Abstract. The concepts of semisimple and semilocal pseudo BL-algebras are investigated. Many facts corresponding with them are considered. Moreover, we give a negative answer to the question from [Di Nola, Georgescu and Iorgulescu (Multiplae Valued Logic 8: 715-750, 2002), Problem 1.33].

1. Introduction

BL-algebras were introduced by Hájek [9] in 1998. The class of BL-algebras contains the MV-algebras introduced by Chang ([1]). Georgescu and Iorgulescu ([5]) introduced pseudo MV-algebras which are a non-commutative generalization of MV-algebras. In 2000, there were introduced pseudo BL-algebras as a natural generalization of BL-algebras and of pseudo MV-algebras. Georgescu and Iorgulescu ([7]) made the connection between pseudo BL-algebras and pseudo BCK-algebras. Kühr ([13]) proved that pseudo BL-algebras are equivalent to certain bounded DRℓ-monoids. Iorgulescu ([12]) showed that the category of pseudo Iséki algebras is equivalent to the category of pseudo BL-algebras. Pseudo BL-algebras correspond to a pseudo-basic fuzzy logic (see [10] and [11]). The paper [2] contains definition and basic properties of pseudo BL-algebras.

In [8], there are characterized and defined some classes of pseudo BL-algebras: local, good, perfect, peculiar and bipartite pseudo BL-algebras. In this paper there are given characterizations of other classes of pseudo BL-algebras: semisimple and semilocal pseudo BL-algebras. In particular, we show that the class of semisimple pseudo BL-algebras is not a quasivariety (and therefore it is not a variety). From this we obtain that representable pseudo BL-algebras are not semisimple in general. Thus Problem 1.33 of [3] is solved.

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2. Preliminaries

**Definition 2.1.** Let \((A; \vee, \wedge, \odot, \to, \sim, 0, 1)\) be an algebra of type \((2, 2, 2, 2, 2, 0, 0)\). \(A\) is called a pseudo BL-algebra if it satisfies the following axioms, for any \(x, y, z \in A:\)

(C1) \((A; \vee, \wedge, 0, 1)\) is a bounded lattice,
(C2) \((A; \odot, 1)\) is a monoid,
(C3) \(x \odot y \leq z \iff x \leq y \to z \iff y \leq x \sim z\),
(C4) \(x \wedge y = (x \to y) \odot x = x \odot (x \sim y)\),
(C5) \((x \to y) \lor (y \to x) = (x \sim y) \lor (y \sim x) = 1\).

In this sequel, we shall agree that the operations \(\vee, \wedge, \odot\) have priority towards the operations \(\to, \sim\). For any pseudo BL-algebra \((A; \vee, \wedge, \odot, \to, \sim, 0, 1)\), the reduct \(\mathcal{L}(A) = (A; \vee, \wedge, 0, 1)\) is a bounded distributive lattice. A pseudo BL-chain is a pseudo BL-algebra such that its lattice order is linear.

Throughout this paper \(A\) will denote a pseudo BL-algebra. For any \(x \in A\) and \(n = 0, 1, \ldots\), we put \(x^0 = 1\) and \(x^{n+1} = x^n \odot x\).

**Proposition 2.2.** ([2]) The following properties hold in \(A\) (for any \(x, y, z \in A\)):

(a) \(x \leq y \iff x \to y = 1\);
(b) \(y \leq x \to y, y \leq x \sim y\);
(c) \(x \odot y \leq x, x \odot y \leq y\);
(d) \(0 \odot x = x \odot 0 = 0\);
(e) \(x \lor z \to y \lor z \geq x \to y\);
(f) \(x \odot (y \lor z) = (x \odot y) \lor (x \odot z)\);
(g) \((x \lor y) \odot z = (x \odot z) \lor (y \odot z)\).

For any \(x \in A\), we define \(x^- = x \to 0\) and \(x^\sim = x \sim 0\).

**Proposition 2.3.** ([2]) The following properties hold in \(A\) (for any \(x, y \in A\)):

(a) \(y \leq x^- \iff y \odot x = 0\);
(b) \(y \leq x^\sim \iff x \odot y = 0\);
(c) \(x \leq y\) implies \(y^- \leq x^-\) and \(y^\sim \leq x^\sim\);
(d) \(x \leq (x^-)^\sim, x \leq (x^\sim)^-\);
(e) \(x \odot x^\sim = x^- \odot x = 0\).

**Definition 2.4.** A nonempty set \(F\) is called a filter of \(A\) if the following conditions hold:

(F1) If \(x, y \in F\), then \(x \odot y \in F\);
(F2) If \(x \in F, y \in A\) and \(x \leq y\), then \(y \in F\).
Under this definition, \{1\} and \(A\) are the simple examples of filters. A filter \(F\) of \(A\) is proper if \(F \neq A\). We denote by \(\text{Fil}(A)\) the set of all filters of \(A\).

**Proposition 2.5.** ([2]) If \(F \in \text{Fil}(A)\), then:

(a) \(1 \in F\);
(b) If \(x, y \in F\), then \(x \land y \in F\);
(c) If \(x \in F, y \in A\), then \(y \rightarrow x \in F, y \rightarrow x \in F\).

