in \mathbb{N}^2$,
6. there exist finite positive integers $A$ and $B$ such that $\sum_{m,n> B} |a_{j,k,m,n}| < A$ holds for every $(j, k) \in \mathbb{N}^2$.

Now let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then, a double sequence $\{x_{m,n}\}$ of fuzzy numbers is said to be $A$-statistically convergent to a fuzzy number $L \in \mathbb{R}_F$ if, for every $\varepsilon > 0$,

$$P - \lim_{j,k} \sum_{(m,n) \in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m, n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon\}.$$

In this case we write $st_{2}^{A} - \lim_{m,n} x_{m,n} = L$.

We should note that if we take $A = C(1;1) := [c_{j,k,m,n}]$, the double Cesáro matrix, defined by

$$c_{j,k,m,n} = \begin{cases} \frac{1}{jk}, & \text{if } 1 \leq m \leq j \text{ and } 1 \leq n \leq k, \\ 0, & \text{otherwise,} \end{cases}$$

then $C(1;1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [14], [15]. Finally, if we replace the matrix $A$ by the identity matrix for four-dimensional matrices, then $A$-statistical convergence reduces to the Pringsheim convergence [16].

2. $A$-statistical fuzzy Korovkin type approximation

Let us choose the real numbers $a; b; c; d$ so that $a < b, c < d$, and $U := [a; b] \times [c; d]$. Let $C(U)$ denote the space of all real valued continuous functions on $U$ endowed with the supremum norm

$$\|f\| = \sup_{(x,y) \in U} |f(x, y)|, (f \in C(U)).$$

Assume that $f : U \to \mathbb{R}_F$ be a fuzzy number valued function. Then $f$ is said to be fuzzy continuous at $x^0 := (x_0, y_0) \in U$ whenever $P - \lim_{m,n} x_{m,n} = x^0$. 

**A-statistical fuzzy Korovkin type approximation**

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then \( P - \lim D(f(x_{m,n}), f(x^0)) = 0 \). If it is fuzzy continuous at every point \((x, y) \in U\), we say that \( f \) is fuzzy continuous on \( U \). The set of all fuzzy continuous functions on \( U \) is denoted by \( C_F(U) \). Note that \( C_F(U) \) is a vector space. Now let \( L : C_F(U) \to C_F(U) \) be an operator. Then \( L \) is said to be fuzzy linear if, for every \( \lambda_1, \lambda_2 \in \mathbb{R} \) having the same sign and for every \( f_1, f_2 \in C_F(U) \), and \((x, y) \in U\),

\[
L(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2; x, y) = \lambda_1 \odot L(f_1; x, y) \oplus \lambda_2 \odot L(f_2; x, y)
\]

holds. Also \( L \) is called fuzzy positive linear operator if it is fuzzy linear and, the condition \( L(f; x, y) \leq L(g; x, y) \) is satisfied for any \( f, g \in C_F(U) \) and all \((x, y) \in U\) with \( f(x, y) \leq g(x, y) \). Also, if \( f, g : U \to \mathbb{R}_F \) are fuzzy number valued functions, then the distance between \( f \) and \( g \) is given by

\[
D^*(f, g) = \sup_{(x,y)\in U} \sup_{r \in [0,1]} \max\{|f_r^--g_r^-|, |f_r^+-g_r^+|\}
\]

(see for details, [1], [2], [3], [4], [9], [10]). Throughout the paper we use the test functions given by

\[
f_0(x, y) = 1, \quad f_1(x, y) = x, \quad f_2(x, y) = y, \quad f_3(x, y) = x^2 + y^2.
\]

**Theorem 2.1.** Let \( A = [a_{j,k,m,n}] \) be a non-negative RH-regular summability matrix and let \( \{L_{m,n}\}_{(m,n) \in \mathbb{N}^2} \) be a double sequence of fuzzy positive linear operators from \( C_F(U) \) into itself. Assume that there exists a corresponding sequence \( \{\tilde{L}_{m,n}\}_{(m,n) \in \mathbb{N}^2} \) of positive linear operators from \( C(U) \) into itself with the property

\[
\{L_{m,n}(f; x, y)\}_{\pm}^{(r)} = \tilde{L}_{m,n}(f_{\pm}^{(r)}; x, y)
\]

for all \((x, y) \in U\), \( r \in [0,1] \), \((m,n) \in \mathbb{N}^2 \) and \( f \in C_F(U) \). Assume further that

\[
\text{st}_{(A)}^{2} - \lim_{m,n \to \infty} \|\tilde{L}_{m,n}(f_i) - f_i\| = 0 \quad \text{for each } i = 0, 1, 2, 3.
\]

