Certain Class of p-Valent Functions
Associated with the Wright Generalized
Hypergeometric Function

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Abstract
Using the Wright’s generalized hypergeometric function, we introduce
a new class \( W_k (p, q, s; A, B, \lambda) \) of analytic p-valent functions with neg-
avative coefficients. In this paper we investigate coefficients estimates,
distortion theorem, the radii of p-valent starlikeness and p-valent
convexity and modified Hadamard products.

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1 . Introduction
Let \( A_p (p, k) \) denote the class of functions of the form:

\[
f (z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k \quad ; \quad p, k \in N = \{1, 2, 3, \ldots\}),
\]
which are analytic and p-valent in \( U = U (1) \), where \( U (r) = \{ z : z \in C \text{ and } |z| < r \} \). Also let us put \( A_p (p) = A (p, p + 1) \) and \( A = A (1) \). Let the
functions $f(z)$ and $g(z)$ be analytic in $U$. Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$, such that $f(z) = g(w(z)) (z \in U)$. We denote this subordination by $f(z) \prec g(z)$.

A function $f(z)$ belonging to the class $A(p)$ is said to be $p$-valent starlike of order $\alpha$ in $U$ if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \ (z \in U \ (r); 0 < r \leq 1; 0 \leq \alpha < p). \quad (1.2)$$

Also a function $f(z)$ belonging to the class $A(p)$ is said to be $p$-valent convex of order $\alpha$ in $U$ if and only if

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \ (z \in U \ (r); 0 < r \leq 1; 0 \leq \alpha < p). \quad (1.3)$$

We denote by $S_p^*(\alpha)$ the class of all functions in $A(p)$ which are $p$-valent starlike of order $\alpha$ in $U$ and by $S_p^c(\alpha)$ the class of all functions in $A(p)$ which are $p$-valent convex of order $\alpha$ in $U$. We also set $S_p^* = S_p^* (0)$, $S^* (\alpha) = S_1^* (\alpha)$, $S_p^c = S_p^c (0)$ and $C (\alpha) = S_1^c (\alpha)$.

Let $G$ be a subclass of the class $A$. We define the radius of starlikeness $R^*(G)$ and the radius of convexity $R^c (G)$ for the class $G$ by

$$R^*(G) = \inf_{f \in G} (\sup \{ r \in (0, 1] : f \text{ is starlike in } U \ (r) \}),$$

and

$$R^c (G) = \inf_{f \in G} (\sup \{ r \in (0, 1] : f \text{ is convex in } U \ (r) \}),$$

respectively.

For analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} a_n z^n$, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s \ (q, s \in N)$ be positive and real parameters such that
\[ 1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geq 0. \]

The Wright generalized hypergeometric function [27] (see also [11])

\[ q\Psi_s \left[ (\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s); z \right] = q\Psi_s \left[ (\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z \right] \]

is defined by

\[ q\Psi_s \left[ (\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z \right] = \sum_{n=0}^{\infty} \prod_{i=1}^{q} \frac{\Gamma (\alpha_i + nA_i)}{\prod_{i=1}^{s} \Gamma (\beta_i + nB_i)} \frac{z^n}{n!} \quad (z \in U). \]

If \( A_i = 1 (i = 1, \ldots, q) \) and \( B_i = 1 (i = 1, \ldots, s) \), we have the relationship:

\[ \Omega_q\Psi_s \left[ (\alpha_i, 1)_{1,q}; (\beta_i, 1)_{1,s}; z \right] = qF_s \left( \alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z \right), \]

where \( qF_s \left( \alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z \right) \) is the generalized hypergeometric function (see for details [7], [8], [9], [10], [15] and)

\[ \Omega = \prod_{i=1}^{q} \frac{\Gamma (\beta_i)}{\prod_{i=1}^{s} \Gamma (\alpha_i)}. \] \hspace{1cm} (1.4)

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [6], [7], [8], [21], [22] and [23]).


First we define a function \( q\Phi_s^p \left[ (\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z \right] \) by

\[ q\Phi_s^p \left[ (\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z \right] = \Omega z^p q\Psi_s \left[ (\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z \right] \]

and consider the following linear operator
We observe that, for a function defined by the convolution
\[ \theta_{p,q,s} \left[ (\alpha_i, A_i)_{1,q} ; (\beta_i, B_i)_{1,s} \right] : A(p,k) \to A(p,k), \]
defined by the convolution
\[ \theta_{p,q,s} \left[ (\alpha_i, A_i)_{1,q} ; (\beta_i, B_i)_{1,s} \right] f(z) = q \Phi_s \left[ (\alpha_i, A_i)_{1,q} ; (\beta_i, B_i)_{1,s} ; z \right] * f(z). \]