For every subset \(X \subseteq A\), the smallest filter of \(A\) which contains \(X\), i.e., the intersection of all filters \(F \supseteq X\), is called generated by \(X\), and is denoted by \([X]\).

**Remark 2.6.** ([2])

(a) If \(X\) is a filter, then \([X] = X\).
(b) If \(X \subseteq A\), then \([X] = \{y \in A : x_1 \odot x_2 \odot \cdots \odot x_n \leq y\text{ for some } n \geq 1\text{ and } x_1, x_2, \ldots, x_n \in X\}\).
(c) If \(X = \{x\}\), then we shall write \([x]\) instead of \([\{x\}]\) and \([x] = \{y \in A : x^n \leq y\text{ for some } n \geq 1\}\).

**Definition 2.7.** Let \(F\) be a proper filter of \(A\).

(a) \(F\) is called prime if for all \(x, y \in A, x \lor y \in F\) implies \(x \in F\) or \(y \in F\).
(b) \(F\) is called maximal (or ultrafilter) if whenever \(H\) is an filter such that \(F \subseteq H \subseteq A\), then either \(H = F\) or \(H = A\).

**Proposition 2.8.** ([2]) Any ultrafilter of \(A\) is a prime filter of \(A\).

**Proposition 2.9.** ([2]) Any proper filter of \(A\) can be extended to an ultrafilter.

We denote by \(\text{Max}(A)\) the set of all ultrafilters of \(A\). Write \(\mathcal{M}(A) = \bigcap\{F : F \in \text{Max}(A)\}\).

**Definition 2.10.** A filter \(H\) of \(A\) is called normal if for every \(x, y \in A,\)

\[
x \rightarrow y \in H \iff x \sim y \in H.
\]

We denote by \(\text{Max}_n(A)\) the set of normal ultrafilters of \(A\). Suppose that \(A\) possesses at least one ultrafilter which is normal. We define \(\mathcal{M}_n(A) = \bigcap\{F : F \in \text{Max}_n(A)\}\). If \(\text{Max}_n(A) = \emptyset\), we set \(\mathcal{M}_n(A) = A\).

From [4] (p. 499) we get

**Proposition 2.11.** If \(A\) is a pseudo BL-chain, then \(\text{Max}(A) = \text{Max}_n(A) = \{F\}\), where \(F = \{x \in A : x^n > 0\text{ for any } n \in \mathbb{N}\}\).

**Proposition 2.12.** ([3]) If \(H\) is a proper normal filter of \(A\), then \(H\) is an ultrafilter of \(A\) if and only if for any \(x \in A\),
For any \( x; y \in \mathbb{R} \times \mathbb{R} \), we define operations \( \vee \) and \( \wedge \) as follows: \( x \vee y = \max\{x, y\} \) and \( x \wedge y = \min\{x, y\} \). Let \( A = \{\left(\frac{1}{2}, b\right) \in \mathbb{R}^2 : b \geq 0\} \cup \{(a, b) \in \mathbb{R}^2 : \frac{1}{2} < a < 1, b \in \mathbb{R}\} \cup \{(1, b) \in \mathbb{R}^2 : b \leq 0\} \). For any \((a, b), (c, d) \in A\), we put:

\[
(a, b) \cap (c, d) = (\frac{1}{2}, 0) \vee (ac, bc + d),
\]

\[
(a, b) \mapsto (c, d) = (\frac{1}{2}, 0) \vee \left[ \left(\frac{c}{a}, \frac{d-b}{a}\right) \wedge (1, 0) \right],
\]

\[
(a, b) \rightsquigarrow (c, d) = (\frac{1}{2}, 0) \vee \left[ \left(\frac{c}{a}, \frac{ad-bc}{a}\right) \wedge (1, 0) \right].
\]

Then \((A; \vee, \wedge, \cap, \mapsto, \rightsquigarrow, (\frac{1}{2}, 0), (1, 0))\) is a pseudo BL-algebra. Let \( H = \{(1, b) : b \leq 0\} \). We show that it is a normal ultrafilter of \( A \). Obviously, \( H \) is a filter. Suppose that \((a, b), (c, d) \in A\). Then

\[
(a, b) \mapsto (c, d) \in H \iff \left(\frac{1}{2}, 0\right) \vee \left[ \left(\frac{c}{a}, \frac{d-b}{a}\right) \wedge (1, 0) \right] \in H
\]

\[
\iff \frac{c}{a} \geq 1 \iff \left(\frac{1}{2}, 0\right) \vee \left[ \left(\frac{c}{a}, \frac{ad-bc}{a}\right) \wedge (1, 0) \right] \in H
\]

\[
\iff (a, b) \rightsquigarrow (c, d) \in H.
\]

By definition, \( H \) is normal. We now apply Proposition 2.12 to show that \( H \) is maximal. Let \( x = (a, b) \notin H \). Then \( \frac{1}{2} \leq a < 1 \), and we have \( x^n = (\frac{1}{2}, 0) \) for some \( n \in \mathbb{N} \). Hence \((x^n)^- = (\frac{1}{2}, 0)^- = (\frac{1}{2}, 0) \mapsto (\frac{1}{2}, 0) = (1, 0) \in H\). Assume now \( x \in H \), that is, \( x = (1, b) \), \( b \leq 0 \). Then \( x^n = (1, nb) \in H \) for all \( n \in \mathbb{N} \), and therefore \((x^n)^- = (1, nb)^- = (\frac{1}{2}, -nb) \notin H \). It is proved that \( H \) is an ultrafilter.

