A Fixed Point Theorem for Multi-valued Weakly Picard Operators in $b$–Metric Space

M. O. Olatinwo*
Department of Mathematics
Obafemi Awolowo University, Ile-Ife, Nigeria

Abstract

In this paper, we establish a fixed point theorem for multi-valued operators in a complete $b$–metric space using the concept of Berinde and Berinde [9] on multi-valued weak contractions for the Picard iteration in a metric space. Our main result generalizes, extends and improves some of the recent results of Berinde and Berinde [9] as well as those of Daffer and Kaneko [17] and also unifies several classical results pertaining to single and multi-valued contractive mappings in the literature.

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1. Introduction

The notion of the $b$–metric space will be introduced in the sequel. Presently, let $(X, d)$ be a complete metric space and $CB(X)$ denote the family of all nonempty closed and bounded subsets of $X$. For $A, B \subseteq X$, define the distance between $A$ and $B$ by $D(A, B) = \inf \{d(a, b) \mid a \in A, \ b \in B\}$, the diameter of $A$ and $B$ by $\delta(A, B) = \sup \{d(a, b) \mid a \in A, \ b \in B\}$, and the Hausdorff-Pompeiu metric on $CB(X)$ by $H(A, B) = \max \{\sup \{d(a, B) \mid a \in A\}, \ sup \{d(b, A) \mid b \in B\}\}$.

$H(A, B)$ is induced by $d$.

Let $P(X)$ be the family of all nonempty subsets of $X$ and $T : X \to P(X)$ a multi-valued mapping. Then, an element $x \in X$ such that $x \in T(x)$ is called a fixed point.

*e-mail: polatinwo@oauife.edu.ng or molaposi@yahoo.com
of $T$. Denote the set of all the fixed points of $T$ by $\text{Fix}(T)$, that is, $\text{Fix}(T) = \{x \in X \mid x \in T(x)\}$.

Markins [27] and Nadler [29] initiated the study of fixed point theorems for multi-valued operators. The celebrated Banach’s fixed point theorem is extended to the following result of Nadler [29] from the single-valued maps to the multi-valued contractive maps.

**Theorem 1.1 (Nadler [29]):** Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ a set-valued $\alpha$-contraction, that is, a mapping for which there exists a constant $\alpha \in (0, 1)$, such that

$$H(Tx, Ty) \leq \alpha d(x, y), \forall x, y \in X.$$  \hfill (1)

Then $T$ has at least one fixed point.

**Example 1.2:** Let $X = [0, 1] \subset \mathbb{R}$ with the usual metric. Define $g(x) : X \to X$ by

$$g(x) = \begin{cases} 
\frac{1}{3}x + \frac{5}{6}, & x \in [0, \frac{1}{4}) \\
-\frac{1}{3}x + 1, & x \in [\frac{1}{4}, 1]
\end{cases}$$

Define $F : X \to 2^X$ by $F(x) = \{0\} \cup \{g(x)\} \forall x \in X$. Then, $F$ is a multi-valued contraction operator and the fixed point set of $F$ is $\{0, \frac{3}{4}\}$.

For the Banach’s fixed point theorem and its various generalizations in single-valued case, we refer to Agarwal et al [1], Banach [2], Berinde [3, 4, 5, 6, 7] and some other references in the reference section of this paper. For the Banach’s fixed point theorem and its various generalizations in single-valued case, we refer to Agarwal et al [1], Banach [2], Berinde [3, 4, 5, 6, 7] and some other references in the reference section of this paper.

Apart from Markins [27] and Nadler [29], several other papers have been devoted to the treatment of multi-valued operators and these include Berinde and Berinde [9], Cirić [14], Cirić and Ume [15, 16], Daffer and Kaneko [17], Itoh [20], Kaneko [22, 23], Kubiaczyk and Ali [25], Lim [26], Mizoguchi [28] and some others in the reference section.

In Berinde and Berinde [9], the following contractive condition was employed:

**Definition 1.3:** Let $(X, d)$ be a metric space and $T : X \to P(X)$ a multi-valued operator. $T$ is said to be a multi-valued weak contraction or a multi-valued $(\theta, L)$-contraction if and only if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + LD(y, Tx), \forall x, y \in X.$$  \hfill (2)

The following notion of $b$–metric space shall be employed in the sequel.

