EMBEDDING MODES INTO SEMIMODULES, PART I

AGATA PILITOWSKA\textsuperscript{1} AND ANNA B. ROMANOWSKA\textsuperscript{2}

Abstract. By recent results of M. Stronkowski, it is known that not all modes embed as subreducts into semimodules over commutative unital semirings. Related to this problem is the problem of constructing a (commutative unital) semiring defining the variety of semimodules whose idempotent subreducts lie in a given variety of modes. We provide a general construction of such semirings, along with basic examples and some general properties. The second part of the paper will deal with applications of the general construction to some selected varieties of modes, and will provide a description of semirings determining varieties of semimodules having algebras from these varieties as idempotent subreducts.

Modes are idempotent and entropic algebras. (See [10, 13] for a general introduction and basic theory.) One of the main methods of representing modes is given by embedding them as subreducts into modules over commutative rings, and more generally into semimodules over commutative semirings. Since modes are idempotent algebras, the embedding is in fact into the full idempotent reducts of such (semi)modules. In the case of modules, such full idempotent reducts are actually affine spaces (sometimes known as affine modules). For a given ring they form a variety of Mal’cev modes. In the case of semimodules over a given semiring, such full idempotent reducts, known as semi-affine spaces (or sometimes affine semimodules), do not form a quasivariety in general, and may be trivial.

A general characterization of modes in a given variety that embed as subreducts (subalgebras of reducts) into an affine space is given in [13, Theorem 7.2.3]. There are some other results showing that modes satisfying certain additional conditions are subreducts of affine spaces, see [13, Chapter 7], [11, 12, 10]. However, there is no easy general method known of deciding for an arbitrary mode if it is embeddable into an affine space. On the other

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hand, there is a general method of finding a commutative unital ring $R$ such that all modes in a given variety $\mathcal{V}$ that embed into affine spaces (and hence into idempotent reducts of the corresponding modules) in fact embed into affine spaces over this ring $R$. The class of affine spaces over such a ring $R$ is known as the affinization of the variety $\mathcal{V}$, while the corresponding variety of modules over $R$ can be called its linearization.

However, not all modes embed into affine spaces. In particular, it was known for some time that the class of subreducts of semi-affine spaces is much broader than the class of subreducts of affine spaces. For instance, all binary (or groupoid) modes are embeddable into semi-affine spaces [3]. A long-standing problem in the theory of modes asked for a characterization of those modes that embed as subreducts into semimodules over commutative, unital semirings. This problem has now been solved [16, 17, 18, 19]. (See also related papers [2, 3, 4, 5, 12, 14, 15].) In particular, it is known that not all modes embed into such semimodules. Related to the embeddability problem is the problem of constructing a (commutative unital) semiring defining the variety of semimodules whose idempotent subreducts lie in a given variety $\mathcal{V}$ of modes. It is be natural to call such a variety of semimodules the semi-linearization of $\mathcal{V}$, and the corresponding class of semi-affine spaces its semi-affinization.

In this paper, we describe such commutative unital semirings of semi-affinizations, comparing them with the commutative unital rings of affinizations. We investigate their properties, and find the semirings of semi-affinizations for some selected varieties of modes. This part of the paper has a somewhat introductory character. In the first section, we recall the basic definitions and properties of the affinization ring of a variety of modes. We also provide some examples that play an essential role in applications, and in the second part of the paper. The second section provides the main construction of the semi-affinization semiring for any variety of modes, and a discussion of its basic properties compared with properties of the affinization ring. In the third section, we describe the relationship between the ring and the semiring of an irregular and a regular variety of modes, and the relationship between semirings of an irregular mode variety and of its regularization (Theorem 3.3).

The second part of the paper will deal with some applications and discussions concerning semi-affinization and semi-linearization of some selected varieties of modes, in particular of varieties of affine spaces, barycentric algebras, and semilattice modes.

For more information about modes and their representations as subreducts of modules and semimodules, we refer the reader to the monographs [10] and [13], and to the papers in the bibliography. For general information
about semirings and semimodules, see [1]. We usually follow the notation and terminology of [10, 13].

