Quantitative estimates for the Lupaş q-analogue of the Bernstein operator

Zoltán Finta *

Abstract

We establish quantitative results for the approximation properties of the q-analogue of the Bernstein operator defined by Lupaş in 1987 and for the approximation properties of the limit Lupaş operator introduced by Ostrovska in 2006, via Ditzian-Totik modulus of smoothness. Our results are local and global approximation theorems.

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1 Introduction

The Bernstein operators \( B_n : C[0,1] \rightarrow [0,1] \),

\[
(B_n f)(x) \equiv B_n(f, x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k},
\]

\( n = 1, 2, \ldots, f \in C[0,1], x \in [0,1] \), possess many remarkable properties. It is well-known that the sequence \( \{B_n(f, x)\} \) converges uniformly to \( f(x) \) on \( [0,1] \). In [10], Popoviciu estimated the rate of convergence \( \|B_n f - f\| \) using the usual modulus of continuity (\( \| \cdot \| \) denotes the uniform norm on \( C[0,1] \)). We have

\[
|(B_n f)(x) - f(x)| \leq \frac{3}{2} \omega(f, n^{-1/2}), \quad (1.1)
\]

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*Babeş-Bolyai University, Department of Mathematics, 1, M. Kogălniceanu st., 400084 Cluj-Napoca, Romania, e-mail: fzoltan@math.ubbcluj.ro
where \( x \in [0, 1] \), \( f \in C[0, 1] \) and \( n = 1, 2, \ldots \). Later investigation by Freud [4] established: for some constant \( C_1 > 0 \), and all \( f \in C[0,1] \) and \( x \in [0,1] \),

\[
|(B_n f)(x) - f(x)| \leq C_1 \omega^2(f, n^{-1/2} \sqrt{x(1-x)}). \tag{1.2}
\]

If \( f \in C[0,1] \) and \( \delta > 0 \), then the second modulus of smoothness of \( f \) is defined by

\[
\omega^2(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0,1-2h]} |f(x + 2h) - 2f(x + h) + f(x)|. \tag{1.3}
\]

In [2] it was shown for the Bernstein operators that the estimate

\[
|(B_n f)(x) - f(x)| \leq C_2 \omega^2_\varphi(f, n^{-1/2} \varphi^{1-\lambda}(x)) \tag{1.4}
\]

holds true, where \( \lambda \in [0, 1] \), \( \varphi(x) = \sqrt{x(1-x)} \), \( x \in [0,1] \) and the Ditzian-Totik modulus of smoothness of second order is given by

\[
\omega^2_\varphi(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \pm h \varphi(x) \in [0,1]} |f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|,
\]

in which \( f \in C[0,1] \), \( \delta > 0 \) and \( \varphi \) is an admissible step-weight function on \([0,1]\) (for details see [3]). If \( \lambda = 0 \), then (1.3) gives the local estimate, and for \( \lambda = 1 \), the inequality (1.3) gives the global norm estimate developed by Ditzian and Totik. Therefore (1.3) bridges the gap between the local and global approximation theorems for Bernstein operators.

Here we mention that \( C_1, C_2, C_3, \ldots \) will denote some absolute positive constants.

In 1997 Phillips [9] introduced the so-called q-Bernstein operators involving q-integers. They quickly gained the popularity and were studied widely by a number of authors. A survey of the obtained results and references on the subject can be found in [7]. It is worth mentioning that the first generalization of the Bernstein operators based on q-integers was obtained by Lupas [6]. For the introduced q-analogue of the Bernstein operator, he investigated its approximating and shape-preserving properties. Further interesting properties of the Lupas q-analogue of the Bernstein operator have been considered also in [8], [1], [12] and [11].

To present the Lupas operator, we recall some notions of the q-calculus. Let \( q > 0 \). Then for each non-negative integer \( k \), the q-integer \([k] = [k]_q\) and the q-factorial \([k]!\) are defined by

\[
[k] = 1 + q + \ldots + q^{k-1} \quad (k = 1, 2, \ldots), \quad [0] = 0
\]

and

\[
[k]! = [1][2] \ldots [k] \quad (k = 1, 2, \ldots), \quad [0]! = 1.
\]
For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by
\[
\binom{n}{k} = \frac{[n]!}{[k]![n-k]!},
\]

Further, we set
\[
b_{n,k}(q,x) = \binom{n}{k} q^{k(k-1)/2} \frac{x^k(1-x)^{n-k}}{(1-x+xq)\cdots(1-x+xq^{n-1})}
\]
for $k = 0, 1, \ldots, n$.