For a filter \( H \) and \( x \in A \), we denote:

\[
x \circ H = \{x \circ h : h \in H\} \text{ and } H \circ x = \{h \circ x : h \in H\}.
\]

**Proposition 2.14.** ([3]) Let \( H \) be a filter of \( A \). The following conditions are equivalent:

(a) \( H \) is normal;

(b) For each \( x \in A \), \( x \circ H = H \circ x \).

As a consequence of Remark 2.6 and Proposition 2.14 we have:

**Proposition 2.15.** Let \( H_1 \) and \( H_2 \) be normal ideals of \( A \). Then
\[ [H_1 \cup H_2] = \{ x \in A : h_1 \circ h_2 \leq x \text{ for some } h_1 \in H_1 \text{ and } h_2 \in H_2 \}. \]

**Proposition 2.16. ([8])** Let \( H \) be a normal filter of \( A \) and let \( x \in A \). Then:

(a) \( x/H = 1/H \iff x \in H \);
(b) \( x/H = 0/H \iff x^- \in H \iff x^- \in H \).

Following [3], for any normal filter \( H \) of \( A \), we define a congruence \( \equiv_H \) on \( A \) by

\[ x \equiv_H y \iff (x \to y) \circ (y \to x) \in H. \]

We also have \( x \equiv_H y \iff (x \bowtie y) \circ (y \bowtie x) \in H \). Applying Proposition 2.2 (c) we get

(1) \[ x \equiv_H y \iff x \to y, y \to x \in H \iff x \bowtie y, y \bowtie x \in H. \]

In [3] it is proved that the map \( H \to \equiv_H \) is an isomorphism between the lattice of normal filters and the lattice of congruences of \( A \). We denote by \( x/H \) the congruence class of an element \( x \in A \), that is, \( x/H = x/\equiv_H \). On the set \( A/H = \{ x/H : x \in A \} \) we define the natural operations induced from those of \( A \). The resulting quotient algebra \( (A/H; \lor, \land, \odot, \rightarrow, \bowtie, 0/H, 1/H) \) becomes a pseudo BL-algebra, called the quotient algebra of \( A \) by the normal filter \( H \). The map \( \varphi : A \to A/H \), defined by \( \varphi(x) = x/H \) for all \( x \in A \), is a homomorphism from \( A \) onto the quotient pseudo BL-algebra \( A/H \).

If \( \varphi : A \to B \) is a homomorphism of pseudo BL-algebras, then the kernel of \( \varphi \) is the set \( \ker(\varphi) = \{ x \in A : \varphi(x) = 1 \} \). The following propositions are easily obtained:

**Proposition 2.17.** Let \( \varphi : A \to B \) be a homomorphism of pseudo BL-algebras. Then:

(a) \( \ker(\varphi) \) is a normal filter of \( A \);
(b) \( A/\ker(\varphi) \cong B \).

**Proposition 2.18.** Let \( H \) be a normal filter of \( A \). Then there is a bijection between the filters of \( A \) containing \( H \) and the filters of \( A/H \).

**Proposition 2.19.** Let \( H_1, \ldots, H_m \) be normal filters of \( A \) such that \( [H_i \cup H_j] = A \) for \( i, j = 1, \ldots, m \) and \( i \neq j \). Let \( x_1, \ldots, x_m \in A \). Then there is \( x \in A \) such that \( x \equiv_{H_i} x_i \) for \( i = 1, \ldots, m \).

Proof. First, let \( m = 2 \). Since \( [H_1 \cup H_2] = A \), by Proposition 2.15 there exist \( h_{12} \in H_1 \) and \( h_{21} \in H_2 \) such that \( h_{12} \circ h_{21} = 0 \). From Proposition 2.3
(a) we have $h_{12} \leq h_2^\sim$. Then $h_2^\sim \in H_1$, and hence $h_{21} \equiv_{H_1} 0$ by Proposition 2.16. Since $h_{12} \leq h_2^\sim$, applying Proposition 2.3 (c) we get $(h_2^\sim)^\sim \leq h_2^\sim$. From Proposition 2.3 (d) we obtain $h_2^\sim \leq (h_2^\sim)^\sim$. Therefore, $h_{21} \leq h_2^\sim$, and consequently, $h_2^\sim \in H_2$. Proposition 2.16 now shows that $h_{12} \equiv_{H_2} 0$.

Pick $x = (h_{12} \odot x_1) \vee (h_{21} \odot x_2)$, where $x_1, x_2 \in A$. Note that using Proposition 2.2 (d) we have $x = (h_2^\sim \odot x_1) \vee (0 \odot x_2) = x_1 / H_1$.

