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RADICAL TRANSVERSAL LIGHTLIKE SUBMANIFOLDS
OF INDEFINITE PARA-SASAKIAN MANIFOLDS

Abstract. In this paper, we study radical transversal lightlike submanifolds and screen slant radical transversal lightlike submanifolds of indefinite para-Sasakian manifolds giving some non-trivial examples of these submanifolds. Integrability conditions of distributions $D$ and $RadTM$ on radical transversal lightlike submanifolds and screen slant radical transversal lightlike submanifolds of indefinite para-Sasakian manifolds, have been obtained. We also study totally contact umbilical radical transversal lightlike submanifolds of indefinite para-Sasakian manifolds.

1. Introduction

The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu [2]. A submanifold $M$ of a semi-Riemannian manifold $\mathcal{M}$ is said to be lightlike submanifold if the induced metric $g$ on $M$ is degenerate, i.e. there exists a non-zero $X \in \Gamma(TM)$ such that $g(X,Y) = 0$, $\forall Y \in \Gamma(TM)$. In 2003, Duggal and Jin [3] studied the geometry of totally umbilical lightlike submanifolds of a semi-Riemannian manifold. The notion of totally contact umbilical lightlike submanifolds of a semi-Riemannian manifold was considered by several geometers ([7], [8], [15]).

In 2006, Duggal and Sahin [5] studied invariant lightlike submanifolds of an indefinite Sasakian manifold. In 2009, Sahin [10] studied screen slant lightlike submanifolds. In 2010, Yildirim and Sahin [15] defined and studied radical transversal lightlike submanifolds of an indefinite Sasakian manifold. In [12], authors introduced the concept of an $\epsilon$-para-Sasakian structure with some examples. The value of $\epsilon$ is not definite, it is either 1 or -1, according as the structure vector field $V$ on $\overline{M}$ is spacelike or timelike.

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In this paper, we study radical transversal lightlike submanifolds of an indefinite para-Sasakian manifold. The paper is arranged as follows. There are some basic results in section 2. In section 3, we study radical transversal lightlike submanifolds of an indefinite para-Sasakian manifold, giving some examples. Section 4 is devoted to the study of totally contact umbilical radical transversal lightlike submanifolds of an indefinite para-Sasakian manifold. In section 5, we study screen slant radical transversal lightlike submanifolds of an indefinite para-Sasakian manifold and obtain integrability conditions of distributions $D$ and $RadTM$.

2. Preliminaries

A submanifold $(M^m, g)$ immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [2] if the metric $g$ induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank $r$, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM).$$

Now consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to $TM$ in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}),$$

$$T\overline{M}|_M = TM \oplus tr(TM),$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM).$$

Following are four cases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$:

**Case.1** r-lightlike if $r < \min (m, n)$,

**Case.2** co-isotropic if $r = n < m$, $S(TM^{\perp}) = \{0\}$,

**Case.3** isotropic if $r = m < n$, $S(TM) = \{0\}$,

**Case.4** totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

$$\nabla_X V = -A_V X + \nabla^*_X V, \quad \forall V \in \Gamma(tr(TM)), $$
where \( \{\nabla_X Y, A_V X\} \) and \( \{h(X, Y), \nabla_X^V\} \) belong to \( \Gamma(TM) \) and \( \Gamma(tr(TM)) \) respectively. \( \nabla \) and \( \nabla^t \) are linear connections on \( M \) and on the vector bundle \( tr(TM) \) respectively. The second fundamental form \( h \) is a symmetric \( F(M) \)-bilinear form on \( \Gamma(TM) \) with values in \( \Gamma(tr(TM)) \) and the shape operator \( A_V \) is a linear endomorphism of \( \Gamma(TM) \).

From (2.5) and (2.6), we have

\[
(2.7) \quad \nabla_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM),
\]
\[
(2.8) \quad \nabla_X N = -A_N X + \nabla^l_X (N) + D^s(X, N), \quad \forall N \in \Gamma(ltr(TM)),
\]
\[
(2.9) \quad \nabla_X W = -A_W X + \nabla^s_X (W) + D^l(X, W), \quad \forall W \in \Gamma(S(TM^\perp)),
\]

where \( h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l(X, V) = L(\nabla^l_X V), D^s(X, V) = S(\nabla^s_X V) \). \( L \) and \( S \) are the projection morphisms of \( tr(TM) \) on \( ltr(TM) \) and \( S(TM^\perp) \) respectively. \( \nabla^l \) and \( \nabla^s \) are linear connections on \( ltr(TM) \) and \( S(TM^\perp) \) called the lightlike connection and screen transversal connection on \( M \) respectively.

Now for any vector field \( X \) tangent to \( M \), we put

\[
(2.10) \quad \phi X = PX + FX,
\]

where \( PX \) and \( FX \) are tangential and transversal parts of \( \phi X \) respectively.

By using (2.5), (2.7)–(2.9) and metric connection \( \nabla \), we obtain

\[
(2.11) \quad \bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y),
\]
\[
(2.12) \quad \bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X).
\]

Denote the projection of \( TM \) on \( S(TM) \) by \( \bar{P} \). Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

\[
(2.13) \quad \nabla_X \bar{P}Y = \nabla^s_X \bar{P}Y + h^*(X, \bar{P}Y), \quad \forall X, Y \in \Gamma(TM),
\]
\[
(2.14) \quad \nabla_X \xi = -A^{*}_\xi X + \nabla^t_X \xi, \quad \xi \in \Gamma(RadTM).
\]

By using above equations, we obtain

\[
(2.15) \quad \bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^{*}_\xi X, \bar{P}Y),
\]
\[
(2.16) \quad \bar{g}(h^*(X, \bar{P}Y), N) = g(A_N X, \bar{P}Y),
\]
\[
(2.17) \quad \bar{g}(h^l(X, \xi), \xi) = 0, \quad A^{*}_\xi \xi = 0.
\]

It is important to note that in general \( \nabla \) is not a metric connection. Since \( \nabla \) is metric connection, by using (2.7), we get

\[
(2.18) \quad (\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y).
\]

A semi-Riemannian manifold \((M, \bar{g})\) is called an \( \epsilon \)-almost paracontact metric manifold \([12]\) if there exists a \((1, 1)\) tensor field \( \phi \), a vector field \( V \) called
characteristic vector field and a 1-form $\eta$, satisfying
\begin{align}
\phi^2 X &= X - \eta(X)V, \quad \eta(V) = \epsilon, \quad \eta \circ \phi = 0, \quad \phi(V) = 0, \\
\overline{g}(\phi X, \phi Y) &= \overline{g}(X, Y) - \epsilon \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(T\overline{M}),
\end{align}
where $\epsilon = 1$ or $-1$.