1. Affinization of mode varieties

For a commutative ring \( R \) with identity, \textit{affine spaces} over \( R \) or \textit{affine \( R \)-spaces} are defined algebraically as full idempotent reducts of unital \( R \)-modules, or equivalently as reducts \( (A, P, R) \) of \( R \)-modules, where \( P \) is the ternary Mal’cev operation \( x - y + z \) and \( R \) consists of the binary operations \( r \), for each \( r \in R \), defined by \( x y r := x(1 - r) + yr \). The class \( R \) of all (isomorphic copies) of affine \( R \)-spaces forms a variety of Mal’cev modes. It is also known that each variety of Mal’cev modes is equivalent to a variety of affine spaces over a certain ring \( R \) [13, Chapter 6]. The ring \( R \) is built on the free affine \( R \)-space on two generators.

The \textit{affinization} of a variety \( V \) of modes \( (A, \Omega) \) of type \( \tau \) is defined to be the variety of Mal’cev modes having \( V \)-algebras as reducts and additionally equipped with an (abstract) ternary Mal’cev operation \( P \) defined by the identities

\[
(1.1) \quad x y y P = x = y y x P.
\]

This variety is the variety \( R(V) \) of affine spaces over a certain commutative ring \( R(V) \), also called the \textit{affinization ring} of the variety \( V \). The ring itself may be considered as an affine \( R(V) \)-space, and in fact is a free affine \( R(V) \)-space on two free generators 0 and 1. Then \( V \)-algebras that embed as subreducts into affine spaces are subreducts of affine \( R(V) \)-spaces [13, Chapter 7]. As affine \( R(V) \)-spaces are idempotent reducts of \( R(V) \)-modules, \( V \)-modes are also subreducts of such modules. The variety of \( R(V) \)-modules is sometimes called the \textit{linearization} of the variety \( V \).

The ring \( R(V) \) is constructed as follows [13, Chapter 7]. The free commutative ring over \( \{X_{\omega i} \mid \omega \in \Omega, 1 \leq i \leq \omega \tau\} \) is isomorphic to the integral polynomial ring \( \mathbb{Z}[X_{\omega i} \mid \omega \in \Omega, 1 \leq i \leq \omega \tau] \) over a set of \( \sum_{\omega \in \Omega} \omega \tau \) commuting indeterminates. Then the affinization ring \( R(\tau) = R(\mathcal{M}_\tau) \) of the variety \( \mathcal{M}_\tau \) of modes of type \( \tau \) is the quotient ring

\[
\mathbb{Z}[X_{\omega i} \mid \omega \in \Omega, 1 \leq i \leq \omega \tau] / \left\langle 1 - \sum_{i=1}^{\omega \tau} X_{\omega i} \mid \omega \in \Omega \right\rangle.
\]

For \( n \)-ary \( \omega \in \Omega \), the corresponding operation on an affine \( R(\tau) \)-space is defined by

\[
(1.2) \quad x_1 \ldots x_n \omega = \sum_{i=1}^{n} x_i X_{\omega i}
\]
for the indeterminates $X_{\omega i}$ pertaining to $\omega$. The $\Omega$-subreducts of $R(\tau)$-spaces are obviously modes, and they form a quasivariety. (See [6] for a more general result concerning the class of subreducts of a given type of algebras in a given quasivariety.) For a subvariety $\mathcal{V}$ of $\mathcal{M_\tau}$, the affinization ring $R(\mathcal{V})$ is obtained as a quotient of $R(\tau)$. The quotient is determined by the identities defining $\mathcal{V}$. Let $s = t$ be one such identity, and let $\{x_1, \ldots, x_n\}$ be the set of its variables. Then the $\Omega$-subreducts of $R(\mathcal{V})$-spaces that are members of $\mathcal{V}$ must satisfy this identity, and the equation $s = t$ may be written in the linear form