Then, for all \( f \in C_F(U) \), we have

\[
\text{st}_{(A)}^{2} - \lim_{m,n \to \infty} D^*(L_{m,n}(f), f) = 0.
\]

**Proof.** Let \( f \in C_F(U) \), \((x, y) \in U\) and \( r \in [0,1] \). By the hypothesis, since \( f_{\pm}^{(r)} \in C(U) \), we can write, for every \( \varepsilon > 0 \), that there exists a number \( \delta > 0 \) such that \( \|f_{\pm}^{(r)}(u, v) - f_{\pm}^{(r)}(x, y)\| < \varepsilon \) holds for every \((u, v) \in U\) satisfying \(|u - x| < \delta \) and \(|v - y| < \delta \). Then we immediately get for all \((u, v) \in U\), that

\[
|f_{\pm}^{(r)}(u, v) - f_{\pm}^{(r)}(x, y)| \leq \varepsilon + \frac{2M_{\pm}^{(r)}}{\delta^2}\{(u - x)^2 + (v - y)^2\},
\]
where $M_{\pm}^{(r)} := \|f_{\pm}^{(r)}\|$. Now, using the linearity and the positivity of the operators $\tilde{L}_{m,n}$, we have, for each $(m,n) \in \mathbb{N}^2$, that

$$\left| \tilde{L}_{m,n} \left( f_{\pm}^{(r)} ; x, y \right) - f_{\pm}^{(r)} (x, y) \right|$$

$$\leq \tilde{L}_{m,n} \left( |f_{\pm}^{(r)} (u, v) - f_{\pm}^{(r)} (x, y)| ; x, y \right) + M_{\pm}^{(r)} \left| \tilde{L}_{m,n} (f_0 ; x, y) - f_0 (x, y) \right|$$

$$\leq \tilde{L}_{m,n} \left( \varepsilon + \frac{2M_{\pm}^{(r)}}{\delta^2} \{(u-x)^2+(v-y)^2\} ; x, y \right) + M_{\pm}^{(r)} \left| \tilde{L}_{m,n} (f_0 ; x, y) - f_0 (x, y) \right|$$

$$\leq \varepsilon + (\varepsilon + M_{\pm}^{(r)}) \left| \tilde{L}_{m,n} (f_0 ; x, y) - f_0 (x, y) \right| + \frac{2M_{\pm}^{(r)}}{\delta^2} \left| \tilde{L}_{m,n} ((u-x)^2+(v-y)^2 ; x, y) \right|$$

$$\leq \varepsilon + (\varepsilon + M_{\pm}^{(r)}) \left| \tilde{L}_{m,n} (f_0 ; x, y) - f_0 (x, y) \right| + \frac{2M_{\pm}^{(r)}}{\delta^2} \left\{ \left| \tilde{L}_{m,n} (f_3 ; x, y) - f_3 (x, y) \right| + 2|x| \left| \tilde{L}_{m,n} (f_1 ; x, y) - f_1 (x, y) \right| + 2|y| \left| \tilde{L}_{m,n} (f_2 ; x, y) - f_2 (x, y) \right| + (x^2 + y^2) \left| \tilde{L}_{m,n} (f_0 ; x, y) - f_0 (x, y) \right| \right\}$$

$$\leq \varepsilon + \left( \varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}}{\delta^2} (x^2 + y^2) \right) \left| \tilde{L}_{m,n} (f_0 ; x, y) - f_0 (x, y) \right|$$

$$+ \frac{4M_{\pm}^{(r)}}{\delta^2} |x| \left| \tilde{L}_{m,n} (f_1 ; x, y) - f_1 (x, y) \right| + \frac{4M_{\pm}^{(r)}}{\delta^2} |y| \left| \tilde{L}_{m,n} (f_2 ; x, y) - f_2 (x, y) \right|$$

$$+ \frac{2M_{\pm}^{(r)}}{\delta^2} \left| \tilde{L}_{m,n} (f_3 ; x, y) - f_3 (x, y) \right|$$