Following Lupaş [6], the positive linear operators $R_{n,q} : C[0,1] \to C[0,1]$ defined by
\[
(R_{n,q}f)(x) \equiv R_{n,q}(f,x) = \sum_{k=0}^{n} f \left( \binom{n}{k} \right) b_{n,k}(q,x)
\]
are called the q-analogue of the Bernstein operators. For $q = 1$, we recover the classical Bernstein operators.

The paper [8] deals with the convergence properties of the sequence $\{R_{n,q}f\}$ and with the limit Lupaş operator $R_{\infty,q} : C[0,1] \to C[0,1]$ defined by

\[
(R_{\infty,q}f)(x) \equiv R_{\infty,q}(f,x)
\]

\[
= \begin{cases} 
\tilde{R}_{\infty,q}(f,x), & \text{if } q \in (0,1) \\
\tilde{R}_{\infty,1/q}(\tilde{f},1-x), & \text{if } q \in (1,\infty)
\end{cases}
\]

where

\[
\tilde{R}_{\infty,q}(f,x) \equiv (\tilde{R}_{\infty,q}f)(x)
\]

\[
= \sum_{k=0}^{\infty} f(1-q^k) b_{\infty,k}(q,x), \quad \text{if } x \in [0,1)
\]

\[
= f(1), \quad \text{if } x = 1,
\]

\[
b_{\infty,k}(q,x) = \frac{q^{k(k-1)/2}(x/(1-x))^k}{(1-q)^k[k!] \prod_{j=0}^{\infty} (1+q^j(x/(1-x)))}, \quad x \in [0,1)
\]

and

\[
\tilde{f}(x) = f(1-x)
\]

for $f \in C[0,1]$ and $x \in [0,1]$. Ostrovska proved in [8] for $q \neq 1$ and $f \in C[0,1]$ that the sequence $\{R_{n,q}(f,x)\}$ converges uniformly to $R_{\infty,q}(f,x)$.
on $[0, 1]$, as $n \to \infty$. The rate of convergence $\|R_{n,q}f - R_{\infty,q}f\|$ has been studied by Wang and Zhang [12]. It is worth mentioning that the concept of limit operator via $q$-Bernstein operators was introduced for the first time in [5].

The goal of the paper is to establish quantitative estimates for the operators $R_{n,q}f$ and $R_{\infty,q}f$, where the errors $|(R_{n,q}f)(x) - f(x)|$ and $|(R_{\infty,q}f)(x) - f(x)|$ will be estimated with the aid of the modulus of smoothness (1.4). In this way we obtain direct local and global approximation theorems similar to (1.3).

2 Main results

Theorem 2.1 Let $R_{n,q}f$ be defined as in (1.5) and let $q = q_n$ such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Then there exists $C_3 > 0$ such that

$$|(R_{n,q_n}f)(x) - f(x)| \leq C_3\omega_{\phi^2}(f, [n]^{-1/2}x^{1-\lambda}(x))$$

(2.1)

for all $x \in [0, 1]$, $f \in C[0, 1]$ $\lambda \in [0, 1]$ and $n = 1, 2, \ldots$

Proof. The corresponding K-functional to (1.4) is defined by

$$K_{2,\phi^2}(f, \delta) = \inf_{g \in W^2(\phi)} \left\{ \|f - g\| + \delta \|\phi^2g''\| \right\},$$

where $W^2(\phi) = \{g \in C[0, 1] : g' \in AC_{loc}[0, 1], \phi^2g'' \in C[0, 1]\}$ and $g' \in AC_{loc}[0, 1]$ means that $g$ is differentiable such that $g'$ is absolutely continuous on every interval $[a, b] \subset [0, 1]$. Due to [3, Theorem 2.1.1], $K_{2,\phi^2}(f, \delta)$ and $\omega_{\phi^2}(f, \sqrt{\delta})$ are equivalent, i.e. there exists $C_4 > 0$ such that

$$C_4^{-1}\omega_{\phi^2}(f, \sqrt{\delta}) \leq K_{2,\phi^2}(f, \delta) \leq C_4\omega_{\phi^2}(f, \sqrt{\delta}).$$