We observe that, for a function \( f(z) \) of the form (1.1), we have
\[ \theta_{p,q,s} \left[ (\alpha_i, A_i)_{1,q} ; (\beta_i, B_i)_{1,s} \right] f(z) = z^p + \sum_{n=k}^{\infty} \Omega \sigma_{n,p} (\alpha_1) a_n z^n, \tag{1.5} \]
where \( \Omega \) is given by (1.4) and \( \sigma_{n,p}(\alpha_1) \) is defined by
\[ \sigma_{n,p}(\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(n-p)) \cdots \Gamma(\alpha_q + A_q(n-p))}{\Gamma(\beta_1 + B_1(n-p)) \cdots \Gamma(\beta_s + B_s(n-p))(n-p)!}. \tag{1.6} \]

If, for convenience, we write
\[ \theta_{p,q,s} [\alpha_1, A_1, B_1] f(z) = \theta_{p,q,s} [(\alpha_1, A_1), \ldots, (\alpha_q, A_q) ; (\beta_1, B_1), \ldots, (\beta_s, B_s)] f(z), \]
then one can easily verify from the definition (1.5) that
\[ z A_1 (\theta_{p,q,s} [\alpha_1, A_1, B_1] f(z))' = \alpha_1 \theta_{p,q,s} [\alpha_1 + 1, A_1, B_1] f(z) \]
\[ - (\alpha_1 - p A_1) \theta_{p,q,s} [\alpha_1, A_1, B_1] f(z). \tag{1.7} \]

We note that for \( A_i = 1 \) (\( i = 1, 2, \ldots, q \)) and \( B_i = 1 \) (\( i = 1, 2, \ldots, s \)), we obtain \( \theta_{p,q,s} [\alpha_1, 1, 1] f(z) = H_{p,q,s}[\alpha_1] f(z) \), which was introduced and studied by Dziok and Srivastava [8]. Also for \( f(z) \in A \), the linear operator \( \theta_{1,q,s} [\alpha_1, A_1, B_1] = \theta [\alpha_1] \) was introduced by Dziok and Riana [6] and studied by Aouf and Dziok [2].

We note that, for \( f(z) \in A(p) \), \( A_i = 1 \) (\( i = 1, \ldots, q \)), \( B_i = 1 \) (\( i = 1, \ldots, s \)), \( q = 2 \) and \( s = 1 \), we have:

(i) \( \theta_{p,2,1} [a, 1; c] f(z) = L_p(a,c) f(z) \) (\( a > 0; c > 0 \)) (see Saitoh [24]);
(ii) \( \theta_{p,2,1} [\mu + p, 1; 1] f(z) = D^{\mu+p-1} f(z) \) (\( \mu > -p \)), where \( D^{\mu+p-1} f(z) \) is the \( (\mu + p - 1) \)–the order Ruscheweyh derivative of a function \( f(z) \in A(p) \) (see Kumer and Shukla [12] and [13]);
(iii) \( \theta_{p,2,1} [1 + p, 1; 1 + p - \mu] f(z) = \Omega_z^{(\mu,p)} f(z) \), where the operator \( \Omega_z^{(\mu,p)} \)
is defined by (see Srivastava and Aouf [26])

\[
\Omega_z^{(\mu,p)} f(z) = \frac{\Gamma(1 + p - \mu)}{\Gamma(1 + p)} z^\mu D_z^p f(z) \quad (0 \leq \mu < 1; p \in \mathbb{N}),
\]
where \( D_z^p \) is the fractional derivative operator (see, for details, [18] and [20]).

(iv) \( \theta_{p,2,1} [\nu + p, 1; \nu + p + 1] f(z) = J_{\nu,p} (f)(z) \), where \( J_{\nu,p} (f)(z) \) is the
generalized Bernardi-Libera-Livingston-integral operator (see [4], [14] and [17]),
deﬁned by

\[
J_{\nu,p} (f)(z) = \left( 1 + \frac{\mu}{\nu + p} \right) \int_0^z t^{\nu - 1} f(t) \, dt \quad (\nu > -p; p \in \mathbb{N});
\]

(v) \( \theta_{p,2,1} [p + 1, 1; n + p] f(z) = I_{n,p} f(z) \) (\( n \in \mathbb{Z}; n > -p \)), where the operator \( I_{n,p} \) was considered by Liu and Noor [16];

(vi) \( \theta_{p,2,1} [\lambda + p, c; a] f(z) = I_p^\lambda (a;c) f(z) \) (\( a, c \in \mathbb{R} | \Re; \lambda > -p \)), where \( I_p^\lambda (a,c) \) is the Cho-Kwon-Srivastava operator [5].