$$\leq \varepsilon + K_{\pm}^{(r)} (\varepsilon) \left\{ \left| \tilde{L}_{m,n} (f_0 ; x, y) - f_0 (x, y) \right| + \left| \tilde{L}_{m,n} (f_1 ; x, y) - f_1 (x, y) \right| + \left| \tilde{L}_{m,n} (f_2 ; x, y) - f_2 (x, y) \right| + \left| \tilde{L}_{m,n} (f_3 ; x, y) - f_3 (x, y) \right| \right\}$$

where $K_{\pm}^{(r)} (\varepsilon) := \max \left\{ \varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}}{\delta^2} (A^2 + B^2), \frac{4M_{\pm}^{(r)}}{\delta^2} A, \frac{4M_{\pm}^{(r)}}{\delta^2} B, \frac{2M_{\pm}^{(r)}}{\delta^2} \right\}$,

$A := \max \{ |a|, |b| \}$, $B := \max \{ |c|, |d| \}$. Also taking supremum over $(x, y) \in U$, the above inequality implies that

$$\left| \tilde{L}_{m,n} (f_{\pm}^{(r)} ) - f_{\pm}^{(r)} \right|$$

$$\leq \varepsilon + K_{\pm}^{(r)} (\varepsilon) \left\{ \left| \tilde{L}_{m,n} (f_0 ) - f_0 \right| + \left| \tilde{L}_{m,n} (f_1 ) - f_1 \right|$$

$$+ \left| \tilde{L}_{m,n} (f_2 ) - f_2 \right| + \left| \tilde{L}_{m,n} (f_3 ) - f_3 \right| \right\}.$$
Now, it follows from (2.1) that
\[ D^* (L_{m,n} (f) , f) = \sup_{(x,y) \in U} D (L_{m,n} (f; x, y) , f (x, y)) \]
\[ = \sup_{(x,y) \in U} \sup_{r \in [0,1]} \max \{| \tilde{L}_{m,n} (f_-^r; x, y) - f_-^r (x, y) |, \]
\[ | \tilde{L}_{m,n} (f_+^r; x, y) - f_+^r (x, y) | \} \]
\[ = \sup_{r \in [0,1]} \max \{| \tilde{L}_{m,n} (f_-^r) - f_-^r ||, | \tilde{L}_{m,n} (f_+^r) - f_+^r || \}. \]

Combining the above equality with (2.3), we have
\[ (2.4) \quad D^* (L_{m,n} (f) , f) \leq \varepsilon + K (\varepsilon) \left\{ | \tilde{L}_{m,n} (f_0) - f_0 || + | \tilde{L}_{m,n} (f_1) - f_1 || + | \tilde{L}_{m,n} (f_2) - f_2 || + | \tilde{L}_{m,n} (f_3) - f_3 || \right\} \]
where \( K (\varepsilon) \) := \sup_{r \in [0,1]} \max \{ K_-^r (\varepsilon) , K_+^r (\varepsilon) \} \).

Now, for a given \( r > 0 \), choose \( \varepsilon > 0 \) such that \( 0 < \varepsilon < r \), and also define the following sets:
\[ G := \{ (m,n) \in \mathbb{N}^2 : D^* (L_{m,n} (f) , f) \geq r \} , \]
\[ G_0 := \{ (m,n) \in \mathbb{N}^2 : | \tilde{L}_{m,n} (f_0) - f_0 || \geq \frac{r - \varepsilon}{4K (\varepsilon)} \} , \]
\[ G_1 := \{ (m,n) \in \mathbb{N}^2 : | \tilde{L}_{m,n} (f_1) - f_1 || \geq \frac{r - \varepsilon}{4K (\varepsilon)} \} , \]
\[ G_2 := \{ (m,n) \in \mathbb{N}^2 : | \tilde{L}_{m,n} (f_2) - f_2 || \geq \frac{r - \varepsilon}{4K (\varepsilon)} \} , \]
\[ G_3 := \{ (m,n) \in \mathbb{N}^2 : | \tilde{L}_{m,n} (f_3) - f_3 || \geq \frac{r - \varepsilon}{4K (\varepsilon)} \} . \]

Then inequality (2.4) gives
\[ G \subset G_0 \cup G_1 \cup G_2 \cup G_3 \]
which guarantees that, for each \((j,k) \in \mathbb{N}^2\)
\[ (2.5) \quad \sum_{(m,n) \in G} a_{j,k,m,n} \leq \sum_{(m,n) \in G_0} a_{j,k,m,n} + \sum_{(m,n) \in G_1} a_{j,k,m,n} \]
\[ + \sum_{(m,n) \in G_2} a_{j,k,m,n} + \sum_{(m,n) \in G_3} a_{j,k,m,n} . \]

If we take the limit as \( j,k \to \infty \) on the both sides of inequality (2.5) and
use the hypothesis (2.2), we immediately see that
\[ \lim_{j,k} \sum_{(m,n) \in G} a_{j,k,m,n} = 0 \]
whence the result. ■

If \( A = I \), the identity matrix, then we obtain the following new fuzzy Korovkin theorem in Pringsheim’s sense.

