STABILITY OF \( n \)-DIMENSIONAL ARUN-ADDITIVE FUNCTIONAL EQUATION IN GENERALIZED 2-NORMED SPACE

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\textbf{Abstract.} In this paper, the author established the general solution and generalized Ulam - Hyers - Rassias stability of \( n \)-dimensional Arun-additive functional equation

\[ f \left( nx_0 \pm \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} f \left( x_0 \pm x_i \right) \]

in Generalized 2-normed space.

1. Introduction and Preliminaries

In 1940, the stability of functional equations had been first raised by S.M. Ulam \cite{32}. In 1941, D. H. Hyers \cite{18} gave an affirmative answer to the question of S.M. Ulam for Banach spaces.

In 1950, T. Aoki \cite{2} was the second author to treat this problem for additive mappings. In 1978, Th.M. Rassias \cite{25} succeeded in extending Hyers’ Theorem by weakening the condition for the Cauchy difference controlled by \((||x||^p + ||y||^p)\), \( p \in [0, 1) \), to be unbounded.

In 1982, J.M. Rassias \cite{23} replaced the factor \( ||x||^p + ||y||^p \) by \( ||x||^p ||y||^q \) for \( p, q \in \mathbb{R} \). A generalization of all the above stability results was obtained by P. Gavruta \cite{16} in 1994 by replacing the unbounded Cauchy difference by a general control function \( \varphi(x, y) \).

In 2008, a special case of Gavruta’s theorem for the unbounded Cauchy difference was obtained by Ravi etal., \cite{30} by considering the summation of both the sum and the product of two \( p \)-norms. The stability problems of several functional

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equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 4, 5, 6, 7, 8, 9, 21, 24, 30]) and reference cited there in.


The solution and stability of the following additive functional equations

\[
\begin{align*}
 f(x + y) &= f(x) + f(y), \\
 f(2x - y) + f(x - 2y) &= 3f(x) - 3f(y), \\
 f(x + y - 2z) + f(2x + 2y - z) &= 3f(x) + 3f(y) - 3f(z), \\
 f(m(x + y) - 2mz) + f(2m(x + y) - mz) &= 3m[f(x) + f(y) - f(z)] \quad \text{for } m \geq 1, \\
 f\left(a \sum_{i=1}^{n-1} x_i - 2ax_n\right) + f\left(2a \sum_{i=1}^{n-1} x_i - ax_n\right) &= 3a \left(\sum_{i=1}^{n-1} f(x_i) - f(x_n)\right) \quad \text{for } n \geq 3,
\end{align*}
\]

were discussed by J. Aczel [1], D.O. Lee [12], K. Ravi, M. Arunkumar [29, 31].

Very recently, the general solution and generalized Hyers-Ulam-Rassias stability of the Arun-additive functional equation

\[
f(2x \pm y \pm z) = f(x \pm y) + f(x \pm z) \quad (1.6)
\]

was studied by M. Arunkumar [5]. Also M. Arunkumar et al [6] investigate the generalized Arun-additive functional equation

\[
f(qx \pm y \pm z) = f(x \pm y) + f(x \pm z) + (q - 2)f(x), \quad q \geq 2
\]

in the setting of intuitionistic fuzzy normed spaces.

**Definition 1.1.** *n-* Dimensional Arun-Additive Functional Equation.* Let *X* and *Y* be real vector spaces. A functional equation is said to *n-* dimensional Arun-Additive functional equation if \( f : X \rightarrow Y \) satisfies the functional equation

\[
f\left(nx_0 \pm \sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_0 \pm x_i) \quad (1.7)
\]

for all \( x_0, x_1, \ldots, x_n \in X \).

In this paper, the author investigate the solution and stability of the *n-* Dimensional Arun-Additive functional equation (1.7) in Generalized 2-normed space.
In Section 2, some basic definitions on Generalized 2-Normed Space is present.