Thus $x \equiv_{H_1} x_1$. Similarly, $x \equiv_{H_2} x_2$. Now let $m$ be arbitrary. For $i, j = 1, \ldots, m$ and $i \neq j$, there exist $h_{ij} \in H_i$ and $h_{ji} \in H_j$ such that $h_{ij} \odot h_{ji} = 0$.

Considering $x = \bigvee_{i=1}^{m} (h_{i_1} \odot \cdots \odot h_{i_{i-1}} \odot h_{i_{i+1}} \odot \cdots \odot h_{i_m} \odot x_i)$ and reasoning as above we see that $x \equiv_{H_i} x_i$ for $i = 1, \ldots, m$.

Let $I$ be a nonempty set. The direct product of the pseudo BL-algebras $A_i$, $i \in I$, denoted by $\prod (A_i : i \in I)$, is the pseudo BL-algebra obtained by endowing the set theoretical cartesian product of $A_i$, $i \in I$, with the pseudo BL-operations defined pointwise.

The map $\pi_i : \prod (A_i : i \in I) \to A$, defined by $\pi_i(x) = x(i)$ for all $x \in \prod (A_i : i \in I)$, is a homomorphism onto $A_i$ and it is called the $i$-th projection function.

**PROPOSITION 2.20.** Let $A_1, \ldots, A_k$ be pseudo BL-algebras and let $A = A_1 \times \cdots \times A_k$. Then

$$\text{Fil}(A) = \text{Fil}(A_1) \times \cdots \times \text{Fil}(A_k).$$

**Proof.** If $F_i \subseteq \text{Fil}(A_i)$ for $i = 1, \ldots, k$, then $F_1 \times \cdots \times F_k$ is a filter of $A$. Conversely, if $F$ is a filter of $A$, then for $i = 1, \ldots, k$, $F_i = \pi_i(F)$ is a filter of $A_i$ and $F = F_1 \times \cdots \times F_k$. From this we conclude that the assertion follows.

**PROPOSITION 2.21.** ([3]) Let $H$ be a proper normal filter of $A$. Then $A/H$ is a pseudo BL-chain if and only if $H$ is a prime filter of $A$.

An algebra $A$ is *simple* if $A$ has exactly two congruences: $0_A = \{(x, x) : x \in A\}$ and $1_A = A^2$. Clearly, a pseudo BL-algebra $A$ is simple if and only if $\text{Fil}_n(A) = \{\{1\}, A\}$. 
Proposition 2.22. A normal filter \( H \) of \( A \) is maximal if and only if \( A/H \) is a simple pseudo BL-chain.

Proof. Let \( H \) be a normal ultrafilter of \( A \). By Propositions 2.8 and 2.21, \( A/H \) is a pseudo BL-chain. From Proposition 2.18 we conclude that \( |\text{Fil}(A/H)| = 2 \). Hence \( A/H \) is simple.

Let \( B = A/H \) be a simple pseudo BL-chain. Then \( \text{Fil}_n(B) = \{1/H\}, B\). Let \( F \) be a proper filter of \( B \). By Proposition 2.9, \( F \) can be extended to an ultrafilter \( M \). From Proposition 2.11 we see that \( M \) is normal. Therefore, \( F = M = \{1/H\} \), and hence \( \text{Fil}(B) = \{1/H\}, B\). From Proposition 2.18 it follows that \( H \) is an ultrafilter of \( A \).

Let \( B(A) \) be the Boolean algebra of all complemented elements in the distributive lattice \( \mathcal{L}(A) = (A; \lor, \land, 0, 1) \).

Proposition 2.23. ([8]) If \( e \in B(A) \), then \( [e] = \{x \in A : e \leq x\} \) and \(([e]; \lor, \land, \odot, \rightarrow, \leftarrow, e, 1)\) is a pseudo BL-algebra.

Proposition 2.24. ([8]) If \( e \in B(A) \) and \( x \in A \), then:
(a) \( e \odot x = e \land x \);
(b) \( e \lor e^- = 1 \) and \( e \land e^- = 0 \);
(c) \( e^- = e^- \) is the complement of \( e \).

Proposition 2.25. ([8]) If \( A = A_1 \times A_2 \), then there is \( e \in B(A) \) such that \( A_1 \cong [e] \) and \( A_2 \cong [e^-] \).

Proposition 2.26. If \( x \in A \) and \( e \in B(A) \), then \((x \lor e) \odot (x \lor e^-) = x \).

Proof. Applying Propositions 2.2 (f, g) and 2.24 we have
\[
(x \lor e) \odot (x \lor e^-) = [(x \lor e) \odot x] \lor [(x \lor e) \odot e^-] \\
= [(x \odot x) \lor (e \odot x)] \lor [(x \odot e^-) \lor (e \odot e^-)] \\
= (x \odot x) \lor (x \land e) \lor (x \land e^-) \\
= (x \odot x) \lor (x \land (e \lor e^-)) = (x \odot x) \lor x = x. \quad \blacksquare
\]

3. Semisimple pseudo BL-algebras

Definition 3.1. A pseudo BL-algebra \( A \) is semisimple if the intersection of all maximal congruences of \( A \) is the congruence \( 0_A \).