**Theorem 2.2.** Let \( \{L_{m,n}\}_{(m,n) \in \mathbb{N}^2} \) be a double sequence of fuzzy positive linear operators from \( C_F(U) \) into itself. Assume that there exists a corresponding sequence \( \{\tilde{L}_{m,n}\}_{(m,n) \in \mathbb{N}^2} \) of positive linear operators from \( C(U) \) into itself with the property (2.1). Assume further that
\[
P - \lim_{m,n \to \infty} \|\tilde{L}_{m,n}(f_i) - f_i\| = 0 \quad \text{for each } i = 0, 1, 2, 3.
\]
Then, for all \( f \in C_F(U) \), we have
\[
P - \lim_{m,n \to \infty} D^*(L_{m,n}(f), f) = 0.
\]

We will now show that our result Theorem 2.1 is stronger than its classical (Theorem 2.2) version.

**Example 2.3.** Take \( A = C(1,1) := [c_{j,k,m,n}] \), the double Cesáro matrix, and define the double sequence \( \{u_{m,n}\} \) by
\[
u_{m,n} = \begin{cases} \sqrt{mn}, & \text{if } m \text{ and } n \text{ are square,} \\ 0, & \text{otherwise.} \end{cases}
\]
We observe that, \( st_{C(1,1)}^{(2)} - \lim_{m,n \to \infty} u_{m,n} = 0 \). But \( \{u_{m,n}\} \) is neither \( P \)-convergent nor bounded. Then consider the fuzzy Bernstein-type polynomials as follows:

\[
B^{(F)}_{m,n}(f; x, y) = (1 + u_{m,n}) \odot \bigoplus_{s=0}^{m} \bigoplus_{t=0}^{n} \binom{m}{s} \binom{n}{t} x^s y^t (1 - x)^{m-s} (1 - y)^{n-t} \odot f \left( \frac{s}{m}, \frac{t}{n} \right),
\]
where \( f \in C_F(U) \), \( (x, y) \in U \), \( (m, n) \in \mathbb{N}^2 \). In this case, we write
\[
\{ B^{(F)}_{m,n}(f; x, y) \}_{r=1}^{r} = \tilde{B}_{m,n}(f_{r}; x, y)
\]
\[
= (1 + u_{m,n}) \sum_{s=0}^{m} \sum_{t=0}^{n} \binom{m}{s} \binom{n}{t} x^s y^t (1 - x)^{m-s} (1 - y)^{n-t} f_{r} \left( \frac{s}{m}, \frac{t}{n} \right),
\]
where \( f^{(r)}_{\pm} \in C(U) \). Then, we get
\[
\begin{align*}
\tilde{B}_{m,n}(f_0; x, y) &= (1 + u_{m,n}) f_0(x, y), \\
\tilde{B}_{m,n}(f_1; x, y) &= (1 + u_{m,n}) f_1(x, y), \\
\tilde{B}_{m,n}(f_2; x, y) &= (1 + u_{m,n}) f_2(x, y), \\
\tilde{B}_{m,n}(f_3; x, y) &= (1 + u_{m,n}) \left( f_3(x, y) + \frac{x - x^2}{m} + \frac{y - y^2}{n} \right).
\end{align*}
\]
So we conclude that
\[
st_{C(1,1)}^2 \lim_{m,n \to \infty} \| \tilde{B}_{m,n}(f_i) - f_i \| = 0 \quad \text{for each } i = 0, 1, 2, 3.
\]
By Theorem 2.1, we obtain for all \( f \in C_F(U) \), that
\[
st_{C(1,1)}^2 \lim_{m,n \to \infty} D^*(B^{(f)}_{m,n}(f), f) = 0.
\]
However, since the sequence \( \{u_{m,n}\} \) is not convergent (in the Pringsheim’s sense), we conclude that Theorem 2.2 do not work for the operators \( \{B^{(f)}_{m,n}(f; x, y)\} \) in (2.6) while our Theorem 2.1 still works.