$$x_1s_1 + \cdots + x_ns_n = x_1t_1 + \cdots + x_nt_n,$$

where all the coefficients $s_i$ and $t_i$ are functions of some indeterminates $X_{\omega i}$. Note that some of the coefficients $s_i$ and $t_i$ may be 0 or 1. Equating coefficients of each variable $x_i$ in (1.3) provides a set of relations $s_i = t_i$, for $i = 1, \ldots, n$, that should be satisfied in the ring $R(\mathcal{V})$. If the equational basis of $\mathcal{V}$ consists of identities $s_j = t_j$ for $j \in J$, then each provides the relations $s_j^i = t_j^i$ for $i = 1, \ldots, n_j$ satisfied in $R(\mathcal{V})$. In particular, the $\Omega$-reduct of the ring $R(\mathcal{V})$ itself also satisfies all these relations.

By [8, Lemma 6.1], it follows that the $\Omega$-subreduct generated by a subset $A$ of the free affine $R(\mathcal{V})$-space over $A$ is also free on $A$ in the quasivariety $\Omega R(\mathcal{V})$ of $\Omega$-subreducts of affine $R(\mathcal{V})$-spaces. In the case $|A| = 2$, this is the subreduct generated by 0 and 1 of the affine space $R(\mathcal{V})$. If the variety $\mathcal{V}$ is the smallest variety containing $\Omega R(\mathcal{V})$, then both $\mathcal{V}$ and $\Omega R(\mathcal{V})$ have the same free algebras [13, Proposition 3.3.8]. In particular, it follows that neither the $\Omega$-subreduct of $R(\mathcal{V})$ nor the affine space $R(\mathcal{V})$ satisfy any additional relations, and thus

$$R(\mathcal{V}) \cong R(\tau)/\langle s_j^i - t_j^i \mid j \in J, i = 1, \ldots, n_j \rangle.$$

Note also that if $\mathcal{V}$ is generated by $\Omega R(\mathcal{V})$, the free $\mathcal{V}$-algebras embed into affine spaces. If the quasivariety $\Omega R(\mathcal{V})$ generates a variety $\mathcal{W}$ that is different from $\mathcal{V}$, then $R(\mathcal{W}) = R(\mathcal{V})$, and all $\mathcal{V}$-algebras that embed as subreducts into affine spaces belong to $\mathcal{W}$.

To reduce the number of indeterminates, one often represents one of the $X_{\omega i}$ in (1.2) as the difference between 1 and the sum of the remaining $X_{\omega j}$. For the purposes of this paper, it will be more convenient to avoid such a reduction. Below are some examples illustrating the procedure.

**Example 1.1.** For the variety $\mathcal{B M}$ of binary (or groupoid) modes, the affinization ring $R(\mathcal{B M})$ is the ring

$$\mathbb{Z}[X, Y]/(1 - X - Y) = \mathbb{Z}[X, Y]/cg(X + Y, 1).$$

The groupoid operation is defined on any $R(\mathcal{B M})$-module by $x \cdot y = xY + yX$. (More precisely, one should write the congruence classes of $X$ and $Y$...
instead of $X$ and $Y$. This ring is isomorphic to the ring $\mathbb{Z}[X]$ with the
groupoid operation defined by $x \cdot y = x(1 - X) + yX = xyX$.

**Example 1.2.** The affinization ring of the variety $\mathcal{NB}$ of normal bands
(semigroup modes) may be computed as follows. The associativity of nor-
mal bands implies that

$$xy \cdot z = xY^2 + yXY + zX = xY + yX + zX^2 = x \cdot yz.$$  

Equating coefficients of $x$ and $z$ on both sides of the inner equality shows
that $X^2 = X$ and $Y^2 = Y$. The ring $R(\mathcal{NB})$ is a quotient of the ring

$$\mathbb{Z}[X,Y]/\langle 1 - X - Y, X - X^2, Y - Y^2 \rangle$$  

(1.4)

isomorphic to $\mathbb{Z}[X]/\langle X - X^2 \rangle \cong \mathbb{Z} \times \mathbb{Z}$. As each affine space over this ring
is a semigroup mode under the operation $x \cdot y = xy + yX$, it follows that
$R(\mathcal{NB})$ coincides with (1.4). The ring is also isomorphic to the ring

$$\mathbb{Z}[d,e]$$  

with $d + e = 1$, $d^2 = d$ and $e^2 = e$.