Let us denote by \( V_k (p, q, s; A, B, \lambda) \) the class of functions of the form (1.1) and satisfy the following subordination:

\[
\frac{1}{p - \lambda} \left( \alpha_1 + \frac{\alpha_1 + 1, A_1, B_1}{\alpha_1, A_1, B_1} f(z) + A_1 (p - \lambda) - \alpha_1 \right) < A_1 \frac{1 + Az}{1 + Bz}
\]

\[
(0 \leq B \leq 1; -B \leq A < B; 0 \leq \lambda < p; p \in \mathbb{N}),
\]
or, by using (1.7), if it satisfies the following subordination:

\[
\frac{1}{p - \lambda} \left( \frac{z (\alpha_1 A_1, B_1) f(z) - \alpha_1 A_1 B_1 f(z) - \lambda}{\alpha_1 A_1 B_1 f(z)} \right) < \frac{1 + Az}{1 + Bz}
\]
or, equivalently, if

\[
\left| \frac{z (\alpha_1 A_1, B_1) f(z) - \alpha_1 A_1 B_1 f(z) - \lambda}{\alpha_1 A_1 B_1 f(z)} \right| < 1 \quad (z \in U). \quad (1.8)
\]

Let \( T(p, k) \) denote the subclass of \( A(p, k) \) consisting of functions of the form:
\[ f(z) = z^p - \sum_{n=k}^{\infty} a_n z^n \quad (a_n \geq 0). \]  

Further, we define the class \( W_k(p, q, s; A, B, \lambda) \) by

\[ W_k(p, q, s; A, B, \lambda) = V_k(p, q, s; A, B, \lambda) \cap T(p, k). \]

In particular, for \( q = s + 1 \) and \( \alpha_{s+1} = A_{s+1} = 1 \), we write

\[ W_k(p, s; A, B, \lambda) = W_k(p, s+1; s; A, B, \lambda). \]

We note that:

(i) The class \( W_2(1, q, s; A, B, 0) = W(q, s; A, B) \) was studied by Dziok and Riana [6];

(ii) The class \( W_2(1, q, s; A, B, \lambda) = W(q, s; A, B, \lambda) \) was studied by Aouf and Dziok [2].

Also we note that, if \( A_i = 1 \) \((i = 1, 2, ..., q)\) and \( B_i = 1 \) \((i = 1, ..., s)\), we have:

(i) \( W_k(p, q, s; A, B, 0) = V_k^p(q, s; A, B) \) (Dziok and Srivastava [8]);

(ii) \( W_k(p, q, s; A, B, \lambda) = V_k^p(q, s; A, B, \lambda) \) (Aouf [1]).

(iii) \( W_{p+1}(p, 2, 1; A, B, \lambda) = W_{p+1}(p, a, 1; c; A, B, \lambda) = T_{\lambda,c}(\lambda; p, A, B) \) (Aouf et al. [3]).

Also we note that:

\[ W_k(p, q, s; -\beta, \beta, \lambda) = W_k(p, q, s; \beta, \lambda) \]

\[ \left\{ f: f(z) \in T(p, k) \text{ and } \left| \frac{z^{(\theta_{p,q,a}[\alpha_1,A_1,B_1])f(z)}}{z^{(\theta_{p,q,a}[\alpha_1,A_1,B_1])f(z)}} - p \right| < \beta, \right. \]

\[ \left. (z \in U; 0 \leq \lambda < p; 0 < \beta \leq 1; p \in N \right\}. \]  

\[ \sum_{n=k}^{\infty} \Omega \delta_{n,p} a_n \leq (B - A)(p - \lambda), \]  

2. Coefficient estimates

**Theorem 1.** Let a function \( f(z) \) of the form (1.9) belongs to the class \( A(p, k) \) and let \( \Omega \) and \( \sigma_{n,p} (\alpha_1) \) be defined by (1.4) and (1.6), respectively. If

\[ \sum_{n=k}^{\infty} \Omega \delta_{n,p} a_n \leq (B - A)(p - \lambda), \]  

where
\[ \delta_{n,p} = [(1 + B)(n - p) + (B - A)(p - \lambda)] \sigma_{n,p}(\alpha_1), \tag{2.2} \]

then \( f(z) \in W_k(p, q, s; A, B, \lambda). \)