In Section 3, the general solution of the functional equation \((1.7)\) is given.

In Section 4, the generalized Ulam - Hyers - Rassias stability of the additive functional equation \((1.7)\) is proved.

Examples for non stability cases of the additive functional equation \((1.7)\) is discussed in Section 5.

### 2. Basic Definition on Generalized 2-Normed Space

In this section, some basic definitions related to Generalized 2-normed spaces is present.

**Definition 2.1.** \([7]\) Let \(X\) be linear space. A function \(N(.,.) : X \times X \rightarrow [0, \infty)\) is called a generalized 2-normed space if it satisfies the followings

- (M1) \(N(x, y) = 0\) if and only if \(x\) and \(y\) are linearly independent vectors.
- (M2) \(N(x, y) = N(y, x)\) for all \(x, y \in X\),
- (M3) \(N(\lambda x, y) = |\lambda|N(x, y)\) for all \(x, y \in X\) and \(X = \varphi, \varphi\) is a real or complex field,
- (M4) \(N(x + y, z) \leq N(x, z) + N(y, z)\) for all \(x, y, z \in X\).

The generalized 2-normed space is denoted by \((X, N(.,.))\).

**Definition 2.2.** \([7]\) A sequence \(\{x_n\}\) in a generalized 2-normed space \((X, N(.,.))\) is called convergent if there exist \(x \in X\) such that \(\lim_{n \to \infty} N(x_n - x, y) = 0\) then \(\lim_{n \to \infty} N(x_n, y) = N(x, y)\) for all \(y \in X\).

**Definition 2.3.** \([7]\) A sequence \(\{x_n\}\) in a generalized 2-normed space \((X, N(.,.))\) is called Cauchy sequence if there exist two linearly independent elements \(y\) and \(z\) in \(X\) such that \(\{N(x_n, y)\}\) and \(\{N(x_n, z)\}\) are real Cauchy sequences.

**Definition 2.4.** \([7]\) A generalized 2-normed space \((X, N(.,.))\) is called generalized 2-Banach space if every Cauchy sequence is convergent.

Recently, the solution and stability of additive and quadratic functional equation in generalized 2-normed space was first discussed in \([7]\).

### 3. General Solution of the Functional equation \((1.7)\)

In this section, the general solution of the functional equation \((1.7)\) is given.

**Theorem 3.1.** Let \(X\) and \(Y\) be real vector spaces. The mapping \(f : X \rightarrow Y\) satisfies the functional equation

\[
f(x + y) = f(x) + f(y)
\]

(3.1)
for all \( x, y \in X \) if and only if \( f : X \to Y \) satisfies the functional equation

\[
f \left( n x_0 \pm \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} f(x_0 \pm x_i) \tag{3.2}
\]

for all \( x_0, x_1, \ldots, x_n \in X \).

**Proof.** Let \( f : X \to Y \) satisfies the functional equation \((3.1)\). Setting \( x = y = 0 \) in \((3.1)\), we get \( f(0) = 0 \). Set \( x = -y \) in \((3.1)\), we get \( f(-y) = -f(y) \) for all \( y \in X \). Therefore \( f \) is an odd function. Replacing \( y \) by \( x \) and \( y \) by \( 2x \) in \((3.1)\), we obtain

\[
f(2x) = 2f(x) \quad \text{and} \quad f(3x) = 3f(x) \tag{3.3}
\]

for all \( x \in X \). In general for any positive integer \( a \), we have

\[
f(ax) = af(x), \quad \text{for all } x \in X. \tag{3.4}
\]

One can easy to verify that the equation \((3.1)\) can be transformed into

\[
f \left( \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} f(x_i) \tag{3.5}
\]

for all \( x_1, \ldots, x_n \in X \). Now, if we replace \((x_1, \ldots, x_n) \) by \((x_0 \pm x_1, \ldots, x_0 \pm x_n) \) in \((3.5)\), we derive \((3.2)\) for all \( x_0, x_1, \ldots, x_n \in X \).