By [8, Lemma 1], we have

$$R_{n,q_n}(1, x) = 1,$$  

(2.3)

$$R_{n,q_n}(t, x) = x$$  

(2.4)

and

$$0 \leq R_{n,q_n}((t - x)^2, x) = R_{n,q_n}(t^2, x) - x^2$$

$$= \frac{x(1 - x)}{[n]} - \frac{x^2(1 - x)(1 - q_n)}{1 - x + xq_n} \left(1 - \frac{1}{[n]}\right)$$

$$= \frac{x(1 - x)}{[n]} \frac{1 - x + xq_n}{1 - x + xq_n}$$

$$\leq \frac{1}{[n]} \phi^2(x)$$

(2.5)

for $x \in [0, 1]$.

4
Let \( x \in (0, 1) \) and \( g \in W^2(\varphi^\lambda) \). Taking into account Taylor’s formula
\[
g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u) \, du, \quad t \in (0, 1),
\]
(2.3) and (2.4), we find that
\[
R_{n,q_n}(g, x) = g(x) + R_{n,q_n} \left( \int_x^t (t-u)g''(u) \, du, x \right).
\]
Hence, by the estimate
\[
\left| \int_x^t (t-u)g''(u) \, du \right| \leq (t-x)^2 \varphi^{-2\lambda}(x) \| \varphi^{2\lambda}g'' \| \quad (2.6)
\]
(see [3, Lemma 9.6.1]) and (2.5), we get
\[
| R_{n,q_n}(g, x) - g(x) | \leq R_{n,q_n} \left( \left| \int_x^t (t-u)g''(u) \, du \right|, x \right)
\leq R_{n,q_n}((t-x)^2, x) \varphi^{-2\lambda}(x) \| \varphi^{2\lambda}g'' \|
\leq \frac{1}{[n]} \varphi^{2(1-\lambda)}(x) \| \varphi^{2\lambda}g'' \|. \quad (2.7)
\]
On the other hand, (1.5) and (2.3) imply for \( x \in [0, 1] \) and \( f \in C[0, 1] \) that
\[
| R_{n,q_n}(f, x) | \leq R_{n,q_n}(1, x) \| f \| = \| f \|, \quad \text{i.e.}
\|
R_{n,q_n}f \| \leq \| f \| \quad (2.8)
\]
for all \( f \in C[0, 1] \). By combining (2.8) and (2.7), we have
\[
| R_{n,q_n}(f, x) - f(x) | \leq | R_{n,q_n}(f, x) - R_{n,q_n}(g, x) |
+ | R_{n,q_n}(g, x) - g(x) | + | g(x) - f(x) |
\leq 2 \| f - g \| + \frac{1}{[n]} \varphi^{2(1-\lambda)}(x) \| \varphi^{2\lambda}g'' \|
\leq 2 \left\{ \| f - g \| + \frac{1}{[n]} \varphi^{2(1-\lambda)}(x) \| \varphi^{2\lambda}g'' \| \right\}.
\]
Taking the infimum on the right-hand side over all \( g \in W^2(\varphi^\lambda) \), by (2.2), we arrive at the estimate (2.1), which was to be proved.

**Remark 2.2** If \( q_n \in (0, 1) \) and \( q_n \to 1 \) as \( n \to \infty \), then \([n] \equiv [n]_{q_n} \to \infty \) when \( n \to \infty \). In conclusion (2.1) implies for \( \lambda = 1 \) that \( \{ R_{n,q_n}(f, x) \} \) converges uniformly to \( f(x) \) on \([0, 1] \), as \( n \to \infty \).

**Remark 2.3** In view of \( \varphi^\lambda(x) \leq 1, \ x, \lambda \in [0, 1] \) and [3, Theorems 2.1.1, 3.2.1 and 4.1.1], we have
\[
\omega_n^2(\varphi^\lambda, f, \delta) \leq C_5\omega^2(f, \delta) \leq 2C_5\omega(f, \delta).
\]
Hence, by (2.1), we obtain a Popoviciu-type estimate for $\lambda = 1$ (see (1.1)):

$$|(R_{n,q_n},f)(x) - f(x)| \leq 2C_3C_5\omega(f,[n]^{-1/2})$$

(2.9)

and a Freud-type estimate for $\lambda = 0$ (see (1.2)):

$$|(R_{n,q_n},f)(x) - f(x)| \leq C_3C_5\omega^2(f,[n]^{-1/2}\sqrt{x(1-x)})$$

(2.10)

respectively. Thus the degree of approximation in (2.1) is better than that in (2.9) and (2.10).