It follows that
\begin{align}
\overline{g}(V, V) &= \epsilon, \\
\overline{g}(X, V) &= \eta(X), \\
\overline{g}(X, \phi Y) &= \overline{g}(\phi X, Y), \quad \forall X, Y \in \Gamma(T\overline{M}).
\end{align}

Then $(\phi, V, \eta, \overline{g})$ is called an $\epsilon$-almost paracontact metric structure on $\overline{M}$.

An $\epsilon$-almost paracontact metric structure $(\phi, V, \eta, \overline{g})$ is called an indefinite para-Sasakian structure [12] if
\begin{align}
(\nabla_X \phi)Y &= -\overline{g}(\phi X, \phi Y)V - \epsilon \eta(Y)\phi^2 X, \quad \forall X, Y \in \Gamma(T\overline{M}),
\end{align}
where $\nabla$ is Levi-Civita connection with respect to $\overline{g}$.

A semi-Riemannian manifold endowed with an indefinite para-Sasakian structure is called an indefinite para-Sasakian manifold.

From (2.24), we get
\begin{align}
(\nabla_X V) &= \phi X, \quad \forall X \in \Gamma(T\overline{M}).
\end{align}

Let $(\overline{M}, \overline{g}, \phi, V, \eta)$ be an $\epsilon$-almost paracontact metric manifold. If $\epsilon = 1$, then $\overline{M}$ is said to be a spacelike almost paracontact metric manifold and if $\epsilon = -1$, then $\overline{M}$ is called a timelike almost paracontact metric manifold. In this paper we consider indefinite para-Sasakian manifolds with spacelike characteristic vector field $V$.

### 3. Radical transversal lightlike submanifolds

**Definition 3.1.** Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold, tangent to the structure vector field $V$, immersed in an indefinite para-Sasakian manifold $(\overline{M}, \overline{g})$. We say that $M$ is radical transversal lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:
\begin{align}
\phi(Rad TM) &= ltr(TM), \\
\phi(D) &= D,
\end{align}
where $S(TM) = D \perp \{V\}$ and $D$ is complementary non-degenerate distribution to $\{V\}$ in $S(TM)$.

Let $(\mathbb{R}^{2m+1}_q, \overline{g}, \phi, \eta, V)$ denote the manifold $\mathbb{R}^{2m+1}_q$ with its usual para-Sasakian structure given by
\[ \eta = \frac{1}{2} \left( dz - \sum_{i=1}^{m} y^i dx^i \right), \quad V = 2 \partial z, \]
\[ \bar{g} = \eta \otimes \eta + \frac{1}{4} \left( - \sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i \right), \]
\[ \phi \left( \sum_{i=1}^{m} (X_i \partial x_i + Y_i \partial y_i) + Z \partial z \right) = \sum_{i=1}^{m} (Y_i \partial x_i + X_i \partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z, \]

where \((x^i, y^i, z)\) are the cartesian coordinates on \(\mathbb{R}^{2m+1}_p\).

**Example 1.** Let \((\mathbb{R}^7, \bar{g}, \phi, \eta, V)\) be an indefinite para-Sasakian manifold, where \(\bar{g}\) is of signature \((-+, +, -, +, +, +, +)\) with respect to the canonical basis 
\{\partial x_1, \partial x_2, \partial x_3, \partial y_1, \partial y_2, \partial y_3, \partial z\}. Suppose \(M\) is a submanifold of \(\mathbb{R}^7\) given by 
\[-x^1 = y^2 = u_1, \quad x^2 = y^1 = u_2, \quad x^3 = u_3, \quad y^3 = u_4 \quad \text{and} \quad z = u_5.\]

The local frame of \(TM\) is given by \(\{Z_1, Z_2, Z_3, Z_4, Z_5\}\), where
\[ Z_1 = 2(\partial x_1 + \partial y_2 - y^1 \partial z), \quad Z_2 = 2(\partial x_2 + \partial y_1 + y^2 \partial z), \]
\[ Z_3 = 2(\partial x_3 + y^3 \partial z), \quad Z_4 = 2\partial y_3 \quad \text{and} \quad Z_5 = V = 2\partial z. \]

Hence \(\text{Rad}TM = \text{span} \{Z_1, Z_2\}, \; S(TM) = \text{span} \{Z_3, Z_4, V\} \) and \(\text{ltr}(TM)\) is spanned by \(N_1 = \partial x_1 - \partial y_2 + y^1 \partial z, \; N_2 = \partial x_2 - \partial y_1 + y^2 \partial z.\)

It follows that \(\phi Z_1 = 2N_2, \; \phi Z_2 = 2N_1, \; \phi Z_3 = Z_4, \; \phi Z_4 = Z_3. \) Thus \(\phi \text{Rad}TM = \text{ltr}(TM)\) and \(\phi D = D. \) Hence \(M\) is a radical transversal 2-lightlike submanifold of \(\mathbb{R}^7\).