Conversely, let \( f : X \to Y \) satisfies the functional equation \((3.2)\). Setting \((x_0, x_1, \ldots, x_n) \) by \((0, 0, \ldots, 0) \) in \((3.2)\), we get \( f(0) = 0 \). Replacing \((x_0, x_1, \ldots, x_n) \) by \((x, 0, \ldots, 0) \) in \((3.2)\), we obtain

\[
f(ax) = af(x), \quad \text{for all } x \in X. \tag{3.6}
\]

Substituting \((x_0, x_1, x_2, x_3, \ldots, x_n) \) by \((0, x, \mp x, 0 \ldots, 0) \) in \((3.2)\), we get \( f(-x) = -f(x) \) for all \( x \in X \). Letting \((x_0, x_1, x_2, x_3, \ldots, x_n) \) by \((0, \pm x, \pm y, 0 \ldots, 0) \) in \((3.2)\), we prove \((3.1)\) for all \( x, y \in X \). \(\square\)

### 4. Stability of the Functional Equation \((1.7)\)

In this section, the generalized Ulam - Hyers - Rassias stability of the additive functional equation \((1.7)\) is provided. Through out this section, let us consider \( X \) be a generalized 2-normed space and \( Y \) be generalized 2-Banach space.

**Theorem 4.1.** Let \( j \in \{-1, 1\} \). Let \( \alpha : X^{n+1} \to [0, \infty) \) be a function such that

\[
\sum_{m=0}^{\infty} \alpha \left( n^{m} x_0, n^{m} x_1, \ldots, n^{m} x_n \right) n^{m} \quad \text{converges and} \quad \lim_{m \to \infty} \alpha \left( n^{m} x_0, n^{m} x_1, \ldots, n^{m} x_n \right) n^{m} = 0 \tag{4.1}
\]

for all \( x_0, x_1, \ldots, x_n \in X \). Suppose a function \( f : X \to Y \) satisfies the inequality

\[
N \left( f \left( n x_0 \pm \sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} f(x_0 \pm x_i), u \right) \leq \alpha \left( x_0, x_1, \ldots, x_n \right) \tag{4.2}
\]
for all \(x_0, x_1, \cdots, x_n \in X\) and all \(u \in X\). Then there exists a unique additive function \(A : X \to Y\) such that

\[
N(f(x) - A(x), u) \leq \frac{1}{n} \sum_{k=\frac{1}{2}}^{\infty} \alpha \left( \frac{n^{kj}x, 0, \cdots, 0}{n}\right) \tag{4.3}
\]

for all \(x \in X\) and all \(u \in X\). The mapping \(A(x)\) is defined by

\[
\lim_{m \to \infty} N \left( A(x) - \frac{f(n^{mj}x)}{n^{mj}}, u \right) = 0 \tag{4.4}
\]

for all \(x \in X\) and all \(u \in X\).

Proof. Assume \(j = 1\). Letting \((x_0, x_1, \cdots, x_n)\) by \(\left( x, 0, \cdots, 0 \right)\) in (4.2), we get

\[
N \left( \frac{f(nx)}{n} - f(x), u \right) \leq \frac{\alpha \left( x, 0, \cdots, 0 \right)}{n} \tag{4.5}
\]

for all \(x \in X\) and all \(u \in X\). Replacing \(x\) by \(nx\) and divided by \(n\) in (4.5), we obtain

\[
N \left( \frac{f(n^2x)}{n^2} - \frac{f(nx)}{n}, u \right) \leq \frac{\alpha \left( nx, 0, \cdots, 0 \right)}{n^2} \tag{4.6}
\]

for all \(x \in X\) and all \(u \in X\). Combining (4.5) and (4.6) using (M4), we arrive

\[
N \left( \frac{f(n^2x)}{n^2} - f(x), u \right) \leq N \left( \frac{f(n^2x)}{n^2} - \frac{f(nx)}{n}, u \right) + N \left( \frac{f(nx)}{n} - f(x), u \right)
\]