**Definition 1.4 (Czerwik [12, 13]):** Let $X$ be a (nonempty) set and $s \geq 1$ a real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a $b$–metric if $\forall x, y, z \in X$,

(i) $d(x, y) = 0$ iff $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair $(X, d)$ is called a $b$–metric space.

In fact, the class of $b$–metric spaces is effectively larger than that of metric spaces, since a $b$–metric is a metric when $s = 1$. 

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Definition 1.5 (Berinde and Berinde [9]): Let \((X, d)\) be a metric space and \(T : X \to P(X)\) a multi-valued operator. \(T\) is said to be a \textit{multi-valued weakly Picard (MWP) Operator} if and only if for each \(x \in X\) and any \(y \in T(x)\), there exists a sequence \(\{x_n\}_{n=0}^{\infty}\) such that

(i) \(x_0 = x, \; x_1 = y\);
(ii) \(x_{n+1} \in T(x_n)\) for all \(n = 0, 1, \ldots\);
(iii) the sequence \(\{x_n\}_{n=0}^{\infty}\) is convergent and its limit is a fixed point of \(T\).

Remark 1.6: A sequence \(\{x_n\}_{n=0}^{\infty}\) satisfying conditions (i) and (ii) in Definition 1.4 will be called a \textit{sequence of successive approximations} of \(T\), starting from \((x, y)\) or a \textit{Picard iteration} associated to \(T\) or a (Picard) orbit of \(T\) at the initial point \(x_0\).

Examples 1.7 (MWP Operators): Several examples including Examples 1.7 (a) and (b) are contained in Rus et al [39]:

(a) (Nadler [29]): Let \((X, d)\) be a complete metric space and \(T : X \to CB(X)\) a multi-valued \(\alpha\)-contraction \((0 < \alpha < 1)\). Then \(T\) is a MWP operator.

(b) (Rus [37]): Let \((X, d)\) be a complete metric space and \(T : X \to CB(X)\) a multi-valued operator for which there exist \(\alpha, \beta \in IR^+\), \(\alpha + \beta < 1\) such that

(i) \(H(Tx, Ty) \leq \alpha d(x, y) + \beta D(y, Tx)\), \(\forall \; x \in X\) and \(\forall \; y \in Tx\);
(ii) \(T\) is a closed multi-valued operator.

Then \(T\) is a MWP operator.

(c) (Berinde and Berinde [9]): Let \((X, d)\) be a complete metric space and \(T : X \to CB(X)\) a multi-valued operator for which there exist two constants \(\theta \in (0, 1)\) and \(L \geq 0\) such that

\[H(Tx, Ty) \leq \theta d(x, y) + LD(y, Tx), \; \forall \; x, \; y \in X.\]

Then \(T\) is a MWP operator.

(d) (Berinde and Berinde [9]): Let \((X, d)\) be a complete metric space and \(T : X \to CB(X)\) a multi-valued operator for which there exist a constant \(L \geq 0\) and a function \(\alpha : [0, \infty) \to [0, 1)\) satisfying

\[\lim_{r \to t^+} \alpha(r) < 1, \; \text{for every} \; t \in [0, \infty),\]

such that

\[H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + LD(y, Tx), \; \forall \; x, \; y \in X.\]

Then \(T\) is a MWP operator.

A more general class of MWP operators will be presented as our main result in this paper.

In this paper, we obtain a more general result than one of the results of Berinde and Berinde [9] using the following general contractive definition:

Definition 1.8: Let \((X, d)\) be a \(b\)–metric space and \(T : X \to P(X)\) a multi-valued operator. Then, \(T\) will be called a \textit{multi-valued (\(\theta_n, \phi\))–weak contraction} if and only if there exist a sequence \(\{\theta_n\}_{n=0}^{\infty} \subset (0, 1)\) and a continuous monotone increasing function \(\phi : IR^+ \to IR^+\) with \(\phi(0) = 0\) such that

\[H(Tx, Ty) \leq \theta_n d(x, y) + \phi(D(y, Tx)), \; \forall \; x, \; y \in X, \; n = 0, 1, 2, \ldots \] \((*)\)
Remark 1.9: If in condition $(\star)$, $\theta_n = \theta$, $0 < \theta < 1$ and $\phi(u) = Lu$, $L \geq 0$, $\forall u \in \mathbb{R}^+$, then we obtain the $(\delta, L)$–weak contraction condition in the multi-valued setting employed by Berinde and Berinde [9] defined in (2). The condition $(\star)$ is also a generalization and extension of several others in the literature.