**Corollary 2.4** Let $R_{n,q}f$ be defined as in (1.5) and let $q = q_n$ such that $q_n \in (1, \infty)$ and $q_n \to 1$ as $n \to \infty$. Then there exists $C_6 > 0$ such that

$$|(R_{n,q_n},f)(x) - f(x)| \leq C_6\omega^2_{\varphi,\lambda}(f,\sqrt{[n]^{-1} + q_n - 1}\varphi^{1-\lambda}(x))$$

(2.11)

for all $x \in [0,1]$, $f \in C[0,1]$ $\lambda \in [0,1]$ and $n = 1,2,\ldots$

**Proof.** For $x, \lambda \in [0,1]$ and $h > 0$, we have $\varphi^{\lambda}(x) = \varphi^{\lambda}(1-x)$ and $x \pm h\varphi^{\lambda}(x) \in [0,1]$ if and only if $1 - x \pm h\varphi^{\lambda}(1-x) \in [0,1]$. Then, by (1.8), we obtain

$$\sup_{x \pm h\varphi^{\lambda}(x) \in [0,1]} |f(x + h\varphi^{\lambda}(x)) - 2f(x) + f(x - h\varphi^{\lambda}(x))|$$

$$= \sup_{x \pm h\varphi^{\lambda}(x) \in [0,1]} |f(1 - x - h\varphi^{\lambda}(1-x)) - 2f(1 - x) + f(1 - x + h\varphi^{\lambda}(1-x))|$$

$$= \sup_{1 - x \pm h\varphi^{\lambda}(1-x) \in [0,1]} |f(1 - x + h\varphi^{\lambda}(1-x)) - 2f(1 - x) + f(1 - x - h\varphi^{\lambda}(1-x))|.$$

Hence, in view of (1.4),

$$\omega^2_{\varphi,\lambda}(\tilde{f},\delta) = \omega^2_{\varphi,\lambda}(f,\delta).$$

(2.12)

On the other hand, for $q > 1$,

$$\frac{1}{[n]_{1/q}} \leq \frac{1}{[n]_q} + q - 1.$$

(2.13)

By reduction formula [8, (10)], we have

$$R_{n,q}(f,x) = R_{n,1/q}(\tilde{f},1-x),$$

(2.14)

where $q > 1$ and $\tilde{f}$ is defined by (1.8). Now combining (2.1), (2.14), (2.12) and (2.13), we find

$$|(R_{n,q_n},f)(x) - f(x)| = |(R_{n,1/q_n},\tilde{f})(1-x) - \tilde{f}(1-x)|$$

$$\leq C_3\omega^2_{\varphi,\lambda}(\tilde{f},[n]_{1/q_n}^{-1/2}\varphi^{1-\lambda}(x))$$

$$= C_3\omega^2_{\varphi,\lambda}(f,[n]_{1/q_n}^{-1/2}\varphi^{1-\lambda}(x))$$

$$\leq C_3\omega^2_{\varphi,\lambda}(f,([n]_q^{-1} + q_n - 1)^{1/2}\varphi^{1-\lambda}(x)).$$
which is the estimate (2.11) with $C_6 = C_3$.

The next results are in connection with the approximation properties of the limit Lupaş operator (1.6).

**Theorem 2.5** Let $R_{\infty,q}f$ be defined as in (1.6). Then there exists $C_7 > 0$ such that

$$|(R_{\infty,q}f)(x) - f(x)| = |(\tilde{R}_{\infty,q}f)(x) - f(x)| \leq C_7 \omega_2^2(f, \sqrt{1-q} \varphi^{1-\lambda}(x))$$

(2.15)

for all $x \in [0,1]$, $f \in C[0,1]$ $\lambda \in [0,1]$ and $q \in (0,1)$.

**Proof.** Due to [8, (17)-(18)], we have $R_{\infty,q}(1,x) = 1$, $R_{\infty,q}(t,x) = x$ and $0 \leq R_{n,q}((t - x)^2, x) \leq \frac{1-q}{1-q^n} \varphi^2(x)$ (see (2.5)). Letting $n \to \infty$ in the last relations, by [8, Theorem 2], we get

$$\tilde{R}_{\infty,q}(1,x) = 1,$$

(2.16)

$$\tilde{R}_{\infty,q}(t,x) = x$$

(2.17)

and

$$0 \leq \tilde{R}_{\infty,q}((t - x)^2, x) \leq (1-q) \varphi^2(x).$$

(2.18)

Further, (1.7) and (2.16) imply that

$$\|\tilde{R}_{\infty,q}f\| \leq \|f\|$$

(2.19)

for all $f \in C[0,1]$.