**Example 2.** Let \((\mathbb{R}^9, \bar{g}, \phi, \eta, V)\) be an indefinite para-Sasakian manifold, where \(\bar{g}\) is of signature \((-+, +, +, -, +, +, +, +, +)\) with respect to the canonical basis 
\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}. Suppose \(M\) is a submanifold of \(\mathbb{R}^9\) given by 
\[ x^1 = y^2 = u_1, \quad -x^2 = y^1 = u_2, \quad x^3 = y^4 = u_3, \quad x^4 = y^3 = u_4 \quad \text{and} \quad z = u_5. \]

The local frame of \(TM\) is given by \(\{Z_1, Z_2, Z_3, Z_4, Z_5\}\), where
\[ Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z), \quad Z_2 = 2(-\partial x_2 + \partial y_1 - y^2 \partial z) \]
\[ Z_3 = 2(\partial x_3 + \partial y_4 + y^3 \partial z), \quad Z_4 = 2(\partial x_4 + \partial y_3 + y^4 \partial z), \]
\[ Z_5 = V = 2\partial z. \]

Hence \(\text{Rad}TM = \text{span} \{Z_1, Z_2\} \) and \(S(TM) = \text{span} \{Z_3, Z_4, V\}. \)

Now \(\text{ltr}(TM)\) is spanned by \(N_1 = \partial x_1 - \partial y_2 + y^1 \partial z, \; N_2 = \partial x_2 + \partial y_1 + y^2 \partial z \) and \(S(TM^\perp)\) is spanned by \(W_1 = 2(\partial x_3 - \partial y_4 + y^3 \partial z), W_2 = 2(-\partial x_4 + \partial y_3 - y^4 \partial z)\).
It follows that \( \phi Z_1 = 2N_2, \phi Z_2 = 2N_1, \phi Z_3 = Z_4, \phi Z_4 = Z_3, \phi W_1 = W_2 \) and \( \phi W_2 = W_1 \). Thus \( \phi \text{RadTM} = \text{ltr}(TM), \phi D = D \) and \( \phi S(TM^\perp) = S(TM^\perp) \). Hence \( M \) is a radical transversal 2-lightlike submanifold of \( \mathbb{R}^2 \).

**Theorem 3.1.** Let \( M \) be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). Then the distribution \( S(TM^\perp) \) is invariant with respect to \( \phi \), i.e. \( \phi(S(TM^\perp)) \subseteq S(TM^\perp) \).

**Proof.** Let \( M \) be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). Then from (2.23), we have

\[
(3.3) \quad g(\phi W, \xi) = g(W, \phi \xi) = 0, \quad \forall W \in \Gamma(S(TM^\perp)) \text{ and } \forall \xi \in \Gamma(\text{RadTM}),
\]

\[
(3.4) \quad g(\phi W, N) = g(W, \phi N) = 0, \quad \forall W \in \Gamma(S(TM^\perp)) \text{ and } \forall N \in \text{ltr}(TM).
\]

From (3.3) and (3.4), we get

\[
\phi(S(TM^\perp)) \cap \text{RadTM} = \{0\} \quad \text{and} \quad \phi(S(TM^\perp)) \cap \text{ltr}(TM) = \{0\}.
\]

From (3.2), we have

\[
(3.5) \quad g(\phi W, X) = g(W, \phi X) = 0, \quad \forall X \in (S(TM)),
\]

which shows that \( \phi(S(TM^\perp)) \cap S(TM) = \{0\} \). Therefore the distribution \( S(TM^\perp) \) is invariant with respect to \( \phi \). This completes the proof. 

Let \( M \) be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). Let \( P_1 \) and \( P_2 \) be the projection morphisms on \( \text{RadTM} \) and \( D \) respectively. Then, for \( X \in \Gamma(TM) \), we have

\[
(3.6) \quad X = P_1 X + P_2 X + \eta(X)V;
\]

where \( P_1 X \in \Gamma(\text{RadTM}) \) and \( P_2 X \in \Gamma(D) \).

Applying \( \phi \) to (3.6), we obtain

\[
(3.7) \quad \phi X = \phi P_1 X + \phi P_2 X,
\]

where \( \phi P_1 X \in \Gamma(\text{ltr}(TM)) \) and \( \phi P_2 X \in \Gamma(D) \).

From (2.24), we have

\[
(3.8) \quad \nabla_X \phi Y - \phi \nabla_X Y = -g(\phi X, \phi Y)V - \eta(Y)\phi^2 X.
\]

In view of (2.7), (2.8), (3.7) and (3.8), we obtain

\[
(3.9) \quad -g(\phi X, \phi Y)V - \eta(Y)\phi^2 X
= \nabla_X \phi P_2 Y + h^l(X, \phi P_2 Y) + h^s(X, \phi P_2 Y)
- A_{\phi P_2 Y} X + \nabla_Y (\phi P_1 Y) + D^s(X, \phi P_1 Y)
- \phi P_2 (\nabla_X Y) - \phi P_1 (\nabla_X Y) - \phi h^l(X, Y) - \phi h^s(X, Y).
\]
Now equating tangential, screen transversal and lightlike transversal components in both sides in equation (3.9) respectively, we obtain

\begin{align}
(3.10) & \quad g(\phi X, \phi Y)V + \eta(Y)\phi^2 X = \phi P_2(\nabla_X Y) + \phi h^l(X, Y) \\
& \quad \quad \quad \quad \quad + A_\phi p_1 Y X - \nabla_X \phi P_2 Y, \\
(3.11) & \quad h^l(X, \phi P_2 Y) + \nabla^l_X (\phi P_1 Y) + \phi P_1 (\nabla_X Y) = 0, \\
(3.12) & \quad h^s(X, \phi P_2 Y) + D^s(X, \phi P_1 Y) - \phi h^s(X, Y) = 0.
\end{align}

**Lemma 3.2.** Let $M$ be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then we have

(i) $g(\nabla_X Y, V) = \overline{g}(Y, \phi X), \forall X, Y \in \Gamma(TM) - \{V\},$

(ii) $g([X, Y], V) = 0, \forall X, Y \in \Gamma(TM) - \{V\}.$

**Proof.** Let $M$ be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then (2.7), we have

\begin{align}
(3.13) & \quad g(\nabla_X Y, V) = \overline{g}(\nabla_X Y, V), \quad \forall X, Y \in \Gamma(TM) - \{V\}.
\end{align}

Since $\nabla$ is a metric connection, from (3.13) we get

\begin{align}
(3.14) & \quad g(\nabla_X Y, V) = \nabla_X g(Y, V) - \overline{g}(Y, \nabla_X V),
\end{align}

which implies

\begin{align}
(3.15) & \quad g(\nabla_X Y, V) = -\overline{g}(Y, \nabla_X V), \quad \forall X, Y \in \Gamma(TM) - \{V\}.
\end{align}