\[
\leq \frac{1}{n} \left[ \alpha \left( x, 0, \cdots, 0 \right) + \frac{\alpha \left( nx, 0, \cdots, 0 \right)}{n^{n\times n-m}} \right] \tag{4.7}
\]
for all \( x \in X \) and all \( u \in X \). In general for any positive integer \( m \), we have

\[
N \left( \frac{f(n^m x)}{n^m} - f(x), u \right) \leq \frac{1}{n} \sum_{k=0}^{m-1} \alpha \left( \frac{n^k x}{n^m}, 0, \ldots, 0 \right) \tag{4.8}
\]

\[
\leq \frac{1}{n} \sum_{k=0}^{\infty} \alpha \left( \frac{n^k x}{n^m}, 0, \ldots, 0 \right)
\]

for all \( x \in X \) and all \( u \in X \). In order to prove the convergence of the sequence \( \left\{ \frac{f(n^m x)}{n^m} \right\} \), replace \( x \) by \( n^l x \) and divided by \( n^l \) in (4.8), for any \( m, l > 0 \), we arrive

\[
N \left( \frac{f(n^{m+l} x)}{n^{m+l}} - \frac{f(n^l x)}{n^l}, u \right) = \frac{1}{n^l} N \left( \frac{f(n^m \cdot n^l x)}{n^m} - f(n^l x), u \right)
\]

\[
\leq \frac{1}{n^l} \sum_{k=0}^{\infty} \alpha \left( \frac{n^{k+l} x}{n^{m+l}}, 0, \ldots, 0 \right)
\]

\[
\to 0 \text{ as } l \to \infty \tag{4.9}
\]

for all \( x \in X \) and all \( u \in X \). Also

\[
N \left( \frac{f(n^{m+l} x)}{n^{m+l}} - \frac{f(n^l x)}{n^l}, v \right) = \frac{1}{n^l} N \left( \frac{f(n^m \cdot n^l x)}{n^m} - f(n^l x), u \right)
\]

\[
\leq \frac{1}{n^l} \sum_{k=0}^{\infty} \alpha \left( \frac{n^{k+l} x}{n^{m+l}}, 0, \ldots, 0 \right)
\]

\[
\to 0 \text{ as } l \to \infty \tag{4.10}
\]

for all \( x \in X \) and all \( v \in X \).

Hence there exists two linearly independent elements \( u \) and \( v \) in \( X \) such that \( \left\{ N \left( \frac{f(n^m x)}{n^m}, u \right) \right\} \) and \( \left\{ N \left( \frac{f(n^m x)}{n^m}, v \right) \right\} \) are real Cauchy sequences. Thus the sequence \( \left\{ \frac{f(n^m x)}{n^m} \right\} \) is Cauchy sequence. Since \( Y \) is complete, there exists a mapping \( A : X \to Y \) such that

\[
\lim_{m \to \infty} N \left( A(x) - \frac{f(n^m x)}{n^m}, u \right) = 0 \tag{4.11}
\]
for all \(x \in X\) and all \(u \in X\). To prove \(A\) satisfies (1.7) replacing \((x_0, x_1, \cdots, x_n)\) by \((n^m x_0, n^m x_1, \cdots, n^m x_n)\) and divided by \(n^m\) in (4.2), we get

\[
N \left( \frac{1}{n^m} \left[ f \left( n^m \left( nx_0 \pm \sum_{i=1}^n x_i \right) \right) - \sum_{i=1}^n f \left( n^m (x_0 \pm x_i) \right) \right], u \right)
\leq \frac{1}{n^m} \cdot \alpha \left( n^m x_0, n^m x_1, \cdots, n^m x_n \right)
\]  