**Proof.** Let \( z \in U. \) If (1.2) holds, we find from (1.9) that

\[
|z(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' - p\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)| - |Bz(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'|
\]

\[
= \left| - \sum_{n=1}^{\infty} (n - p) \Omega \sigma_{n,p}(\alpha_1) a_n z^n \right| - (B - A)(p - \lambda) z^p
\]

\[
- \sum_{n=1}^{\infty} [B(n - p) + (B - A)(p - \lambda)] \Omega \sigma_{n,p}(\alpha_1) a_n z^n
\]

\[
\leq \sum_{n=1}^{\infty} (n - p) \Omega \sigma_{n,p}(\alpha_1) a_n r^n - \left\{ (B - A)(p - \lambda) r^p
\right\}
\]

\[
- \sum_{n=1}^{\infty} [B(n - p) + (B - A)(p - \lambda)] \Omega \sigma_{n,p}(\alpha_1) a_n r^n
\]

\[
= r^p \left\{ \sum_{n=1}^{\infty} [(1 + B)(n - p) + (B - A)(p - \lambda)] \Omega \sigma_{n,p}(\alpha_1) a_n r^{n-p}
\right\}
\]

\[
- (B - A)(p - \lambda)
\]

\[
< \sum_{n=1}^{\infty} \Omega \sigma_{n,p}(\alpha_1) a_n - (B - A)(p - \lambda) \leq 0.
\]

Thus we have the condition (1.8) and hence, \( f(z) \in W_k(p, q, s; A, B, \lambda). \)

**Theorem 2.** A function \( f(z) \) of the form (1.9) belongs to the class \( W_k(p, q, s; A, B, \lambda) \) if and only if

\[
\sum_{n=1}^{\infty} \Omega \delta_{n,p} a_n \leq (B - A)(p - \lambda), \tag{2.3}
\]
where $\Omega$ and $\delta_{n,p}$ are defined by (1.4) and (2.2), respectively.

**Proof.** By Theorem 1 we have that (2.3) is the sufficient condition for the class $W_k(p, q, s; A, B, \lambda)$. Let now $f(z) \in W_k(p, q, s; A, B, \lambda)$ be given by (1.9). Then, from (1.8) and (1.9), we have

$$
\frac{z^{(\theta_{p,q,s}[1,A_1, B_1])}(z)}{\theta_{p,q,s}[1,A_1, B_1]} - p
$$

$$
B \frac{z^{(\theta_{p,q,s}[1,A_1, B_1])}(z)}{\theta_{p,q,s}[1,A_1, B_1]} - [pB + (A - B)(p - \lambda)]
$$

$$
= \left| \sum_{n=k}^{\infty} (n - p) \Omega \sigma_{n,p} (\alpha) a_n z^{n-p} 
\right| < 1 \quad (z \in U),
$$

(2.4)

where $\Omega$ and $\sigma_{n,p} (\alpha)$ are defined by (1.4) and (1.6), respectively. Putting $z = r \quad (0 \leq r < 1)$, we obtain

$$
\sum_{n=k}^{\infty} (n - p) \Omega \sigma_{n,p} (\alpha) a_n r^{n-p} < (B - A)(p - \lambda)
$$

$$
- \sum_{n=k}^{\infty} [B(n - p) + (B - A)(p - \lambda)] \Omega \sigma_{n,p} (\alpha) a_n r^{n-p},
$$

which, upon letting $r \to 1^-$, readily yields the assertion (2.3). This completes the proof of Theorem 2.

Since the expression $\delta_{n,p}$ defined by (2.2) is a decreasing function with respect to $\beta_r$, $B_r$ ($r = 1, \ldots, s$) and an increasing function with respect to $\alpha_\varrho, A_\varrho$ ($\varrho = 1, \ldots, q$), from Theorem 2, we have:

**Corollary 1.** If $\varrho \in \{1, \ldots, q\}$, $j \in \{1, \ldots, s\}$, $0 \leq \alpha_\varrho \leq \alpha_\varrho^\prime$, $0 < A_\varrho^\prime \leq A_\varrho$ and $0 \leq \beta_j \leq \beta_j^\prime$, $0 < B_j \leq B_j^\prime$, then the class $W_k(p, q, s; A, B, \lambda)$ for the parameters $(\alpha_1, A_1), \ldots, (\alpha_{\varrho-1}, A_{\varrho-1}), (\alpha_\varrho^\prime, A_\varrho^\prime), (\alpha_{\varrho+1}, A_{\varrho+1}), \ldots, (\alpha_q, A_q)$ and $(\beta_1, B_1), \ldots, (\beta_{j-1}, B_{j-1}), (\beta_j^\prime, B_j^\prime), (\beta_{j+1}, B_{j+1}), \ldots, (\beta_s, B_s)$ for the parameters

$$
(\alpha_1, A_1), \ldots, (\alpha_{\varrho-1}, A_{\varrho-1}), (\alpha_\varrho^\prime, A_\varrho^\prime), (\alpha_{\varrho+1}, A_{\varrho+1}), \ldots, (\alpha_q, A_q)
$$

and

$$
(\beta_1, B_1), \ldots, (\beta_{j-1}, B_{j-1}), (\beta_j^\prime, B_j^\prime), (\beta_{j+1}, B_{j+1}), \ldots, (\beta_s, B_s).
$$
From Theorem 2, we also have the following corollary.