Now let $x \in (0,1)$ and $g \in W^2(\varphi^\lambda)$. By Taylor’s formula

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u)g''(u)du, \quad t \in (0,1),$$

(2.16) and (2.17), we obtain

$$\tilde{R}_{\infty,q}(g,x) = g(x) + \tilde{R}_{\infty,q} \left( \int_x^t (t - u)g''(u)du, x \right).$$

Hence, in view of (2.6) and (2.18), we find

$$|\tilde{R}_{\infty,q}(g,x) - g(x)| \leq (1-q)\varphi^{2(1-\lambda)}(x)\|\varphi^{2\lambda}g''\|.$$ 

Then, by (2.19),

$$|R_{\infty,q}(f,x) - f(x)| = |\tilde{R}_{\infty,q}(f,x) - f(x)|$$

$$\leq |\tilde{R}_{\infty,q}(f,x) - \tilde{R}_{\infty,q}(g,x)| + |\tilde{R}_{\infty,q}(g,x) - g(x)| + |g(x) - f(x)|$$

$$\leq 2 \left\{ \|f - g\| + (1-q)\varphi^{2(1-\lambda)}(x)\|\varphi^{2\lambda}g''\| \right\}.$$ 

Taking the infimum on the right-hand side over all $g \in W^2(\varphi^\lambda)$ and using (2.2), we arrive at (2.15).
Remark 2.6 For $\lambda = 1$, (2.15) implies that $\tilde{R}_{\infty,q}(f,x)$ converges uniformly to $f(x)$ on $[0,1]$, as $q \nrightarrow 1$.

Corollary 2.7 Let $R_{\infty,q}f$ be defined as in (1.6). Then there exists $C_8 > 0$ such that

$$|(R_{\infty,q}f)(x) - f(x)| \leq C_8 \omega_{\varphi,\lambda}^2(f, \sqrt{1/q - 1} \varphi^{1-\lambda}(x))$$

(2.20)

for all $x \in [0,1]$, $f \in C[0,1]$, $\lambda \in [0,1]$ and $q \in (1, \infty)$.

Similar to the proof of Corollary 2.4, the inequalities (2.15) and $1 - \frac{1}{q} \leq q - 1$ when $q > 1$, imply (2.20). We omit the details.

Theorem 2.8 Let $\tilde{f}$ be defined by (1.8), where $f \in C[0,1]$.

a) There exists $C_9 > 0$ such that

$$|(R_{\infty,q}f)(x) - \tilde{f}(1-x)| \leq C_9 \omega_{\varphi,\lambda}^2(\tilde{f}, \sqrt{1/q - 1} \varphi^{1-\lambda}(x))$$

for all $x \in [0,1]$, $f \in C[0,1]$, $\lambda \in [0,1]$ and $q \in (1, \infty)$.

b) There exists $C_{10} > 0$ such that

$$|(R_{\infty,q}\tilde{f})(x) - f(1-x)| \leq C_{10} \omega_{\varphi,\lambda}^2(f, \sqrt{1/q - 1} \varphi^{1-\lambda}(x))$$

for all $x \in [0,1]$, $f \in C[0,1]$, $\lambda \in [0,1]$ and $q \in (1, \infty)$.

Proof. a) In view of (1.6) and Theorem 2.5, we have

$$|(R_{\infty,q}f)(x) - \tilde{f}(1-x)| = |(R_{\infty,1/q}\tilde{f})(1-x) - \tilde{f}(1-x)|$$

$$\leq C_9 \omega_{\varphi,\lambda}^2(\tilde{f}, \sqrt{1/(1/q) - 1} \varphi^{1-\lambda}(x)) \leq C_9 \omega_{\varphi,\lambda}^2(\tilde{f}, \sqrt{q - 1} \varphi^{1-\lambda}(x))$$

because $q > 1$.

b) It is a consequence of a), taking into account (1.8) and $(\tilde{f})(x) = \tilde{f}(1-x) = f(x)$, $x \in [0,1]$.

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References