The semigroup operation on an $R(\mathcal{NB})$-space is defined as

$$x \cdot y = xe + yd.  \tag{1.5}$$

Note also that since $Y = 1 - X$, it follows that

$$XY = X(1 - X) = X - X = 0,$$

whence

$$xy \cdot z = xY + zX = xz.  \tag{1.6}$$

This shows that $R(\mathcal{NB})$ coincides with the affinization ring $R(\mathcal{RB})$ of the
variety $\mathcal{RB}$ of rectangular bands.

**Remark 1.3.** The affinization ring $R(\mathcal{NB})$ is usually just described as
the extension $\mathbb{Z}[d]$ of the integers by an idempotent $d$, and the semigroup
operation (1.5) is given as $x \cdot y = x(1 - d) + yd$. Our current procedures are
designed to adapt easily to subsequent computations of semi-affinization
rings.

More generally, as shown in [13, Section 7.1], the affinization rings of a
given variety of modes and of its regularization coincide.

**Example 1.4.** In particular, this holds for two subvarieties of $\mathcal{NB}$, the
variety $\mathcal{LN}$ of left-normal bands and the variety $\mathcal{LZ}$ of left-zero bands. Indeed, the left-normal law implies that

$$xy \cdot z = xY + yXY + zX = xY + zXY + yX = xz \cdot y,$$
whence $X = XY$ or $d = de$ for the ring $\mathbb{Z}[d,e]$. Together with (1.6) this shows that $xy \cdot z = xY + yX + zX = xY + zX = xz$, whence $X = 0$ and $Y = 1$. Consequently, $R(\mathcal{LN}) = R(\mathcal{LZ}) \cong \mathbb{Z}$ [13, Section 7.1].

**Example 1.5.** The affinization ring of the variety $\mathcal{SL}$ of semilattices would require the conditions implied by the associativity from Example 1.2 and additionally a condition determined by the commutativity of the semilattice operation. Commutativity implies

$$xy = xY + yX = yY + xX = yx,$$

whence $X = Y$. Consequently, $1 = X + X = X^2 + X^2 = X = Y$, so

$$1 = X = Y = 1 - X = 0.$$ The ring $R(\mathcal{SL})$ is the trivial one-element ring.

The preceding example indicates that the affinization ring of any variety of semilattice modes (modes with a semilattice reduct) has a trivial affinization ring. In particular, no non-trivial semilattice mode can embed into an affine space. This agrees with the known fact that varieties of semilattice modes are independent of varieties of affine spaces.

2. **Semi-affinization of mode varieties**

In this section, we adapt the concepts of affinization and linearization by considering unital semimodules over commutative, unital semirings instead of modules over commutative rings. For the purposes of this paper, unital commutative semirings and semimodules are defined similarly to unital commutative rings and modules. We merely replace the additive structure of abelian groups with commutative monoids, and demand that the semimodules $(\mathcal{M}, +, 0, \mathcal{S})$ over a semiring $(\mathcal{S}, +, \cdot, 1)$ be zero-preserving in the sense that $x o = 0$ for each semimodule element $x$. In particular, the semiring $\mathcal{S}$ is said to have an absorbing zero if the semimodule reduct $(\mathcal{S}, +, o, \cdot)$ is zero-preserving.