3. \( A \)-statistical fuzzy rates

Various ways of defining rates of convergence in the \( A \)-statistical sense for two-dimensional summability matrices were introduced in [7]. In a similar way, we obtain fuzzy approximation theorems based on \( A \)-statistical rates for four-dimensional summability matrices.

**Definition 3.1.** Let \( A = [a_{j,k,m,n}] \) be a non-negative RH-regular summability matrix and let \( \{\alpha_{m,n}\} \) be a non-increasing double sequence of positive real numbers. A double sequence \( x = \{x_{m,n}\} \) of fuzzy numbers is \( A \)-statistically convergent to a fuzzy number \( L \) with the rate of \( o(\alpha_{m,n}) \) if for every \( \varepsilon > 0 \),
\[
P - \lim_{j,k \to \infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n) \in K(\varepsilon)} a_{j,k,m,n} = 0,
\]
where
\[
K(\varepsilon) := \{(m, n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon\}.
\]
In this case, we write
\[
D(x_{m,n}, L) = st_{(A)}^2 - o(\alpha_{m,n}) \quad \text{as } m, n \to \infty.
\]

**Definition 3.2.** Let \( A = [a_{j,k,m,n}] \) and \( \{\alpha_{m,n}\} \) be the same as in Definition 3.1. Then, a double sequence \( x = \{x_{m,n}\} \) of fuzzy numbers is \( A \)-statistically
convergent to a fuzzy number $L$ with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$, 
\[ P - \lim_{j,k \to \infty} \sum_{(m,n) \in M(\varepsilon)} a_{j,k,m,n} = 0, \]
where 
\[ M(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : D(x_m,n,L) \geq \varepsilon \alpha_{m,n}\}. \]
In this case, we write 
\[ D(x_m,n,L) = st^2_{(A)} - o_{m,n}(\alpha_{m,n}) \text{ as } m,n \to \infty. \]

Note that the rate of convergence given by Definition 3.1 is more controlled by the entries of the summability matrices rather than the terms of the sequence $x = \{x_m,n\}$. However, according to the statistical rate given by Definition 3.2, the rate is mainly controlled by the terms of the fuzzy sequence $x = \{x_m,n\}$.

Also, we can give the corresponding $A$-statistical rates of real sequence $\{x_m\}$.

**Definition 3.3.** [6] Let $A = [a_{j,k,m,n}]$ be a non-negative $RH$-regular summability matrix and let $\{\alpha_{m,n}\}$ be a non-increasing double sequence of positive real numbers. A double sequence $x = \{x_m,n\}$ is $A$-statistically convergent to a number $L$ with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon > 0$, 
\[ P - \lim_{j,k \to \infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n) \in K(\varepsilon)} a_{j,k,m,n} = 0, \]
where 
\[ K(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_m,n - L| \geq \varepsilon \}. \]
In this case, we write 
\[ x_m,n - L = st^2_{(A)} - o(\alpha_{m,n}) \text{ as } m,n \to \infty. \]

**Definition 3.4.** [6] Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 3.3. Then, a double sequence $x = \{x_m,n\}$ is $A$-statistically convergent to a number $L$ with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$, 
\[ P - \lim_{j,k \to \infty} \sum_{(m,n) \in M(\varepsilon)} a_{j,k,m,n} = 0, \]
where 
\[ M(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_m,n - L| \geq \varepsilon \alpha_{m,n}\}. \]
In this case, we write 
\[ x_m,n - L = st^2_{(A)} - o_{m,n}(\alpha_{m,n}) \text{ as } m,n \to \infty. \]

Then we have the following.
Theorem 3.5. Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{L_{m,n}\}_{(m,n) \in \mathbb{N}^2}$ be a double sequence of fuzzy positive linear operators from $C_F(U)$ into itself. Assume that there exists a corresponding sequence $\{\tilde{L}_{m,n}\}_{(m,n) \in \mathbb{N}^2}$ of positive linear operators from $C(U)$ into itself with the property (2.1). Assume further that $\{\alpha_{i,m,n}\}_{(m,n) \in \mathbb{N}^2}, i = 0,1,2,3$ are non-increasing sequences of positive real numbers. If, for each $i = 0,1,2,3$

\begin{equation}
\|\tilde{L}_{m,n}(f_i) - f_i\| = 0 \quad \text{as} \quad m,n \to \infty
\end{equation}

then, for all $f \in C_F(U)$, we have

\begin{equation}
D^*(L_{m,n}(f), f) = \frac{1}{2} - o(\alpha_{i,m,n}) \quad \text{as} \quad m,n \to \infty
\end{equation}

where $\gamma_{m,n} := \max_{0 \leq i \leq 3} \{\alpha_{i,m,n}\}$ for every $(m,n) \in \mathbb{N}^2$.