Since, in a pseudo BL-algebra \( A \), the congruences are in bijective correspondence with the normal filters, it follows that \( A \) is semisimple if and only if \( \mathcal{M}_n(A) = \{1\} \). Obviously, every simple pseudo BL-algebra is semisimple.
Definition 3.2. A is a subdirect product of pseudo BL-algebras $A_i$, $i \in I$, if there exists an injective homomorphism $\varphi : A \to \prod (A_i : i \in I)$ such that $\pi_i \circ \varphi$ maps $A$ onto $A_i$ for all $i \in I$.

Theorem 3.3. Let $A$ be a pseudo BL-algebra. The following are equivalent:

(a) $A$ is semisimple;
(b) There is a family $\{H_i : i \in I\}$ of normal ultrafilters of $A$ with $\bigcap \{H_i : i \in I\} = \{1\}$;
(c) $A$ is a subdirect product of simple pseudo BL-chains.

Proof. (a) $\Rightarrow$ (b): Follows from definition.

(b) $\Rightarrow$ (c): Let $\{H_i : i \in I\}$ be a family of normal ultrafilters of $A$ such that $\bigcap \{H_i : i \in I\} = \{1\}$. Write $A_i = A/H_i$ for $i \in I$. From Proposition 2.22 we deduce that $A_i$ are simple pseudo BL-chains. Now, define $\varphi : A \to \prod (A_i : i \in I)$ by

$$\varphi(x) = (x/H_i : i \in I)$$

for all $x \in A$.

Evidently, $\varphi$ is a homomorphism. Let $\varphi(x) = \varphi(y)$. Then $x/H_i = y/H_i$ for all $i \in I$. By (1), $x \to y$, $y \to x \in \bigcap \{H_i : i \in I\} = \{1\}$. Therefore, $x \to y = y \to x = 1$. From Proposition 2.22 (a) it follows that $x = y$. Consequently, $\varphi$ is injective. It is easy to see that $\pi_i \circ \varphi$ maps $A$ onto $A_i$. Thus $A$ is a subdirect product of the simple pseudo BL-chains $A_i$, $i \in I$.

(c) $\Rightarrow$ (a): Let $\psi : A \to \prod (A_i : i \in I)$ be an injective homomorphism, where $A_i$ are simple BL-chains, and let $\pi_i \circ \psi : A \to A_i$ be surjective. Set $\text{Ker}(\pi_i \circ \psi) = H_i$ for $i \in I$. From Proposition 2.17 we conclude that $H_i$ is a normal filter of $A$ and $A/H_i \cong A_i$. In consequence, $A/H_i$ is simple. By Proposition 2.22, $H_i$ is maximal. Let $x \in \bigcap \{H_i : i \in I\}$. Then $\pi_i(\psi(x)) = 1$ for all $i \in I$, and hence $\psi(x) = 1$. Since $\psi$ is injective we obtain $x = 1$. Therefore, $\bigcap \{H_i : i \in I\} = \{1\}$. Consequently, $A$ is semisimple. ■

Proposition 3.4. Any subalgebra of semisimple pseudo BL-algebra is semisimple.

Proof. Let $A$ be a semisimple pseudo BL-algebra and let $B$ be a subalgebra of $A$. By Theorem 3.3, there is a family $\{H_i : i \in I\}$ of normal ultrafilters of $A$ such that $\bigcap \{H_i : i \in I\} = \{1\}$. Observe that $H_i \cap B \in \text{Max}_n(B)$ for each $i \in I$. By definition, $H_i \cap B$ is a normal proper filter of $B$ and from Proposition 2.12 we see that it is maximal. Moreover,

$$\bigcap \{H_i \cap B : i \in I\} = (\bigcap \{H_i : i \in I\}) \cap B = \{1\} \cap B = \{1\}.$$
Now, applying Theorem 3.3 we conclude that $B$ is a semisimple pseudo BL-algebra. ■

**Proposition 3.5.** Let $A_1$ and $A_2$ be semisimple pseudo BL-algebras. Then the direct product $A = A_1 \times A_2$ is also semisimple.

**Proof.** By Theorem 3.3 there exist families $\{H_i : i \in I_1\} \subseteq \text{Max}_n(A_1)$ and $\{F_i : i \in I_2\} \subseteq \text{Max}_n(A_2)$ such that $\bigcap\{H_i : i \in I_1\} = \{1\}$ and $\bigcap\{F_i : i \in I_2\} = \{1\}$. Let

$$U_i = \begin{cases} H_i \times A_2 & \text{if } i \in I_1, \\ A_1 \times F_i & \text{if } i \in I_2. \end{cases}$$

We set $I = I_1 \cup I_2$. It is clear that $U_i (i \in I)$ are normal ultrafilters of $A$ and $\bigcap\{U_i : i \in I\} = \{1\}$. Consequently, $A$ is a semisimple pseudo BL-algebra. ■

In a similar way, we get the following more general result.