**Remark 2.1.** Semirings and semimodules may be defined in a more general way, by considering semigroups instead of monoids in the additive structure. However, our assumptions do not form an essential limitation. Let $(\mathcal{M}, +, \mathcal{S})$ be a semimodule over a commutative unital semiring $\mathcal{S}$, with $(\mathcal{S}, +)$ and $(\mathcal{M}, +)$ as commutative semigroups. If $(\mathcal{M}, +)$ does not have a zero element, then first extend $\mathcal{M}$ by adding a zero $0$ to the semigroup $(\mathcal{M}, +)$ to obtain the additive monoid $(\mathcal{M}^0, +, 0)$, and define $0s = 0$ for each $s \in \mathcal{S}$. This defines the $\mathcal{S}$-semimodule $(\mathcal{M}^0, +, 0, \mathcal{S})$ with zero. Then extend the semiring $\mathcal{S}$ by adding a new zero $o$ to the semiring $\mathcal{S}$, to obtain a (commutative) semiring $\mathcal{S}^o$, and define $xo = 0$ for each $x \in \mathcal{M}$. Now $(\mathcal{M}^0, +, 0, \mathcal{S}^o)$ is a (unital) zero-preserving semimodule over a commutative
unital semiring with an absorbing zero. (See e.g. [2].) The idempotent reduct of \((M, +, S)\) is the restriction to \(M\) of the idempotent reduct of \((M^0, +, 0, S^0)\).

From now on, we assume that all semimodules are unital zero-preserving semimodules over commutative semirings with identity and with absorbing zero. For such a semiring \(S\), a semi-affine \(S\)-space (or an affine \(S\)-semimodule) is defined to be the full idempotent reduct of a unital \(S\)-semimodule. Such reducts, and their subalgebras (subreducts), are obviously modes. They all satisfy the so-called Szendrei identities. For a given type \(\tau\), these are identities arising from each word (term) of type \(\tau\) of the form

\[x_{11} \ldots x_{1n} w \ldots x_{n1} \ldots x_{nn} w w,\]

where \(w\) is a derived operator with \(n\) variables defining a basic operation of the reduct in question, by interchanging \(x_{ij}\) and \(x_{ji}\) for fixed \(1 \leq i, j \leq n\) [20]. By results of Stronkowski and Stanovský [16, 17, 18, 19], a mode embeds as a subreduct into a semimodule over a commutative semiring precisely when it satisfies the Szendrei identities.

The class \(R\) of affine \(R\)-spaces is known to be a Mal’cev variety. No comparable characterization of the class \(S\) of semi-affine \(S\)-spaces is known. While the full idempotent reduct of a non-trivial module over a non-trivial commutative ring is always non-trivial, the idempotent reducts of the semimodules in question may actually be trivial [2, 9]. For example, any free semimodule over the semiring of natural numbers has as non-trivial derived (or “term”) operations the linear combinations \(x_1 n_1 + \cdots + x_k n_k\). The only such idempotent operations are projections. We know that the class \(S\) of full idempotent reducts of \(S\)-semimodules does not necessarily form a variety. It is not even necessarily closed under subalgebras [2], [9, Example 7.7]. Thus a characterization of the class of semi-affine \(S\)-spaces, similar to that of affine \(R\)-spaces, is not possible in general. (For some better-behaved cases, see [2].)

Let \(\mathcal{V}\) be a variety of \(\Omega\)-modes. Recall that a mode embeds as a subreduct into a semimodule over a commutative semiring if and only if it satisfies the Szendrei identities. It follows that the class of embeddable \(\mathcal{V}\)-modes forms a subvariety of \(\mathcal{V}\), the subvariety \(Sz(\mathcal{V})\) of Szendrei \(\mathcal{V}\)-modes. We may observe that the variety \(Sz(\mathcal{V})\) is generated by the class of \(\Omega\)-reducts of semi-affine spaces over an appropriate commutative semiring \(S(\mathcal{V})\). The semiring \(S(\mathcal{V})\) will be constructed in a similar way as the ring of an affinization of \(\mathcal{V}\). Each \(\mathcal{V}\)-mode which embeds as a subreduct of a semimodule will be a subreduct of an \(S(\mathcal{V})\)-semimodule. It is then natural to call the variety of such semimodules the semi-linearization of the variety \(\mathcal{V}\), and the corresponding class of semi-affine \(S(\mathcal{V})\)-spaces its semi-affinization.
The construction of the semiring \( S(\mathcal{V}) \) goes as follows. For a fixed type \( \tau \) of \( \Omega \)-modes, we start with the polynomial semiring \( \mathbb{N}[\{X_{\omega i} \mid \omega \in \Omega, 1 \leq i \leq \omega \tau\}] \) over a set of commuting indeterminates, the free commutative semiring over the set \( \{X_{\omega i} \mid \omega \in \Omega, 1 \leq i \leq \omega \tau\} \). Then the quotient \( S(\tau) = S(\mathcal{M}_\tau) \) of this semiring by the congruence \( \theta = cg(\{ \sum_{i=1}^{\omega \tau} X_{\omega i}, 1 \mid \omega \in \Omega \}) \) is the semiring of the semi-affinization of the variety \( \mathcal{M}_\tau \) of all \( \Omega \)-modes. As in the case of affinization, the corresponding \((n\text{-ary})\) operation \( \omega \in \Omega \) on an \( S(\tau) \)-semimodule is defined by