However, we shall require the following Lemma in the sequel.

Lemma 1.10: Let $(X, d)$ be a metric space. Let $A, B \subset X$ and $q > 1$. Then, for every $a \in A$, there exists $b \in B$ such that

$$d(a, b) \leq qH(A, B).$$

Lemma 1.10 is contained in Berinde and Berinde [9], Ciric [14] and Rus [35] in a metric space setting.

2. Main Result

The following main result shows that any multi-valued weak contraction is a MWP operator.

Theorem 2.1: Let $(X, d)$ be a complete $b$–metric space with continuous $b$–metric and $T : X \rightarrow CB(X)$ multi-valued $(\theta_n, \phi)$–weak contraction. Suppose that $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous monotone increasing function such that $\phi(0) = 0$. Then,

(i) Fix $(T) \neq \phi$;

(ii) for any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of $T$ at the point $x_0$ that converges to a fixed point $x^*$ of $T$;

(iii) the a priori and the a posteriori error estimates are respectively given by

$$d(x_n, x^*) \leq sM_1 d(x_0, x_1), \quad s \geq 1, \quad n = 1, 2, \cdots,$$

and

$$d(x_n, x^*) \leq sM_2 d(x_{n-1}, x_n), \quad s \geq 1, \quad n = 1, 2, \cdots,$$

where $M_1 = \sum_{j=0}^{\infty} \prod_{k=0}^{n+j-1} h_k$;

and

$$M_2 = \sum_{j=0}^{\infty} \prod_{k=0}^{n+j-1} h_k,$$

for a certain sequence $\{h_n\}_{n=0}^{\infty} \subset (0, 1)$.

Proof: Let $q > 1$ and $h_n = q\theta_n \in (0, 1)$, $n = 0, 1, 2, \cdots$. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $H(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$.

Let $H(Tx_0, Tx_1) \neq 0$. Then, we have by Lemma 1.10 that there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq qH(Tx_0, Tx_1),$$

so that by $(\star)$ we have

$$d(x_1, x_2) \leq q[\theta_0 d(x_0, x_1) + \phi(D(x_1, Tx_0))] = q\theta_0 d(x_0, x_1) = h_0 d(x_0, x_1),$$
where we take $h_0 = q\theta_0 < 1$. If $H(Tx_1, Tx_2) = 0$, then $Tx_1 = Tx_2$, that is, $x_2 \in Tx_2$. Let $H(Tx_1, Tx_2) \neq 0$. Again, by Lemma 1.10, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq qH(Tx_1, Tx_2),$$

$$\leq q[\theta d(x_1, x_2) + \phi(D(x_2, Tx_1))]$$

$$= q\theta_1 d(x_1, x_2) = h_1 d(x_1, x_2) \leq h_0 h_1 d(x_0, x_1) \quad (7)$$

By induction, we obtain

$$d(x_n, x_{n+1}) \leq \Pi_{k=0}^{n-1} h_k d(x_0, x_1). \quad (8)$$

Therefore, we have by (8) and the property (iii) of the Definition 1.4 that

$$d(x_n, x_{n+p}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})]$$

$$\leq s[\Pi_{k=0}^{p-1} h_k + \Pi_{k=0}^{n} h_k + \cdots + \Pi_{k=0}^{n+p-2} h_k] d(x_0, x_1) \quad (9)$$

$$= s \left( \sum_{j=n-1}^{n+p-2} \Pi_{k=0}^{j} h_k \right) d(x_0, x_1). \quad (10)$$