Proof. Let $f \in C_F(U)$, $(x, y) \in U$ and $r \in [0,1]$. Then, we immediately see from Theorem 2.1’s proof that, for every $\varepsilon > 0$,

\begin{equation}
D^*(L_{m,n}(f), f) \leq \varepsilon + K(\varepsilon) \left\{ \|\tilde{L}_{m,n}(f_0) - f_0\| + \|\tilde{L}_{m,n}(f_1) - f_1\| \\
+ \|\tilde{L}_{m,n}(f_2) - f_2\| + \|\tilde{L}_{m,n}(f_3) - f_3\| \right\}
\end{equation}

where $K(\varepsilon) := \sup_{r \in [0,1]} \max_{\varepsilon} \{K^r(\varepsilon), K_+(^r\varepsilon)\}$.

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < r$, and also define the following sets:

$G := \{(m,n) \in \mathbb{N}^2 : D^*(L_{m,n}(f), f) \geq r\}$,

$G_0 := \{(m,n) \in \mathbb{N}^2 : \|\tilde{L}_{m,n}(f_0) - f_0\| \geq \frac{r - \varepsilon}{4K(\varepsilon)}\}$,

$G_1 := \{(m,n) \in \mathbb{N}^2 : \|\tilde{L}_{m,n}(f_1) - f_1\| \geq \frac{r - \varepsilon}{4K(\varepsilon)}\}$,

$G_2 := \{(m,n) \in \mathbb{N}^2 : \|\tilde{L}_{m,n}(f_2) - f_2\| \geq \frac{r - \varepsilon}{4K(\varepsilon)}\}$,

$G_3 := \{(m,n) \in \mathbb{N}^2 : \|\tilde{L}_{m,n}(f_3) - f_3\| \geq \frac{r - \varepsilon}{4K(\varepsilon)}\}$.

Then inequality (3.3) gives

$G \subset G_0 \cup G_1 \cup G_2 \cup G_3$

which guarantees that, for each $(j,k) \in \mathbb{N}^2$

$$\sum_{(m,n) \in G} a_{j,k,m,n} \leq \sum_{i=0}^{3} \left( \sum_{(m,n) \in G_i} a_{j,k,m,n} \right).$$
Also, by the definition of \((\gamma_{m,n})_{(m,n)\in\mathbb{N}^2}\), we have

\begin{equation}
\frac{1}{\gamma_{j,k}} \sum_{(m,n)\in G} a_{j,k,m,n} \leq \sum_{i=0}^{3} \left( \frac{1}{\alpha_{i,j,k}} \sum_{(m,n)\in G_i} a_{j,k,m,n} \right).
\end{equation}

If we take the limit as \(j, k \to \infty\) on both sides of inequality (3.4) and use the hypothesis (3.1), we immediately see that

\[ D \quad \text{holds for some } \varepsilon' > 0 \]

which gives (3.2). So, the proof is completed. 

We also give the next result.

**Theorem 3.6.** Let \(A = [a_{j,k,m,n}], \ {\alpha_i,m,n}_{(m,n)\in\mathbb{N}^2} (i = 0, 1, 2, 3), \ {\gamma_i}_{(m,n)\in\mathbb{N}^2}, \ {L_m}_{(m,n)\in\mathbb{N}^2} \) and \({\tilde{L}_m}_{(m,n)\in\mathbb{N}^2}\) be the same as in Theorem 3.5 with the property (2.1). If, for each \(i = 0, 1, 2, 3\)

\begin{equation}
||L_m(m,f_i) - f_i|| = st_{(A)}^2 - o_{m,n}(\alpha_i,m,n) \quad \text{as } m, n \to \infty
\end{equation}

then, for all \(f \in C(U)\), we have

\begin{equation}
D^*(L_m, f) = st_{(A)}^2 - o_{m,n}(\gamma_m) \quad \text{as } m, n \to \infty.
\end{equation}