**Corollary 2.** If a function \( f(z) \) of the from (1.9) belong to the class \( W_k(p, q, s; A, B, \lambda) \), then

\[
a_n \leq \frac{(B - A) (p - \lambda)}{\Omega \delta_{n,p}} \quad (n \geq k),
\]

where \( \Omega \) and \( \delta_{n,p} \) are defined by (1.4) and (2.2), respectively. The result is sharp, the functions \( f_n(z) \) of the form:

\[
f_n(z) = z^p - \frac{(B - A) (p - \lambda)}{\Omega \delta_{n,p}} z^n \quad (n \geq k)
\]

being the external functions.

Let \( f(z) \) be defined by (1.9) with \( k = p + 1, p \in N \) and for \( A = -1 \) and \( B = 1 \), the condition (2.3) is equivalent to

\[
\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z) \in T^*_p(\lambda) \quad (0 \leq \lambda < p),
\]

where \( T^*_p(\lambda) \) is the class of \( p \)-valent starlike functions of order \( \lambda \) \((0 \leq \lambda < p)\) with negative coefficients, studied by Owa [19]. Thus we have the following lemma:

**Lemma 1.** If \( \alpha_i = \beta_i \) and \( A_i = B_i \) \((i = 1, ..., s)\), then

\[
W_k(p, s; -1, 1, \lambda) \subset T^*_p(\lambda) \quad (0 \leq \lambda < p)
\]

By the definition of the class \( W_k(p, q, s; A, B, \lambda) \), we have the following lemma.

**Lemma 2.** If \( A_1 \leq A_2, B_1 \geq B_2 \) and \( 0 \leq \lambda_1 \leq \lambda_2 < p \), then

\[
W_k(p, q, s; A_1, B_1, \lambda_2) \subset W_k(p, q, s; A_2, B_2, \lambda_1) \subset W_k(p, q, s; -1, 1, 0).
\]

**Remark 1.** Throughout our paper \( \Omega \) and \( \delta_{n,p} \) are defined by (1.4) and (2.2), respectively.

### 3. Distortion theorem

**Theorem 3.** Let a function \( f(z) \) of the form (1.9) belongs to the class \( W_k(p, q, s; A, B, \lambda) \). If the sequence \( \{\delta_{n,p}\} \) \((n \geq k)\) is nondecreasing, then
\[
\frac{r^p - (B - A)(p - \lambda)}{\Omega \delta_{n,p}} r^k \leq |f'(z)| \leq \frac{r^p + (B - A)(p - \lambda)}{\Omega \delta_{n,p}} r^k \quad (|z| = r < 1).
\]

(3.1)

If the sequence \(\left\{ \frac{\delta_{n,p}}{n} \right\} \quad (n \geq k)\) is nondecreasing, then

\[
pr^{p-1} \frac{k (B - A)(p - \lambda)}{\Omega \delta_{k,p}} r^{k-1} \leq |f'(z)| \leq pr^{p-1} + \frac{k (B - A)(p - \lambda)}{\Omega \delta_{k,p}} r^{k-1} \quad (|z| = r < 1).
\]

(3.2)

The result is sharp, with the external function \(f(z)\) given by

\[
f(z) = z^p - \frac{(B - A)(p - \lambda)}{\Omega \delta_{k,p}} z^k.
\]

(3.3)

**Proof.** Let a function \(f(z)\) of the form (1.9) belongs to the class \(W_k(p, q, s; A, B, \lambda)\).

If the sequence \(\left\{ \frac{\delta_{n,p}}{n} \right\} \quad (n \geq k)\) is nondecreasing and positive, by Theorem 2, we have

\[
\sum_{n=k}^{\infty} a_n \leq \frac{(B - A)(p - \lambda)}{\Omega \delta_{k,p}},
\]

(3.4)

and if the sequence \(\left\{ \frac{\delta_{n,p}}{n} \right\} \) is nondecreasing and positive, by Theorem 2, we have

\[
\sum_{n=k}^{\infty} na_n \leq \frac{k(B - A)(p - \lambda)}{\Omega \delta_{k,p}}.
\]

(3.5)

Making use of the conditions (3.4) and (3.5), in conjunction with the definition (1.9), we readily obtain the assertions (3.1) and (3.2) of Theorem 2.