From (2.25) and (3.15), we obtain

\begin{align}
(3.16) & \quad g(\nabla_X Y, V) = -\overline{g}(Y, \phi X), \quad \forall X, Y \in \Gamma(TM) - \{V\}.
\end{align}

On interchanging $X$ and $Y$ in (3.16), we get

\begin{align}
(3.17) & \quad g(\nabla_Y X, V) = -\overline{g}(X, \phi Y), \quad \forall X, Y \in \Gamma(TM) - \{V\}.
\end{align}

From (2.23), (3.16) and (3.17), we have

$$g([X, Y], V) = 0, \forall X, Y \in \Gamma(TM) - \{V\}.$$ \hfill \blacksquare

**Theorem 3.3.** Let $M$ be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then $D$ is integrable if and only if $h^l(X, \phi Y) = h^l(Y, \phi X), \forall X, Y \in \Gamma(D)$.

**Proof.** Let $M$ be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. On interchanging the role of $X$ and $Y$ in equation (3.11), we obtain

\begin{align}
(3.18) & \quad h^l(Y, \phi P_2 X) + \nabla^l_Y (\phi P_1 X) + \phi P_1 (\nabla_Y X) = 0, \quad \forall X, Y \in \Gamma(D).
\end{align}
Then from (3.11) and (3.18), we get
\begin{equation}
(3.19) \quad h^l(X, \phi Y) - h^l(Y, \phi X) = \phi P_1[X, Y], \quad \forall X, Y \in \Gamma(D).
\end{equation}

Since \( D \) is integrable if and only if \([X, Y] \in \Gamma(D), \forall X, Y \in \Gamma(D)\).

The proof follows from (3.19) and Lemma (3.2). \( \blacksquare \)

Theorem 3.4. Let \( M \) be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). Then \( \text{Rad}TM \) is integrable if and only if \( A_{\phi X}Y = A_{\phi Y}X, \forall X, Y \in \Gamma(\text{Rad}TM) \).

Proof. Let \( M \) be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). Then from (3.10), we have
\begin{equation}
(3.20) \quad A_{\phi Y}X + \phi P_2(\nabla_X Y) + \phi h^l(X, Y) = 0, \quad \forall X, Y \in \Gamma(\text{Rad}TM).
\end{equation}
Interchanging the role of \( X \) and \( Y \) in (3.20), we obtain
\begin{equation}
(3.21) \quad A_{\phi X}Y + \phi P_2(\nabla_Y X) + \phi h^l(Y, X) = 0, \quad \forall X, Y \in \Gamma(\text{Rad}TM).
\end{equation}

Now from (3.20) and (3.21), we get
\[ \phi P_2(\nabla_X Y) - \phi P_2(\nabla_Y X) + \phi h^l(X, Y) - \phi h^l(Y, X) = A_{\phi X}Y - A_{\phi Y}X. \]
Since \( h^l \) is symmetric, from above equation, we obtain
\begin{equation}
(3.22) \quad \phi P_2[X, Y] = A_{\phi X}Y - A_{\phi Y}X, \quad \forall X, Y \in \Gamma(\text{Rad}TM).
\end{equation}

Since \( \text{Rad}(TM) \) is integrable if and only if \([X, Y] \in \Gamma(\text{Rad}TM), \forall X, Y \in \Gamma(\text{Rad}TM)\).

The proof follows from (3.22) and lemma (3.2). \( \blacksquare \)

Theorem 3.5. Let \( M \) be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). Then \( \text{Rad}TM \oplus \{V\} \) defines a totally geodesic foliation on \( M \) if and only if \( \overline{g}(\phi Y, X)\eta(Z) = -\overline{g}(A_{\phi Y}X, \phi Z), \forall X, Y \in \Gamma(\text{Rad}TM) \oplus \{V\} \) and \( Z \in \Gamma(D) \).

Proof. Let \( M \) be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). By definition of radical transversal lightlike submanifold, \( \text{Rad}TM \oplus \{V\} \) defines a totally geodesic foliation if and only if \( \overline{g}(\nabla_X Y, Z) = 0, \forall X, Y \in \Gamma(\text{Rad}TM) \oplus \{V\} \) and \( Z \in \Gamma(S(TM)) \).

Since \( \nabla \) is a metric connection, using (2.7), we have
\begin{equation}
(3.23) \quad \overline{g}(\nabla_X Y, Z) = X\overline{g}(Z, X) - \overline{g}(Y, \nabla_X Z) = -\overline{g}(Y, \nabla_X Z), \quad \forall Z \in \Gamma(D) \) and \( \forall X, Y \in \Gamma(\text{Rad}TM) \oplus \{V\} \).
\end{equation}

Using (2.7), (2.20), (2.24) and (3.23), we get
\begin{equation}
(3.24) \quad \overline{g}(\nabla_X Y, Z) = -\overline{g}(\phi Y, X)\eta(Z) - \overline{g}(\phi Y, \nabla_X \phi Z), \quad \forall Z \in \Gamma(D) \) and \( \forall X, Y \in \Gamma(\text{Rad}TM) \oplus \{V\} \).
\end{equation}
From (2.13), (2.16) and (3.24), we have

\[(3.25) \quad g(\nabla_X Y, Z) = -g(\phi Y, X)\eta(Z) - g(A_{\phi Y} X, \phi Z), \]
\[\forall Z \in \Gamma(D) \text{ and } \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \]

The proof follows from (3.25) and lemma (3.2). □

**Theorem 3.6.** Let \( M \) be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). Then screen distribution defines a totally geodesic foliation if and only if \( A^*_{\phi N} X \) has no components in \( D \), \( \forall N \in \Gamma(ltr(TM)) \) and \( \forall X \in \Gamma(S(TM)). \)