\[
(2.1) \quad x_1 \ldots x_n \omega = \sum_{i=1}^{n} x_i X_{\omega i}
\]

for the indeterminates \( X_{\omega i} \) pertaining to \( \omega \). The \( \Omega \)-subreducts of \( S(\tau) \)-semimodules are obviously Szendrei \( \Omega \)-modes. By a result of Stanovský [16, Theorem 3], the free Szendrei \( \Omega \)-mode on a set \( A \) is isomorphic to the \( \Omega \)-subreduct generated by \( A \) of the free \( S(\tau) \)-semimodule over \( A \). (This fact, as well as the similar one concerning free \( \Omega \)-subreducts of affine spaces mentioned earlier, follows from a more general result [7, Theorem 3.9].)

For a subvariety \( \mathcal{V} \) of the variety \( \mathcal{M}_\tau \), the semiring \( S(\mathcal{V}) \) of the semilinearization of \( \mathcal{V} \) is obtained as the quotient of \( S(\tau) \) determined by the additional identities defining \( \mathcal{V} \), similarly as in the case of affinization. The semiring \( S(\mathcal{V}) \) is called the (semi-affinization) semiring of the variety \( \mathcal{V} \).

The first known class of modes embeddable into semimodules over commutative semirings was the class of groupoid modes. It is well known that varieties of groupoid modes satisfy Szendrei identities, so that all groupoid modes embed as subreducts into affine semimodules over commutative semirings. For the variety \( \mathcal{BM} \) of all groupoid modes, the semiring \( S(\mathcal{BM}) \) may be described as the semiring \( \mathbb{N}[d, e] \), where \( d + e = 1 \). The groupoid operation is defined on each \( S(\mathcal{BM}) \)-semimodule by

\[
x \cdot y := xe + yd.
\]

The semirings of some varieties of semigroup modes are calculated in the following examples.

**Example 2.2.** The semiring of the variety \( \mathcal{NB} \) (calculated as the affinization ring in Example 1.2), is the semiring \( S(\mathcal{NB}) = \mathbb{N}[d, e] \), where \( d + e = 1 \), \( d^2 = d \), and \( e^2 = e \). The first relation implies that

\[
d e + d = de + d^2 = d,
\]

whence

\[
d e + nd = nd
\]

for all positive integers \( n \). Similarly

\[
d e + e = e \text{ and } de + ne = ne.
\]
Moreover,
\[ de + de = de + de^2 = de \text{ and } nde = de. \]

Also
\[ de + n = de + d + e + (n - 1) = 1 + (n - 1) = n. \]

It follows that the typical element of \( S(\mathcal{NB}) \) has the form \( en + dk \) for some \( k, n \in \mathbb{N} \) or is equal to \( ed \). Note that 0 is the zero element of \( S(\mathcal{NB}) \), and that \( \{0, de\} \) forms a subsemiring, in fact the two-element lattice with \( 0 < de \). The semiring \( S(\mathcal{NB}) \) is isomorphic to \( \mathbb{N} \times \mathbb{N} \cup \{de\} \). The semiring \( S(\mathcal{RB}) \) of the variety of rectangular bands satisfies the additional relation \( de = 0 \) determined by the identity \( xy \cdot z = xz \), and is isomorphic to \( \mathbb{N} \times \mathbb{N} \).