**Theorem 3.6.** Any direct product of semisimple pseudo BL-algebras is a semisimple pseudo BL-algebra.

From Proposition 3.4 and Theorem 3.6 we have

**Corollary 3.7.** The class of all semisimple pseudo BL-algebras is closed under the formation of subalgebras and direct products.

**Proposition 3.8.** The class of all semisimple pseudo BL-algebras is not closed under the formation of ultraproducts (and hence it is not a quasivariety).

**Proof.** Let $[0,1]$ be the unit interval of real numbers $\mathbb{R}$. For any $x, y \in \mathbb{R}$, define $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. For $x, y \in [0,1]$ we put

$$x \odot y = (x + y - 1) \vee 0 \quad \text{and} \quad x \rightarrow y = (y - x + 1) \wedge 1.$$ 

Then $A = ([0,1]; \vee, \wedge, \odot, \rightarrow, \rightarrow, 0, 1)$ is a (pseudo) BL-chain. Proposition 2.11 shows that $\mathcal{M}_n(A) = \{x \in A : x^n > 0 \text{ for all } n \in \mathbb{N}\}$. It is easy to see that

$$x^n = [n(x - 1) + 1] \vee 0$$

for $x \in [0,1]$ and $n \in \mathbb{N}$. We have

$$x^n > 0 \iff n(x - 1) + 1 > 0 \iff n(1 - x) < 1.$$
Hence, if $x^n > 0$ for all $n \in \mathbb{N}$, then $x = 1$. Therefore, $\mathcal{M}_n(A) = \{1\}$ and consequently, $A$ is semisimple.

Let $\mathcal{F}$ be an ultrafilter over $\mathbb{N}$ containing all cofinite subsets of $\mathbb{N}$. Let $B$ be the ultrapower of $A$ determined by $\mathcal{F}$, in symbols, $B = A^{\mathbb{N}}/\mathcal{F}$. By the fundamental ultraproduct theorem, $B$ is a (pseudo) BL-algebra. Let $b = (b_k : k \in \mathbb{N})$, where $b_k = 1 - \frac{1}{k}$ for $k \in \mathbb{N}$. We prove that

$$ (b^n)^{-}/\mathcal{F} \leq b/\mathcal{F} \quad \text{for all } n \in \mathbb{N}. $$

Fix $n \in \mathbb{N}$ and let $k > n$. We have $b^n_k = [n(b_k - 1) + 1] \vee 0 = 1 - \frac{n}{k}$, and hence $(b^n_k)^{-} = 1 - b^n_k = \frac{n}{k} \leq 1 - \frac{1}{k} = b_k$. From this we obtain (2).

Observe that $b/\mathcal{F} \notin \mathcal{H}$ for some normal ultrafilter $\mathcal{H}$ of $B$. By Proposition 2.12, there is $m \in \mathbb{N}$ such that $[(b/\mathcal{F})^m]^\sim = (b^n)^{-}/\mathcal{F} \leq b/\mathcal{F}$. Therefore, $b/\mathcal{F} \in \mathcal{H}$. This contadiction shows that $b/\mathcal{F} \in \mathcal{M}_n(B)$. Since $b/\mathcal{F} \neq 1/\mathcal{F}$, $\mathcal{M}_n(B) \neq \{1/\mathcal{F}\}$. Thus $B$ is not semisimple.

We shall say that a pseudo BL-algebra is representable, if it can be represented as a subdirect product of pseudo BL-chains. Kühr [13] proved that $A$ is a representable pseudo BL-algebra if and only if there exists a family $\{P_i : i \in I\}$ of normal prime filters of $A$ such that $\bigcap \{P_i : i \in I\} = \{1\}$. Consequently, if $A$ is semisimple, then $A$ is representable. The converse implication is not true in general, that is, the question of \[3\] (Problem 1.33) has a negative answer. Indeed, the class of representable BL-algebras is a variety (see Theorem 3.4 of [13]) but the class of semisimple BL-algebras is not a variety.

4. Semilocal pseudo BL-algebras

**Definition 4.1.** A pseudo BL-algebra is called semilocal if it has only finitely many normal ultrafilters.

**Theorem 4.2.** Let $A$ be a pseudo BL-algebra. The following are equivalent:

(a) $A$ is semilocal;
(b) $A/\mathcal{M}_n(A)$ is isomorphic to a direct product of finitely many simple pseudo BL-chains;
(c) $A/\mathcal{M}_n(A)$ has finitely many filters.