**Corollary 3.** Let a function \(f(z)\) of the form (1.9) belongs to the class \(W_k(p, s; A, B, \lambda)\). If \(\beta_i \leq \alpha_i, B_i \leq A_i \quad (i = 1, ..., s)\), then the assertions (3.1) and (3.2) hold true.

**Proof.** If \(q = s\) and \(\beta_i \leq \alpha_i, B_i \leq A_i \quad (i = 1, ..., s)\), then the sequences \(\{\delta_{n,p}\}\) and \(\left\{ \frac{\delta_{n,p}}{n} \right\} \quad (n \geq k)\) are nondecreasing. Thus, by Theorem 3, we have Corollary 3.
4 . Radii of convexity and starlikeness

Theorem 4. The radius of \( p \)-valently starlike for the class \( W_k (p, q, s; A, B, \lambda) \) is given by

\[
R^* (W_k (p, q, s; A, B, \lambda)) = \inf_{n \geq k} \left[ \frac{p}{n (B - A) (p - \lambda)} \right]^{\frac{1}{n-p}}. \tag{4.1}
\]

The result is sharp.

Proof. It is sufficient to show that

\[
\left| \frac{zf'(z)}{f(z)} - p \right| < p \quad (z \in U(r); 0 < r \leq 1; 0 \leq \alpha < p). \tag{4.2}
\]

Since

\[
\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{\sum_{n=k}^{\infty} (n-p) a_n z^n}{z^p - \sum_{n=k}^{\infty} a_n z^n} \right| \leq \frac{\sum_{n=k}^{\infty} (n-p) a_n |z|^{n-p}}{1 - \sum_{n=k}^{\infty} a_n |z|^{n-p}}.
\]

Putting \(|z| = r\), the condition (4.2) is true if

\[
\sum_{n=k}^{\infty} \frac{n}{p} a_n r^{n-p} \leq 1. \tag{4.3}
\]

By Theorem 2, we have

\[
\sum_{n=k}^{\infty} \frac{\Omega \delta_{n,p}}{(B - A) (p - \lambda)} a_n \leq 1.
\]

Thus the condition (4.3) is true if

\[
\frac{n}{p} r^{n-p} \leq \frac{\Omega \delta_{n,p}}{(B - A) (p - \lambda)} \quad (n \geq k; k > p),
\]

that is, if

\[
r \leq \left[ \frac{p}{n (B - A) (p - \lambda)} \right]^{\frac{1}{n-p}} \quad (n \geq k; k > p).
\]
It follows that any function \( f(z) \in W_k(p, q; A, B, \lambda) \) is \( p \)-valently starlike in the disc \( U(R^*(W_k(p, q; A, B, \lambda))) \), where \( R^*(W_k(p, q; A, B, \lambda)) \) is defined by (4.1).

**Corollary 4.**

\[
R^*(W_k(p, s; A, B, \lambda)) = \begin{cases} 
1 & (\alpha_i \geq \beta_i; A_i \geq B_i; i = 1, \ldots, s) \\
\min_{n \geq k} \left( \frac{p \Omega \delta_{n,p}}{n (B - A)(p - \lambda)} \right)^{\frac{1}{n-p}} & (\alpha_i < \beta_i; A_i < B_i; i = 1, \ldots, s).
\end{cases}
\]

The result is sharp.

**Proof.** From Corollary 1, Lemma 1 and Lemma 2 we have

\[ W_k(p, s; A, B, \lambda) \subset T_p^*(\lambda) \quad (\alpha_i \geq \beta_i; A_i \geq B_i; i = 1, \ldots, s). \]

By Theorem 4, any function \( f(z) \in W_k(p, s; A, B, \lambda) \) is \( p \)-valent starlike in the disc \( U(r) \), where

\[
r = \inf_{n \geq k} \left( d_n \right)^{\frac{1}{n-p}} = \left( \frac{p \Omega \delta_{n,p}}{n (B - A)(p - \lambda)} \right),
\]

Since, for \( \alpha_i < \beta_i; A_i < B_i (i = 1, \ldots, s) \), we have

\[
\lim_{n \to \infty} d_n = d < 1, \lim_{n \to \infty} \left( d_n \right)^{\frac{1}{n-p}} = 1, \text{ and } d_n > 0 (n \geq k),
\]

the infimum of the set \( \left\{ (d_n)^{\frac{1}{n-p}} : n \geq k \right\} \) is realized for an element of this set for some \( n = n_0 \).