From (10), we have

$$d(x_n, x_{n+p}) \leq s \left( \sum_{j=n-1}^{n+p-2} \Pi_{k=0}^{j} h_k \right) d(x_0, x_1)$$

$$= s \left[ \sum_{j=0}^{n-2} \Pi_{k=0}^{j} h_k + \sum_{j=0}^{n-2} \Pi_{k=0}^{j} h_k \right] d(x_0, x_1) \quad \to 0 \text{ as } n \to \infty. \quad (11)$$

We therefore have from (11), that for any $x_0 \in X$, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $X$. Since $(X, d)$ is a complete $b$–metric space, then $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in X$. That is,

$$\lim_{n \to \infty} x_n = x^*. \quad (12)$$

Therefore, by (*), we have that

$$D(x^*, Tx^*) \leq s[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)]$$

$$\leq s[d(x^*, x_{n+1}) + H(Tx_n, Tx^*)]$$

$$\leq s d(x^*, x_{n+1}) + s[\theta d(x_n, x^*) + \phi(D(x^*, Tx_n))] \quad (13)$$

By using (12), the continuity of the function $\phi$ and the fact that $x_{n+1} \in Tx_n$, then $\phi(D(x^*, Tx_n)) \to 0$ as $n \to \infty$ and also $d(x_n, x^*) \to 0$ as $n \to \infty$. It follows from (13) that, as $n \to \infty$, $D(x^*, Tx^*) = 0$.

Since $Tx^*$ is closed, then $x^* \in Tx^*$.

To prove the a priori error estimate in (5), we have from (10) that

$$d(x_{n+p}, x_n) \leq s \left( \sum_{j=0}^{p-1} \Pi_{k=0}^{n+j-1} h_k \right) d(x_0, x_1),$$

from which it follows by the continuity of the $b$–metric that

$$d(x_n, x^*) = d(x^*, x_n) = \lim_{p \to \infty} d(x_{n+p}, x_n) \leq s \left( \sum_{j=0}^{\infty} \Pi_{k=0}^{n+j-1} h_k \right) d(x_0, x_1),$$
giving the result in (5).

We now prove the a posteriori estimate in (6): Let \( q_\theta = h_n \in (0, 1) \), \( n = 0, 1, \cdots \), we get by condition (\( \star \)) and Lemma 1.10 that

\[
d(x_n, x_{n+1}) \leq qH(Tx_{n-1}, Tx_n) \leq q[\theta_{n-1}d(x_{n-1}, x_n) + \phi(D(x_n, Tx_{n-1}))] = q\theta_{n-1}d(x_{n-1}, x_n) = h_{n-1}d(x_{n-1}, x_n).
\]

Also, we have

\[
d(x_{n+1}, x_{n+2}) \leq h_n d(x_n, x_{n+1}) \leq h_n h_{n-1}d(x_{n-1}, x_n),
\]

so that in general, we obtain

\[
d(x_{n+j}, x_{n+j+1}) \leq \Pi_{k=n-1}^{n+j-1} h_k d(x_{n-1}, x_n), \quad j = 0, 1, \cdots.
\]

Using (14) in (9) yields

\[
d(x_n, x_{n+p}) \leq s \left( \sum_{j=n-1}^{n+p-2} \Pi_{k=n-1}^{j} h_k \right) d(x_{n-1}, x_n)
= s \left( \sum_{j=0}^{p-1} \Pi_{k=n-1}^{n+j-1} h_k \right) d(x_{n-1}, x_n).
\]

(15)

Again, by taking limits in (15) as \( p \to \infty \) and using the continuity of the \( b \)-metric, we have

\[
d(x_n, x^*) = d(x^*, x_n) = \lim_{p \to \infty} d(x_{n+p}, x_n) \leq s \left( \sum_{j=0}^{\infty} \Pi_{k=n-1}^{n+j-1} h_k \right) d(x_{n-1}, x_n),
\]

giving the required a posteriori error estimate.

**Remark 2.2:** Theorem 2.1 is a generalization and extension of Theorem 3 of Berinde and Berinde [9]. It is also a generalization and extension of Theorem 1.1 (which is Theorem 5 of Nadler [29]). Indeed, Theorem 2.1 is a generalization and extension of a multitude of results in the literature pertaining to the single-valued and multi-valued cases. In particular, the error estimates of Theorem 2.1 indeed extend those of Berinde [8].

**References**