**Proof.** For now on throughout our proof, we will let $U$ stand for $\mathcal{M}_n(A)$.

(a) $\Rightarrow$ (b): Assume that $A$ is semilocal. If $\mathcal{M}_n(A) = \emptyset$, then $A/U = A/A$ is a one-element pseudo BL-algebra and so it is the direct product of empty family of algebras. Now, let $\{H_1, \ldots, H_k\}$ be the set of all normal ultrafilters of $A$. Then $U = H_1 \cap \cdots \cap H_k$. By Proposition 2.22, $A/H_i$ are simple pseudo BL-chains. We define the map $\phi : A/U \to A/H_1 \times \cdots \times A/H_k$ by $\phi(x/U) =$
(x/H_1, \ldots, x/H_k). Then \( \varphi \) is clearly a homomorphism. We show that \( \varphi \) is an isomorphism. Let \((x_1/H_1, \ldots, x_k/H_k) \in A/H_1 \times \cdots \times A/H_k. \) Since \([H_i \cup H_j] = A\) for \(i, j = 1, \ldots, k\) and \(i \neq j\), we have by Proposition 2.19, that there exists \(x \in A\) such that \(x/H_i = x_i/H_i\) for \(i = 1, \ldots, k\). Hence \((x_1/H_1, \ldots, x_k/H_k) = (x/H_1, \ldots, x/H_k) = \varphi(x/U)\). Consequently, \(\varphi\) is surjective. Now, it suffices to show that \(\varphi\) is injective. Suppose that \(\varphi(x/U) = \varphi(y/U)\) for \(x, y \in A\). Hence \(x/H_i = y/H_i\) for each \(i = 1, \ldots, k\). Then \(x \rightarrow y \in H_i\) and \(y \rightarrow x \in H_i\) for \(i = 1, \ldots, k\), that is, \(x \rightarrow y \in U\) and \(y \rightarrow x \in U\). Therefore, \(x/U = y/U\). It is proved that \(\varphi\) is an isomorphism.

(b) \(\Rightarrow\) (c): Let \(A/U \cong A_1 \times \cdots \times A_k\), where \(A_i\) are simple pseudo BL-chains for \(i = 1, \ldots, k\). Proposition 2.20 gives \(|\text{Fil}(A/U)| = |\text{Fil}(A_1) \times \cdots \times \text{Fil}(A_k)|\). Since \(\text{Fil}(A_i)\) has two elements for every \(i = 1, \ldots, k\), we have that \(|\text{Fil}(A/U)| = 2^k\). Thus \(A/U\) has finitely many filters.

(c) \(\Rightarrow\) (a): To obtain a contradiction, suppose that \(A\) has infinitely many normal ultrafilters \(F_n, n \in \mathbb{N}\). Obviously, all \(F_n/U\) are filters of \(A/U\). Observe that

\[(3) \quad F/U = F'/U \Rightarrow F = F'
\]

for all \(F, F' \in \text{Max}_n(A)\). Let \(F/U = F'/U\) and let \(x \in F\). Then \(x/U \in F'/U\) and hence \(x/U = y/U\) for some \(y \in F'\). By (1), \(y \rightarrow x \in U \subseteq F'\). Consequently, \(x \land y = (y \rightarrow x) \lor y \in F'\). Therefore, \(x \in F'\). This clearly forces \(F \subseteq F'\). Similarly, \(F' \subseteq F\), and we obtain \(F = F'\). Thus (3) holds. From (3) it follows that \(A/U\) has infinitely many filters \(F_n/U, n \in \mathbb{N}\), which is impossible. \(\blacksquare\)

**Definition 4.3.** Let \(\{a_i : i \in I\}\) be a family of elements of a pseudo BL-algebra \(A\) and \(\{H_i : i \in I\}\) be a family of normal filters of \(A\). We say that the family \(\{(a_i, H_i) : i \in I\}\) has a property \((P)\) if for any finite subset \(J\) of \(I\), there is \(x_J \in A\) with \(x_J \equiv_{H_i} a_i\) for any \(i \in J\).

**Definition 4.4.** \(A\) is called maximal if for any family \(\{(a_i, H_i) : i \in I\}\) with property \((P)\) there exists \(x \in A\) such that \(x \equiv_{H_i} a_i\) for any \(i \in I\).

**Remark 4.5.** If \(A\) has finitely many normal filters, then \(A\) is maximal. Hence any simple pseudo BL-algebra is maximal.

**Lemma 4.6.** A finite direct product of maximal pseudo BL-algebras is a maximal pseudo BL-algebra.

**Proof.** We only need to prove that if \(A_1\) and \(A_2\) are maximal, then \(A = A_1 \times A_2\) is also maximal. By Proposition 2.25, \(A_1 \cong [e]\) and \(A_2 \cong [e^-]\) with \(e \in B(A)\). Let \(H\) be a normal filter of \(A\). From Proposition 2.24 we conclude that \([e]\) is
Since we will show that \( H \cap [e] \in \text{Fil}_n(A) \). Let \( x, y \in A \) and \( x \equiv_H y \). We show that \( x \vee e \equiv_{H \cap [e]} y \vee e \). Since \( x \equiv_H y \), we have \( x \to y, y \to x \in H \).

It suffices to prove that \( x \vee e \to y \vee e, y \vee e \to x \vee e \in (H \cap [e]) \). By Proposition 2.2 (b), \( x \vee e \to y \vee e, y \vee e \to x \vee e \in [e] \), i.e., \( x \vee e \to y \vee e \in [e] \). From Proposition 2.2 (e) we obtain \( x \vee e \to y \vee e \geq x \to y \in H \). Therefore, \( x \vee e \to y \vee e \in H \).

So \( x \vee e \to y \vee e \in (H \cap [e]) \) and similarly, \( y \vee e \to x \vee e \in (H \cap [e]) \). Thus \( x \vee e \equiv_{H \cap [e]} y \vee e \). Likewise, we can prove that \( x \vee e \equiv_{H \cap [e]} y \vee e \).