Morever, the function

\[
f_{n_0}(z) = z^p - \frac{(B - A)(p - \lambda)}{\Omega \delta_{n_0,p}} z^{n_0},
\]

belongs to the class \( W_k(p, s; A, B, \lambda) \), and for \( z = (d_{n_0})^{\frac{1}{n_0-p}} \), we have

\[
\text{Re} \left\{ \frac{z_0 f'_{n_0}(z)}{f_{n_0}(z)} \right\} = 0.
\]

Thus the result is sharp.

**Theorem 5.** The radius of \( p \)-valently convex for the class \( W_k(p, q; A, B, \lambda) \) is given by
\[ R^c (W_k (p, q, s; A, B, \lambda)) = \inf_{n \geq k} \left( \frac{p^2 \Omega \delta_{n,p}}{n^2 (B - A) (p - \lambda)} \right)^{\frac{1}{n-p}}. \]

The result is sharp.

**Proof.** The proof is analogous to that of Theorem 4, and we omit the details.

## 5. Modified Hadamard products

For the functions
\[ f_j(z) = z^p - \sum_{n=k}^{\infty} a_{n,j} z^n (a_{n,j} \geq 0; j = 1, 2; p < k; p, k \in \mathbb{N}), \quad (5.1) \]
we denote by \((f_1 \otimes f_2)(z)\) the modified Hadamard product or convolution of the functions \(f_1(z)\) and \(f_2(z)\), that is,
\[ (f_1 \otimes f_2)(z) = z^p - \sum_{n=k}^{\infty} a_{n,1} a_{n,2} z^n. \quad (5.2) \]

**Theorem 6.** Let the functions \(f_j(z) (j = 1, 2)\) defined by (5.1) be in the class \(W_k (p, q, s; A, B, \lambda)\). If the sequence \(\{\delta_{n,p}\} (n \geq k)\) is nondecreasing, then \((f_1 \otimes f_2)(z) \in W_k (p, q, s; A, B, \gamma)\), where
\[ \gamma = p - \frac{(1 + B) (B - A) (p - \lambda)^2 (k - p)}{\Omega [(1 + B) (k - p) + (B - A) (p - \lambda)]^2 \sigma_{k,p} (\alpha_1) - (B - A)^2 (p - \lambda)^2}. \quad (5.3) \]

The result is sharp.

**Proof.** Employing the technique used earlier by Schild and Silverman [25], we need to find the largest \(\gamma\) such that
\[ \sum_{n=k}^{\infty} \frac{\Omega [(1 + B) (n - p) + (B - A) (p - \gamma)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \gamma)} a_{n,1} a_{n,2} \leq 1. \quad (5.4) \]

Since \(f_j(z) \in W_k (p, q, s; A, B, \lambda) (j = 1, 2)\), we readily see that
\[ \sum_{n=k}^{\infty} \frac{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \lambda)} a_{n,1} \leq 1 \quad (5.5) \]
\[
\sum_{n=k}^{\infty} \frac{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \lambda)} a_{n,2} \leq 1,
\]  
(5.6)

by the Cauchy-Schwarz inequality, we have

\[
\sum_{n=k}^{\infty} \frac{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \lambda)} \sqrt{a_{n,1} a_{n,2}} \leq 1.
\]  
(5.7)

Thus it is sufficient to show that

\[
\frac{[(1 + B) (n - p) + (B - A) (p - \gamma)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \gamma)} a_{n,1} a_{n,2} \leq
\]

\[
\frac{[(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \lambda)} \sqrt{a_{n,1} a_{n,2}} \quad (n \geq k),
\]  
(5.8)

that is, that

\[
\sqrt{a_{n,1} a_{n,2}} \leq \frac{[(1 + B) (n - p) + (B - A) (p - \gamma)] (p - \gamma)}{[(1 + B) (n - p) + (B - A) (p - \gamma)] (p - \lambda)} \quad (n \geq k).
\]  
(5.9)

Note that

\[
\sqrt{a_{n,1} a_{n,2}} \leq \frac{(B - A) (p - \lambda)}{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)} \quad (n \geq k).
\]  
(5.10)

Consequently, we need only to prove that

\[
\frac{(B - A) (p - \lambda)}{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)} \leq
\]

\[
\frac{[(1 + B) (n - p) + (B - A) (p - \lambda)] (p - \gamma)}{[(1 + B) (n - p) + (B - A) (p - \gamma)] (p - \lambda)} \quad (n \geq k),
\]  
(5.11)
or, equivalently, that

\[
\gamma \leq p - \frac{(1 + B)(B - A)(p - \lambda)^2 (n - p)}{\Omega [(1 + B)(n - p) + (B - A)(p - \lambda)]^2 \sigma_{n,p} (\alpha_1) - (B - A)^2 (p - \lambda)^2} \quad (n \geq k).
\]