**Proof.** Let \( M \) be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). By definition of radical transversal lightlike submanifold, \( S(TM) \) defines a totally geodesic foliation if and only if \( g(\nabla_X Y, N) = 0, \forall X, Y \in \Gamma(S(TM)) \) and \( N \in \Gamma(ltr(TM)). \) From (2.7), we have

\[(3.26) \quad g(\nabla_X Y, N) = g(\nabla_X Y, N), \]
\[\forall X, Y \in \Gamma(S(TM)) \text{ and } N \in \Gamma(ltr(TM)). \]

From (2.20), (2.24) and (3.26), we get

\[(3.27) \quad g(\nabla_X Y, N) = g(\nabla_X \phi Y, \phi N), \]
\[\forall X, Y \in \Gamma(S(TM)) \text{ and } N \in \Gamma(ltr(TM)). \]

In view of equations (2.7), (2.15) and (3.27), we obtain

\[(3.28) \quad g(\nabla_X Y, N) = g(A^*_{\phi N} X, \phi Y), \]
\[\forall X, Y \in \Gamma(S(TM)) \text{ and } N \in \Gamma(ltr(TM)). \]

The proof follows from (3.28) and lemma (3.2). □

4. Totally contact umbilical radical transversal lightlike submanifolds

**Definition 4.1.** A lightlike submanifold \( M \), tangent to the structure vector field \( V \), of an indefinite para-Sasakian manifold \( \overline{M} \) is said to be totally contact umbilical radical transversal lightlike submanifold if the second fundamental form \( h \) of \( M \) satisfies:

\[(4.1) \quad h^l(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha_L + \eta(X)h^l(Y, V) + \eta(Y)h^l(X, V), \]
\[\forall X, Y \in \Gamma(TM) \text{ and } \alpha_L \in \Gamma(ltr(TM)). \]

\[(4.2) \quad h^s(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha_S + \eta(X)h^s(Y, V) + \eta(Y)h^s(X, V), \]
\[\forall X, Y \in \Gamma(TM) \text{ and } \alpha_S \in \Gamma(S(TM^\perp)). \]
Theorem 4.1. Let $M$ be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then distribution $D$ is integrable.

Proof. Let $M$ be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then for any $X, Y \in \Gamma(D)$ and $N \in \Gamma(ltr(TM))$, we have

\begin{equation}
\mathcal{g}([X, Y], N) = \mathcal{g}(\nabla_X Y, N) - \mathcal{g}(\nabla_Y X, N).
\end{equation}

Now from (2.7), (2.20) and (4.3), we have

\begin{equation}
\mathcal{g}([X, Y], N) = \mathcal{g}(h^l(X, \phi Y), \phi N) - \mathcal{g}(h^l(Y, \phi X), \phi N).
\end{equation}

Replacing $Y$ by $\phi Y$ in (4.1), we get

\begin{equation}
h^l(X, \phi Y) = [g(X, \phi Y)]\alpha_L + \eta(X)h^l(\phi Y, V), \quad \forall X, Y \in \Gamma(D).
\end{equation}

Similarly, we have

\begin{equation}
h^l(Y, \phi X) = [g(Y, \phi X)]\alpha_L, \quad \forall X, Y \in \Gamma(D).
\end{equation}

Now, from (4.4), (4.5) and (4.6), we get

\begin{equation}
\mathcal{g}([X, Y], N) = \mathcal{g}(g(X, \phi Y)\alpha_L, \phi N) - \mathcal{g}(g(Y, \phi X)\alpha_L, \phi N),
\end{equation}

which implies

\begin{equation}
\mathcal{g}([X, Y], N) = g(Y, \phi X)(\mathcal{g}(\alpha_L, \phi N) - \mathcal{g}(\alpha_L, \phi N)) = 0,
\end{equation}

\begin{equation}
\forall X, Y \in \Gamma(D) \text{ and } N \in \Gamma(ltr(TM)).
\end{equation}

The proof follows from (4.8) and lemma (3.2).

Theorem 4.2. Let $M$ be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then $\alpha_L = 0$ if and only if $h^*(X, \phi Y) = 0$, $\forall X, Y \in \Gamma(D)$.

Proof. Let $M$ be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then from (3.9), we have

\begin{equation}
-g(\phi X, \phi Y)V - \eta(Y)\phi^2 X = \nabla_X \phi Y + h^l(X, \phi Y) + h^s(X, \phi Y)
- \phi P_2(\nabla X Y) - \phi P_1(\nabla X Y) - \phi h^l(X, Y)
- \phi h^s(X, Y), \quad \forall X, Y \in \Gamma(D).
\end{equation}

Now, from (4.9), we get

\begin{equation}
\mathcal{g}(\nabla_X \phi Y, \phi Z) - \mathcal{g}(\phi h^l(X, Y), \phi Z) = 0,
\end{equation}

\begin{equation}
\forall X, Y \in \Gamma(D) \text{ and } \forall Z \in \Gamma(Rad(TM)).
\end{equation}
From (2.13), (2.20) and (4.10), we have

\[ g(h^*(X, \phi Y), \phi Z) - g(h^l(X, Y), Z) = 0, \]
\[ \forall X, Y \in \Gamma(D) \text{ and } \forall Z \in \Gamma(\text{Rad}(TM)). \]

Now using (4.1) in (4.11), we get

\[ g(h^*(X, \phi Y), \phi Z) - g(g(X, Y) \alpha_L, Z) = 0, \]
\[ \forall X, Y \in \Gamma(D) \text{ and } \forall Z \in \Gamma(\text{Rad}(TM)). \]

This completes the proof. \( \blacksquare \)

**Theorem 4.3.** Let \( M \) be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). Then the induced connection \( \nabla \) on \( M \) is a metric connection if and only if \( A_{\phi Y}X = -\eta(X)Y \), for \( X \in \Gamma(TM) \) and \( Y \in \Gamma(\text{RadTM}) \).

**Proof.** Let \( M \) be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \). It is known that the induced connection is metric connection if and only if \( \nabla_X Y \in \Gamma(\text{RadTM}) \), for \( X \in \Gamma(TM) \) and \( Y \in \Gamma(\text{RadTM}) \).