**Example 2.3.** Similarly as the affinization ring in Example 1.4, the semiring of the variety of semilattices satisfies additional relations \( d = e = 1 \).

In particular, \( 1 + 1 = 1 \), and the semiring is in fact a two-element lattice with \( 0 < 1 \). (Compare with the semiring of the variety of semilattices resulting from the general form of semirings of varieties of semilattice modes as calculated in [4].) It follows that each semilattice can be considered as a subreduct of a semilattice semimodule (with the semilattice monoid reduct) over the two-element lattice.

**Example 2.4.** The semirings of the varieties of left-zero bands and of left-normal bands are calculated similarly. The semiring \( S(\mathcal{LN}) \) satisfies the additional relation \( de = d \). Moreover, \( e = de + e = d + e = 1 \), whence the semiring is isomorphic to \( \mathbb{N} \cup \{d\} \). Note also that \( \{0, d\} \) forms the two-element lattice with \( 0 < d \). The semiring \( S(\mathcal{LZ}) \) satisfies the additional relation \( d = 0 \), and is isomorphic to \( \mathbb{N} \). Note that the non-zero elements of the semiring \( S(\mathcal{LN}) \) form a subsemiring (with \( d \) playing the role of zero) isomorphic to the semiring \( S(\mathcal{LZ}) \).

Below are some general properties of semiaffinization semirings of varieties of modes. Consequences of these properties, and some applications, will be discussed in the second part of the paper. We start with a basic though obvious observation.

**Proposition 2.5.** Let \( V \) be a variety of modes, and \( Sz(V) \) its subvariety of Szendrei modes. Then the semirings \( S(V) \) and \( S(Sz(V)) \) coincide.

**Proposition 2.6.** Equivalent varieties of modes have isomorphic semirings.

**Proof.** First note that each variety \( V \) of \( \Omega \)-modes of type \( \tau \) is equivalent to the variety \( V^d \) of the same algebras but considered as algebras of the extended type \( \tau^\prime \) with the set \( \{x_1 < x_2 < \ldots\}\Omega = \mathcal{P}\Omega \) of all derived operators as basic operators. Moreover, the semirings \( S(V) \) and \( S(V^d) \) are isomorphic. Now let \( V \) and \( W \) be equivalent varieties of modes. Then \( V^d \)
and $\mathcal{W}^d$ are also equivalent [13, Section 2.2], and their semi-affinization semirings have the same sets of generators corresponding to the same sets of derived operators. The two varieties satisfy the same identities. These identities determine the same congruences of the semiring

$$\mathbb{N}[\{X_{ti} \mid t \in P\Omega, 1 \leq i \leq t\tau'\}].$$

Hence $S(\mathcal{V}^d)$ and $S(\mathcal{W}^d)$ are isomorphic. Consequently, $S(\mathcal{V})$ and $S(\mathcal{W})$ are also isomorphic. 

If we write the ring and the semiring of a variety $\mathcal{V}$ in the form of quotients of the corresponding free ring and free semiring by an appropriate congruence, then it is easy to observe that the relations determining the ring and the semiring of $\mathcal{V}$ are formally the same. However, they may give different consequences for the structure of the ring and of the semiring in question. For example $R(\mathcal{LN}) = R(\mathcal{LZ}) \cong \mathbb{Z}$, but $S(\mathcal{LN}) \cong \mathbb{N} \cup \{d\}$ and $S(\mathcal{LZ}) \cong \mathbb{N}$ are not isomorphic.

3. Regularization

Recall that each idempotent irregular variety $\mathcal{V}$ is defined by regular identities and one irregular identity of the form

$$(3.1)\quad x \circ y = x,$$

where $x \circ y$ is a term with two variables $x$ and $y$. It defines a (derived) binary operation of a left-zero band in each $\mathcal{V}$-algebra. Let us add the operation $\circ$ to the basic operations of $\mathcal{V}$-modes. The variety we obtain is equivalent to the given one, and by Proposition 2.6, both have the same rings and the same semirings. We may then assume that the operation $\circ$ is defined in the affinization and the semi-affinization of $\mathcal{V}$ by $x \circ y = xe + yd$.