**Proof.** By (3.3), it is clear that, for any \(\varepsilon > 0\),

\begin{equation}
D^*(L_m, f) \leq \varepsilon \gamma_m + B(\varepsilon) \left\{ ||\tilde{L}_m(m,f_0) - f_0|| + ||\tilde{L}_m(m,f_1) - f_1|| + ||\tilde{L}_m(m,f_2) - f_2|| + ||\tilde{L}_m(m,f_3) - f_3|| \right\}
\end{equation}
holds for some \(B(\varepsilon) > 0\). Now, as in the proof of Theorem 3.5, for a given \(\varepsilon' > 0\), choosing \(\varepsilon > 0\) such that \(\varepsilon < \varepsilon'\). Now we define the following sets:

\[ E : = \{(m,n) \in \mathbb{N}^2 : D^*(L_m, f) \geq \varepsilon' \gamma_m \}, \]

\[ E_0 : = \{(m,n) \in \mathbb{N}^2 : ||\tilde{L}_m(m,f_0) - f_0|| \geq \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \alpha_{0,m,n} \}, \]

\[ E_1 : = \{(m,n) \in \mathbb{N}^2 : ||\tilde{L}_m(m,f_1) - f_1|| \geq \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \alpha_{1,m,n} \}, \]

\[ E_2 : = \{(m,n) \in \mathbb{N}^2 : ||\tilde{L}_m(m,f_2) - f_2|| \geq \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \alpha_{2,m,n} \}, \]

\[ E_3 : = \{(m,n) \in \mathbb{N}^2 : ||\tilde{L}_m(m,f_3) - f_3|| \geq \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \alpha_{3,m,n} \}. \]
In this case, we claim that
\[(3.8) \quad E \subset E_0 \cup E_1 \cup E_2 \cup E_3.\]
Indeed, otherwise, there would be an element \((m, n) \in E\) but \((m, n) \notin E_0 \cup E_1 \cup E_2 \cup E_3\). So, we get
\[
(m, n) \notin E_0 \Rightarrow \|\tilde{L}_{m,n}(f_0) - f_0\| < \left( \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \right) \alpha_{0,m,n},
\]
\[
(m, n) \notin E_1 \Rightarrow \|\tilde{L}_{m,n}(f_1) - f_1\| < \left( \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \right) \alpha_{1,m,n},
\]
\[
(m, n) \notin E_2 \Rightarrow \|\tilde{L}_{m,n}(f_2) - f_2\| < \left( \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \right) \alpha_{2,m,n},
\]
\[
(m, n) \notin E_3 \Rightarrow \|\tilde{L}_{m,n}(f_3) - f_3\| < \left( \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \right) \alpha_{3,m,n}.
\]
By the definition of \(\{\gamma_{m,n}\}_{(m,n) \in \mathbb{N}^2}\), we immediately see that
\[(3.9) \quad B(\varepsilon) \sum_{k=0}^{3} \|\tilde{L}_{m,n}(f_k) - f_k\| < (\varepsilon' - \varepsilon) \gamma_{m,n}.
\]
Since \((m, n) \in E\), we have \(D^* (L_{m,n}(f), f) \geq \varepsilon' \gamma_{m,n}\), and hence, by (3.7),
\[
B(\varepsilon) \sum_{k=0}^{3} \|\tilde{L}_{m,n}(f_k) - f_k\| \geq (\varepsilon' - \varepsilon) \gamma_{m,n},
\]
which contradicts with (3.9). So, our claim (3.8) holds true. Now, it follows from (3.8) that
\[(3.10) \quad \sum_{(m,n) \in E} a_{j,k,m,n} \leq \sum_{i=0}^{3} \left( \sum_{(m,n) \in E_i} a_{j,k,m,n} \right).
\]
Letting \(j, k \to \infty\) in (3.10) and using (3.5), we observe that
\[
P - \lim_{j,k \to \infty} \sum_{(m,n) \in E} a_{j,k,m,n},
\]
which means (3.6). The proof is completed.

**Remark 3.7.** If \(\alpha_{i,m,n} \equiv 1\) for each \(i = 0, 1, 2, 3\), then Theorem 3.6 reduced to Theorem 2.1. Also, if \(A = I\), the identity matrix, \(\alpha_{i,m,n} \equiv 1\) for each \(i = 0, 1, 2, 3\), then Theorem 3.6 reduced to Theorem 2.2.

**References**


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