From (3.9), we have

\[ \phi P_2(\nabla_X Y) + \phi P_1(\nabla_X Y) + \phi h^l(X, Y) + \phi h^s(X, Y) = \nabla_X (\phi P_1 Y) - A_{\phi P_1 Y} X + D^s(X, \phi P_1 Y), \forall X \in \Gamma(TM) \text{ and } Y \in \Gamma(\text{RadTM}). \]

Now, using (4.1) and (4.2) in (4.13), we obtain

\[ \phi P_2(\nabla_X Y) + \phi P_1(\nabla_X Y) + \eta(X)\phi h^l(Y, V) + \eta(X)\phi h^s(Y, V) = \nabla_X (\phi P_1 Y) - A_{\phi P_1 Y} X + D^s(X, \phi P_1 Y), \forall X \in \Gamma(TM) \text{ and } Y \in \Gamma(\text{RadTM}). \]

Taking tangential component of above equation, we get

\[ \phi P_2(\nabla_X Y) + \eta(X)\phi h^l(Y, V) = -A_{\phi Y} X, \]
\[ \forall X \in \Gamma(TM) \text{ and } Y \in \Gamma(\text{RadTM}). \]

Also from (2.7) and (2.25), we have

\[ \phi Y = h^l(Y, V), \forall Y \in \Gamma(\text{RadTM}). \]

Now, from (4.15) and (4.16), we get

\[ \phi P_2(\nabla_X Y) = -\eta(X) Y - A_{\phi Y} X, \]
\[ \forall X \in \Gamma(TM) \text{ and } Y \in \Gamma(\text{RadTM}). \]

The proof follows from (4.17) and lemma (3.2). \( \blacksquare \)
5. Screen slant radical transversal lightlike submanifolds

At first, we state the following Lemma for later use:

**Lemma 5.1.** Let \( M \) be a \( 2q \)-lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \), of index \( 2q \) such that \( 2q < \dim(M) \) with structure vector field tangent to \( M \). Then the screen distribution \( S(TM) \) of lightlike submanifold \( M \) is Riemannian.

The proof of above Lemma follows as in Lemma (4.1) of [11], so we omit it.

**Definition 5.1.** Let \( M \) be a \( 2q \)-lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \) of index \( 2q \) such that \( 2q < \dim(M) \) with structure vector field tangent to \( M \). Then we say that \( M \) is screen slant radical transversal lightlike submanifold of \( \overline{M} \) if following conditions are satisfied:

(i) \( \phi(RadTM) = ltr(TM) \),
(ii) For each non-zero vector field \( X \) tangent to \( D \) at \( x \in U \subset M \), the angle \( \theta(X) \) between \( \phi X \) and the vector space \( D_x \) is constant, i.e. it is independent of the choice of \( x \in U \subset M \) and \( X \in D_x \), where \( D \) is complementary non-degenerate distribution to \( \{V\} \) in \( S(TM) \) such that \( S(TM) = D \perp \{V\} \).

This constant angle \( \theta(X) \) is called slant angle of distribution \( D \). A screen slant lightlike submanifold is said to be proper if \( D \neq \{0\} \) and \( \theta \neq 0, \frac{\pi}{2} \).

From the above definition, we have the following decomposition

\[
(5.1) \quad TM = RadTM \perp D \perp \{V\}.
\]

**Theorem 5.2.** Let \( M \) be a screen slant radical transversal lightlike submanifold of \( \overline{M} \). Then \( M \) is radical transversal lightlike submanifold (resp. transversal lightlike submanifold) if and only if \( \theta = 0 \) (resp. \( \theta = \frac{\pi}{2} \)).

The proof of above theorem follows from definitions of radical transversal lightlike submanifolds and transversal lightlike submanifolds.

**Example 3.** Let \((\mathbb{R}^9_2, \overline{g}, \phi, \eta, V)\) be an indefinite para-Sasakian manifold, where \( \overline{g} \) is of signature \((-+-+-+-+-+)\) with respect to the canonical basis \( \{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\} \). Suppose \( M \) is a submanifold of \( \mathbb{R}^9_2 \) given by

\[
x^1 = y^2 = u_1, \quad x^2 = -y^1 = u_2, \quad x^3 = u_3 \cos \theta, \quad x^4 = -u_3 \sin \theta, \\
y^3 = u_4 \cos \theta, \quad y^4 = u_4 \sin \theta, \quad z = u_5.
\]
The local frame of $TM$ is given by \{\(Z_1, Z_2, Z_3, Z_4, Z_5\)\}, where
\[
\begin{align*}
Z_1 &= 2(\partial x_1 + \partial y_2 + y^1 \partial z), \\
Z_2 &= 2(\partial x_2 - \partial y_1 + y^2 \partial z), \\
Z_3 &= 2(\cos \theta \partial x_3 - \sin \theta \partial x_4 + y^3 \cos \theta \partial z - y^4 \sin \theta \partial z), \\
Z_4 &= 2(\cos \theta \partial y_3 + \sin \theta \partial y_4), \\
Z_5 &= V = 2\partial z. \\
\end{align*}
\]
Hence $RadTM = \text{span} \{Z_1, Z_2\}$ and $S(TM) = \text{span} \{Z_3, Z_4, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = \partial x_1 - \partial y_2 + y^1 \partial z$, $N_2 = \partial x_2 + \partial y_1 + y^2 \partial z$ and $S(TM^\perp)$ is spanned by
\[
\begin{align*}
W_1 &= 2(\sin \theta \partial x_3 + \cos \theta \partial x_4 + y^3 \sin \theta \partial z + y^4 \cos \theta \partial z), \\
W_2 &= 2(\sin \theta \partial y_3 - \cos \theta \partial y_4). \\
\end{align*}
\]
It follows that $\phi Z_1 = N_2$, $\phi Z_2 = -N_1$, which implies that $\phi RadTM = ltr(TM)$. On otherhand, we can see that $D = \text{span} \{Z_3, Z_4\}$ is a slant distribution with slant angle $2\theta$. Hence $M$ is screen slant radical transversal 2-lightlike submanifold of $\mathbb{R}^3_2$.