**Proposition 3.1.** If a mode variety $\mathcal{V}$ is irregular, then the semiring $S(\mathcal{V})$ of $\mathcal{V}$ is obtained from the ring $R(\mathcal{V})$ of $\mathcal{V}$ by replacing $\mathbb{Z}$ with $\mathbb{N}$.

**Proof.** The regular identities defining $\mathcal{V}$ provide the same relations determining the ring and the semiring of $\mathcal{V}$. The irregular identity (3.1) shows that the operation $\circ$ is an operation of a left-zero semigroup and hence satisfies the properties listed in Examples 1.4 and 2.4. In particular $e = 1$ and $d = de = 0$, both in the ring and in the semiring of $\mathcal{V}$. 

The relationship between the semirings of a variety of semigroup modes and its regularization observed in Examples 2.2 and 2.4 is an instance of a more general theorem. Before we formulate this theorem, let us first observe the following fact.
Lemma 3.2. Let $V$ be a variety of $\tau$-modes, and let $S(V) = S(\tau)/\varphi$. If $V$ is regular and $S(V)$ is not a ring, then the $\varphi$-class $0^\varphi$ of 0 consists precisely of one element.

Proof. Suppose on the contrary, that for a non-zero element $s \in S(\tau)$, one has $(s, 0) \in \varphi$. Such a relation may come only from an irregular identity true in $V$, where a coefficient of some variable is $s$ on one side and 0 on the other. But a regular variety does not satisfy an irregular identity. \hfill $\Box$

Theorem 3.3. The semiring $S(\tilde{V})$ of the regularisation $\tilde{V}$ of an irregular variety $V$ of modes is the semiring obtained from $S(V)$ by adjoining a new zero element.

Proof. Let us consider $V$-modes as algebras with the operation $\circ$, as above, and represent $x \circ y$ as $xe + yd$ in the semi-affinization of $V$. As we have seen in the proof of Proposition 3.1, this gives $e = 1$ and $d = de = 0$ in $S(V)$.

In the regularisation $\tilde{V}$ of $V$, the operation $x \circ y$ becomes a left-normal band operation. By Lemma 3.2, the elements 0 and $d$ are different. (Compare also Example 2.4.)

We will show that $d$ is the zero of the subsemiring of $S(\tilde{V})$ consisting of non-zero elements. This subsemiring is obviously isomorphic to $S(V)$.

Let $\omega$ be a basic operation of $V$-algebras. Let $x_1 \ldots x_n \omega = x_1 r_1 + \ldots + x_n r_n$, where $r_1, \ldots, r_n \in S(\tilde{V})$ and are different from zero. Then the identity

$$y \circ x_1 \ldots x_n \omega = y \circ x_1 \circ x_2 \ldots \circ x_n,$$

that is known to be satisfied in the regularization $\tilde{V}$, implies the following

$$y + (x_1 r_1 + \cdots + x_n r_n)d = y + x_1 r_1 d + \cdots + x_n r_n d = y + x_1 d + \cdots + x_n d.$$

Similarly as in previous examples, one shows that this implies $r_i d = d$ for each $1 \leq i \leq n$. Consequently, for all non-zero $r \in S(\tilde{V})$, one has

$$rd = d.$$

Since $1 + d = 1$, it follows that $r + rd = r$, whence $r + d = r$. Consequently, $d$ is indeed the zero of the subsemiring in question. And obviously, 0 is the zero of the whole semiring. \hfill $\Box$

The second part of the paper will discuss some applications, and the semi-affinization of several selected types of varieties of modes, in particular affine spaces, barycentric algebras, and semilattice modes.
References


Faculty of Mathematics and Information Science, Warsaw University of Technology, 00-661 Warsaw, Poland

E-mail address: 1apili@alpha.mini.pw.edu.pl
E-mail address: 2aroman@alpha.mini.pw.edu.pl
URL: 1http://www.mini.pw.edu.pl/~apili
URL: 2http://www.mini.pw.edu.pl/~aroman