Since

\[
\Phi(n) = p - \frac{(1 + B)(B - A)(p - \lambda)^2 (n - p)}{\Omega [(1 + B)(n - p) + (B - A)(p - \lambda)]^2 \sigma_{n,p} (\alpha_1) - (B - A)^2 (p - \lambda)^2},
\]

is an increasing function of \( n \) \((n \geq k)\), letting \( n = k \) in (5.13), we obtain

\[
\gamma \leq \Phi(k) = p - \frac{(1 + B)(B - A)(p - \lambda)^2 (k - p)}{\Omega [(1 + B)(k - p) + (B - A)(p - \lambda)]^2 \sigma_{k,p} (\alpha_1) - (B - A)^2 (p - \lambda)^2},
\]

which proves the main assertion of Theorem 5.

Finally, by taking the functions \( f_j(z) \) \((j = 1, 2)\) given by

\[
f_j(z) = z^p - \frac{(B - A)(p - \lambda)}{\Omega [(1 + B)(k - p) + (B - A)(p - \lambda)]^2 \sigma_{k,p} (\alpha_1)} z^k \quad (j = 1, 2),
\]

we can see that the result is sharp.

**Theorem 7.** Let the functions \( f_j(z) \) \((j = 1, 2)\) defined by (5.1) be in the class \( W_k (p, q, s; A, B, \lambda) \). If the sequence \{\( \delta_{n,p} \)\} \((n \geq k)\) is nondecreasing, then the function

\[
h(z) = z^p - \sum_{n=k}^{\infty} \left( a_{n,1}^2 + a_{n,2}^2 \right) z^n
\]

belongs to the class \( W_k (p, q, s; A, B, \tau) \), where

\[
\tau = p - \frac{2(1 + B)(B - A)(p - \lambda)^2 (k - p)}{\Omega [(1 + B)(k - p) + (B - A)(p - \lambda)]^2 \sigma_{k,p} (\alpha_1) - 2(B - A)^2 (p - \lambda)^2}.
\]

The result is sharp for the functions \( f_j(z) \) \((j = 1, 2)\) defined by (5.15).
**Proof.** By virtue of Theorem 1, we obtain

\[
\sum_{n=k}^{\infty} \left\{ \frac{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \lambda)} \right\}^2 a_{n,1}^2 \leq
\]

\[
\sum_{n=k}^{\infty} \left\{ \frac{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \lambda)} a_{n,1} \right\}^2 \leq 1 \quad (5.18)
\]

and

\[
\sum_{n=k}^{\infty} \left\{ \frac{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \lambda)} a_{n,2} \right\}^2 \leq 1. \quad (5.19)
\]

It follows from (5.18) and (5.19) that

\[
\sum_{n=k}^{\infty} \left\{ \frac{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \lambda)} \right\}^2 \left( a_{n,1}^2 + a_{n,2}^2 \right) \leq 1. \quad (5.20)
\]

Therefore, we need to find the largest \( \tau \) such that

\[
\frac{\Omega [(1 + B) (n - p) + (B - A) (p - \tau)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \tau)} \leq
\]

\[
\frac{1}{2} \left\{ \frac{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)] \sigma_{n,p} (\alpha_1)}{(B - A) (p - \lambda)} \right\}^2 (n \geq k), \quad (5.21)
\]

that is, that

\[
\tau \leq p - \frac{2 (1 + B) (B - A) (p - \lambda)^2 (n - p)}{\Omega [(1 + B) (n - p) + (B - A) (p - \lambda)]^2 \sigma_{n,p} (\alpha_1) - 2 (B - A)^2 (p - \lambda)^2} \quad (n \geq k). \quad (5.22)
\]
Since

\[ D(n) = p - \frac{2(1 + B)(B - A)(p - \lambda)^2(n - p)}{\Omega[(1 + B)(n - p) + (B - A)(p - \lambda)]^2 \sigma_{n,p}(\alpha_1) - 2(B - A)^2(p - \lambda)^2}, \]

is an increasing function of \( n \) \( (n \geq k) \), we readily have

\[ \tau \leq D(k) = p - \frac{2(1 + B)(B - A)(p - \lambda)^2(k - p)}{\Omega[(1 + B)(k - p) + (B - A)(p - \lambda)]^2 \sigma_{k,p}(\alpha_1) - 2(B - A)^2(p - \lambda)^2} \]

and Theorem 7 follows at once.

References