Now, we denote the projections on $RadTM$ and $D$ in $TM$ by $P_1$ and $P_2$ respectively. Similarly, we denote the projections on $ltr(TM)$ and $S(TM^\perp)$ in $tr(TM)$ by $Q_1$ and $Q_2$ respectively. Then, we get
\[
(5.2) \quad X = P_1 X + P_2 X + \eta(X)V, \quad \forall X \in \Gamma(TM). 
\]
On applying $\phi$ to (5.2), we have
\[
(5.3) \quad \phi X = \phi P_1 X + \phi P_2 X, 
\]
which gives
\[
(5.4) \quad \phi X = \phi P_1 X + f P_2 X + F P_2 X, \quad \forall X \in \Gamma(TM), 
\]
where $f P_2 X$ (resp. $F P_2 X$) denotes the tangential (resp. transversal) component of $\phi P_2 X$. Thus we get $\phi P_1 X \in ltr(TM), f P_2 X \in \Gamma(D)$ and $F P_2 X \in \Gamma(S(TM^\perp))$. Also, we have
\[
(5.5) \quad W = Q_1 W + Q_2 W, \quad \forall W \in \Gamma(tr(TM)). 
\]
Applying $\phi$ to (5.5), we obtain
\[
(5.6) \quad \phi W = \phi Q_1 W + \phi Q_2 W, 
\]
which gives
\[
(5.7) \quad \phi W = \phi Q_1 W + BQ_2 W + CQ_2 W, 
\]
where $BQ_2 W$ (resp. $CQ_2 W$) denote the tangential (resp. transversal) component of $\phi Q_2 W$. Thus we get $\phi Q_1 W \in RadTM, BQ_2 W \in \Gamma(D)$ and $CQ_2 W \in \Gamma(S(TM^\perp))$. 


Now, by using (2.7), (2.8), (2.9), (2.24), (5.4) and (5.7) and equating tangential, lightlike transversal and screen transversal components, we obtain
\[ \dot{g}(\phi X, \phi Y)V - \eta(Y)\phi^2 X = \nabla_X fP_2 Y - A_{FP_2 Y} X - A_{\phi P_1 Y} X \]
\[ - fP_2 \nabla_X Y + Bh^s(X, Y) + \phi h^l(X, Y), \]
(5.8)
\[ h^l(X, fP_2 Y) + D^l(X, FP_2 Y) + \nabla^l_X \phi P_1 Y = \phi P_1 \nabla_X Y, \]
(5.9)
\[ D^s(X, \phi P_1 Y) + h^s(X, fP_2 Y) = Ch^s(X, Y) - \nabla^s_X FP_2 Y + FP_2 \nabla_X Y. \]

**Theorem 5.3.** Let \( M \) be a 2q-lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \) with structure vector field tangent to \( M \) such that \( \phi \text{Rad} TM = ltr(TM) \). Then \( M \) is screen slant radical transversal lightlike submanifold if and only if there exists a constant \( \lambda \in [0, 1] \) such that \( P^2 X = \lambda(X - \eta(X)V), \forall X \in \Gamma(D). \)

**Proof.** Let \( M \) be a 2q-lightlike submanifold of an indefinite para-Sasakian manifold \( \overline{M} \) with structure vector field tangent to \( M \) such that \( \phi \text{Rad} TM = ltr(TM) \). Suppose there exists a constant \( \lambda \), such that \( P^2 X = \lambda(X - \eta(X)V) = \lambda \phi^2 X, \forall X \in \Gamma(D). \)

Now \( \cos \theta(X) = \frac{g(\phi X, PX)}{|\phi X||PX|} = \frac{g(X, \phi PX)}{|\phi X||PX|} = \frac{g(X, P^2 X)}{|\phi X||PX|} \)
\[ = \lambda \frac{g(X, \phi^2 X)}{|\phi X||PX|} = \lambda \frac{g(\phi X, \phi X)}{|\phi X||PX|}. \]
From above equation, we get
\[ (5.11) \cos \theta(X) = \lambda \frac{|\phi X|}{|PX|}. \]
Also \( |PX| = |\phi X| \cos \theta(X) \), which implies
\[ (5.12) \cos \theta(X) = \frac{|PX|}{|\phi X|}. \]
From (5.11) and (5.12), we get \( \cos^2 \theta(X) = \lambda(\text{constant}) \).

Hence \( M \) is a screen slant radical transversal lightlike submanifold.

Conversely, suppose that \( M \) is a screen slant radical transversal lightlike submanifold. Then \( \cos^2 \theta(X) = \lambda \), where \( \lambda \) is a constant. From (5.12), we have
\[ (5.13) \frac{|PX|^2}{|\phi X|^2} = \lambda. \]
Now \( g(PX, PX) = \lambda g(\phi X, \phi X) \), which gives \( g(X, P^2 X) = \lambda g(X, \phi^2 X) \).
Thus \( g(X, (P^2 - \lambda \phi^2)X) = 0 \). Since \( X \) is non-null vector, we have
\[(P^2 - \lambda \phi^2)X = 0.\] Hence
\[P^2 X = \lambda \phi^2 X = \lambda (X - \eta(X)V), \quad \forall X \in \Gamma(D).\]

**Corollary 5.1.** Let \(M\) be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \(\overline{M}\) with slant angle \(\theta\), then

\[
g(PX, PY) = \cos^2 \theta (g(X, Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D),
\]

\[
g(FX, FY) = \sin^2 \theta (g(X, Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D).
\]

The proof of above corollary follows using the steps as in proof of corollary (3.2) of [10].