Now let \( \{(a_i, H_i) : i \in I\} \) be a family in \( A \) with the property (P). Then the families \( \{(a_i \vee e, H_i \cap [e]) : i \in I\} \) and \( \{(a_i \vee e, H_i \cap [e]) : i \in I\} \) verify the property (P) in maximal pseudo BL-algebras \([e]\) and \([e]^{-}\) respectively. Let \( y \in [e] \) and \( z \in [e]^{-} \) such that \( y \equiv_{H_i \cap [e]} a_i \vee e \) and \( z \equiv_{H_i \cap [e]} a_i \vee e^{-} \) for any \( i \in I \). Hence \( y \circ z \equiv_{F_i} (a_i \vee e) \circ (a_i \vee e^{-}) \), and from Proposition 2.26 we conclude that \( y \circ z \equiv_{H_i} a_i \).

**Theorem 4.7.** If \( A \) is a maximal pseudo BL-algebra, then it is semilocal.

**Proof.** Let \( G = \{(x_H, H) : x_H \in A, H \in \text{Max}_n(A)\} \). Observe that the family \( G \) has the property (P). Indeed, let \( \{H_1, \ldots, H_m\} \subseteq \text{Max}_n(A) \). Since \( [H_i \cup H_j] = A \) for \( i \neq j \), we conclude from Proposition 2.19 that there exists \( x^* \in A \) such that \( x^* \equiv_{H_i} x_H \) for \( i = 1, \ldots, m \). Thus \( G \) satisfies (P).

Let \( F = \{x \in A : \{H \in \text{Max}_n(A) : x \notin H\} \text{ is finite}\} \). It is easily seen that \( F \) is a normal filter of \( A \). Let us consider the family

\[
\mathcal{H} = \{(1, F) \cup (0, H) : H \in \text{Max}_n(A)\}.
\]

We will show that \( \mathcal{H} \) has the property (P). Take a subfamily

\[
\{(1, F), (0, H_1), \ldots, (0, H_m)\}
\]

of \( \mathcal{H} \). It is obvious that

\[
\bigcap\{H : H \in \text{Max}_n(A) - \{H_1, \ldots, H_m\}\} \subseteq F.
\]

Since \( G \) satisfies (P), the family

\[
\{(0, H_1), \ldots, (0, H_m)\} \cup \{(1, H) : H \in \text{Max}_n(A) - \{H_1, \ldots, H_m\}\}
\]

also satisfies (P). By assumption, \( A \) is maximal, and hence there is \( x \in A \) such that \( x/H_i = 0/H_i \) for all \( i = 1, \ldots, m \) and \( x/H = 1/H \) for all \( H \in \text{Max}_n(A) - \{H_1, \ldots, H_m\} \). Proposition 2.16 shows that \( x \in H \) for all \( H \neq H_1, \ldots, H_m \).

We conclude from (4) that \( x \in F \), what implies that \( x/F = 1/F \). Therefore, \( \mathcal{H} \) has the property (P).

By hypothesis, there exists \( y \in A \) such that \( y/F = 1/F \) and \( y/H = 0/H \) for all \( H \in \text{Max}_n(A) \). From this we deduce that \( y \in F \) and \( y^- \in H \) for
any $H \in \text{Max}_n(A)$. Applying Proposition 2.3 (e) we see that $y \notin H$ for all $H \in \text{Max}_n(A)$. It follows that $\text{Max}_n(A) = \{H \in \text{Max}_n(A) : y \notin H\}$. Since $y \in F$, we conclude that $\text{Max}_n(A)$ is finite. Hence $A$ is semilocal. ■

**Theorem 4.8.** For a pseudo BL-algebra $A$, the following are equivalent:
(a) $A$ is semisimple and maximal;
(b) $A$ is semisimple and semilocal;
(c) $A$ is isomorphic to a direct product of finitely many simple pseudo BL-chains;
(d) $|\text{Max}_n(A)| < \aleph_0$ and $\mathcal{M}_n(A) = \{1\}$.

**Proof.** (a) $\Rightarrow$ (b): Let $A$ be semisimple and maximal. By Theorem 4.7 we have (b).
(b) $\Rightarrow$ (c): Follows from Theorem 4.2.
(c) $\Rightarrow$ (d): Let $A \cong B = A_1 \times \cdots \times A_k$, where $A_i$ are simple pseudo BL-chains. It is clear that $F$ is an ultrafilter of $B$ if and only if there is an $i \in \{1, \ldots, k\}$ such that $F = A_1 \times \cdots \times A_{i-1} \times F_i \times A_{i+1} \times \cdots \times A_k$, where $F_i = \{1\}$ is the unique ultrafilter of $A_i$. Hence (d) holds.
(d) $\Rightarrow$ (a): By definition, $A$ is semisimple. From Remark 4.5 we see that $A$ is maximal. ■

From Theorem 4.8 we have

**Corollary 4.9.** Let $A$ be a semisimple pseudo BL-algebra. Then $A$ is maximal if and only if $A$ is semilocal.

**References**


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