**Theorem 5.4.** Let \(M\) be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \(\overline{M}\) with structure vector field tangent to \(M\). Then

(i) the radical distribution \(\text{RadTM}\) is integrable if and only if
\[
D^s(Y, \phi X) = D^s(X, \phi Y) \quad \text{and} \quad A_\phi X Y = A_\phi Y X, \quad \forall X, Y \in \Gamma(\text{RadTM}),
\]

(ii) the distribution \(D\) is integrable if and only if
\[
h^l(X, fY) + D^l(X, FY) = h^l(Y, fX) + D^l(Y, FX), \quad \forall X, Y \in \Gamma(D).
\]

**Proof.** Let \(M\) be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold \(\overline{M}\). From (5.10), we get

\[
D^s(X, \phi Y) = Ch^s(X, Y) + FP_2 \nabla X Y, \quad \forall X, Y \in \Gamma(\text{RadTM}).
\]

Interchanging \(X\) and \(Y\) in (5.16), we get

\[
D^s(Y, \phi X) = Ch^s(Y, X) + FP_2 \nabla Y X, \quad \forall X, Y \in \Gamma(\text{RadTM}).
\]

From (5.16) and (5.17), we get

\[
D^s(X, \phi Y) - D^s(Y, \phi X) = FP_2 (\nabla X Y - \nabla Y X) = FP_2 [X, Y].
\]

From (5.8), we have

\[
A_\phi Y X + fP_2 \nabla X Y = Bh^s(X, Y) + \phi h^l(X, Y), \quad \forall X, Y \in \Gamma(\text{RadTM}).
\]

Interchanging \(X\) and \(Y\) in (5.19), we get

\[
A_\phi X Y + fP_2 \nabla Y X = Bh^s(Y, X) + \phi h^l(Y, X), \quad \forall X, Y \in \Gamma(\text{RadTM}).
\]

From (5.19) and (5.20), we get

\[
A_\phi X Y - A_\phi Y X = fP_2 [X, Y], \quad \forall X, Y \in \Gamma(\text{RadTM}).
\]

The proof of (i) follows from (5.18) and (5.21).

From (5.9), we have

\[
h^l(X, fY) + D^l(X, FY) = \phi P_1 \nabla X Y, \quad \forall X, Y \in \Gamma(D).
\]
Interchanging $X$ and $Y$ in (5.20), we have

\[(5.23) \quad h^l(Y, fX) + D^l(Y, FX) = \phi P_1 \nabla_Y X, \quad \forall X, Y \in \Gamma(D).\]

From (5.22) and (5.23), we get

\[(5.24) \quad h^l(X, fY) - h^l(Y, fX) + D^l(X, FY) - D^l(Y, FX) = \phi P_1 [X, Y], \quad \forall X, Y \in \Gamma(D).\]

Now the proof of (ii) follows from (5.24) and lemma (3.2).

**Theorem 5.5.** Let $M$ be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then the screen distribution $\overline{S}(TM)$ defines a totally geodesic foliation if and only if $\overline{g}(A^*_{\phi N} X, fY) = -\overline{g}(D^l(X, FY), \phi N)$, $\forall X, Y \in \Gamma(\overline{S}(TM))$ and $N \in ltr(TM)$.

**Proof.** Let $M$ be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. By definition of radical transversal lightlike submanifold, $\overline{S}(TM)$ defines a totally geodesic foliation if and only if $\overline{g}(\nabla_X Y, N) = 0$, $\forall X, Y \in \Gamma(\overline{S}(TM))$ and $N \in \Gamma(ltr(TM))$. From (2.7), we have

\[(5.25) \quad \overline{g}(\nabla_X Y, N) = \overline{g}(\nabla_X Y, N), \quad \forall X, Y \in \Gamma(\overline{S}(TM)) \text{ and } N \in ltr(TM).\]

From (2.20), (2.24) and (5.25), we obtain

\[(5.26) \quad \overline{g}(\nabla_X Y, N) = \overline{g}(\nabla_X \phi Y, \phi N), \quad \forall X, Y \in \Gamma(\overline{S}(TM)) \text{ and } N \in ltr(TM).\]

In view of equations (5.4), (2.7), (2.9) and (5.26), we get

\[(5.27) \quad \overline{g}(\nabla_X Y, N) = \overline{g}(h^l(X, fY) + D^l(X, FY), \phi N), \quad \forall X, Y \in \Gamma(\overline{S}(TM)) \text{ and } N \in ltr(TM).\]

From (2.15) and (5.27), we have

\[(5.28) \quad \overline{g}(\nabla_X Y, N) = \overline{g}(A^*_{\phi N} X, fY) + \overline{g}(D^l(X, FY), \phi N), \quad \forall X, Y \in \Gamma(\overline{S}(TM)) \text{ and } N \in ltr(TM).\]

The proof follows from (5.28) and lemma (3.2).

**Theorem 5.6.** Let $M$ be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then $\text{Rad}TM \oplus \{V\}$ defines a totally geodesic foliation on $M$ if and only if $A_{FZ} X = h^*(X, fZ) + \eta(Z)X$, $\forall X \in \Gamma(\text{Rad}TM) \oplus \{V\}$ and $Z \in \Gamma(D)$.

**Proof.** Let $M$ be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. By definition of radical transversal
lightlike submanifold, $RadTM \oplus \{V\}$ defines a totally geodesic foliation if and only if $\bar{g}(\nabla_X Y, Z) = 0$, $\forall X, Y \in \Gamma(RadTM) \oplus \{V\}$ and $Z \in \Gamma(S(TM))$.

Since $\nabla$ is a metric connection, using (2.7), we have

(5.29) \[ \bar{g}(\nabla_X Y, Z) = -\bar{g}(Y, \nabla_X Z), \quad Z \in \Gamma(D) \quad \text{and} \quad \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \]

In view of equations (2.7), (2.20), (2.24) and (5.29), we obtain

(5.30) \[ \bar{g}(\nabla_X Y, Z) = -\bar{g}(\phi Y, X)\eta(Z) - \bar{g}(\phi Y, \nabla_X \phi Z), \quad Z \in \Gamma(D) \quad \text{and} \quad \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \]

From (2.7), (2.9), (2.13), (5.4) and (5.26), we have

(5.31) \[ \bar{g}(\nabla_X Y, Z) = -\bar{g}(\phi Y, X)\eta(Z) - \bar{g}(\phi Y, h^*(X, fZ) + \bar{g}(\phi Y, A_{FZ} X), \quad Z \in \Gamma(D) \quad \text{and} \quad \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \]

(5.32) \[ \bar{g}(\nabla_X Y, Z) = \bar{g}(\phi Y, A_{FZ} X - h^*(X, fZ) - \eta(Z)X), \quad Z \in \Gamma(D) \quad \text{and} \quad \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \]

The proof follows from (5.32) and lemma (3.2).

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