(4.12)

for all \(x_0, x_1, \cdots, x_n \in X\) and all \(u \in X\). Now

\[
N \left( A \left( nx_0 \pm \sum_{i=1}^n x_i \right) - \sum_{i=1}^n A(x_0 \pm x_i), u \right)
\]

\[
= N \left( A \left( nx_0 \pm \sum_{i=1}^n x_i \right) - \frac{1}{n^m} f \left( n^m \left( nx_0 \pm \sum_{i=1}^n x_i \right) \right), u \right)
\]

\[
+ N \left( \sum_{i=1}^n A(x_0 \pm x_i) - \frac{1}{n^m} \sum_{i=1}^n f \left( n^m (x_0 \pm x_i) \right), u \right)
\]

\[
+ N \left( \frac{1}{n^m} \left[ f \left( n^m \left( nx_0 \pm \sum_{i=1}^n x_i \right) \right) - \sum_{i=1}^n f \left( n^m (x_0 \pm x_i) \right) \right], u \right)
\]

(4.13)

for all \(x_0, x_1, \cdots, x_n \in X\) and all \(u \in X\). Hence it follows from (4.11), (4.12) and (4.13), we arrive

\[
N \left( A \left( nx_0 \pm \sum_{i=1}^n x_i \right) - \sum_{i=1}^n A(x_0 \pm x_i), u \right) = 0 + 0 + \frac{1}{n^m} \cdot \alpha \left( n^m x_0, n^m x_1, \cdots, n^m x_n \right)
\]

(4.14)

for all \(x_0, x_1, \cdots, x_n \in X\) and all \(u \in X\). Letting \(m \to \infty\) in (4.14) and using (4.1), we see that

\[
N \left( A \left( nx_0 \pm \sum_{i=1}^n x_i \right) - \sum_{i=1}^n A(x_0 \pm x_i), u \right) = 0.
\]
Using (M1), we see that \( A \) satisfies (1.7). To prove \( A(x) \) is unique, let \( B(x) \) be another additive mapping satisfying (1.7) and (4.3), we arrive

\[
N(A(x) - B(x), u) = \frac{1}{n^m}N(A(n^mx) - B(n^mx), u)
\leq \frac{1}{n^m}\{N(A(n^mx) - f(n^mx), u) + N(f(n^mx) - B(n^mx), u)\}
\leq \sum_{k=0}^{m-1} \frac{2\alpha(n^mx_0, n^mx_1, \ldots, n^mx_n)}{n^{(k+m)}}
\leq \sum_{k=0}^{\infty} \frac{2\alpha(n^mx_0, n^mx_1, \ldots, n^mx_n)}{n^{(k+m)}}
\rightarrow 0 \text{ as } m \rightarrow \infty
\]

for all \( x \in X \) and all \( u \in X \). Hence \( A \) is unique.

For \( j = -1 \), we can prove the similar stability result. This completes the proof of the theorem. \( \square \)

The following corollary is a immediate consequence of Theorem 4.1 concerning the stability of (1.7).

**Corollary 4.2.** Let \( \lambda \) and \( s \) be nonnegative real numbers. If a function \( f : X \rightarrow Y \) satisfies the inequality

\[
N\left( f\left( nx_0 \pm \sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} f(x_0 \pm x_i), u \right) \leq \begin{cases} 
\lambda, & s < 1 \text{ or } s > 1; \\
\lambda \sum_{i=0}^{n} ||x_i||^s, & s < \frac{1}{n+1} \text{ or } s > \frac{1}{n+1}; \\
\lambda\left( \prod_{i=0}^{n} ||x_i||^s + \sum_{i=0}^{n} ||x_i||^{(n+1)s} \right), & s < \frac{1}{n+1} \text{ or } s > \frac{1}{n+1}; 
\end{cases} 
\]

(4.15)

for all \( x_0, x_1, \ldots, x_n \in X \) and all \( u \in X \). Then there exists a unique additive function \( A : X \rightarrow Y \) such that

\[
N(f(x) - A(x), u) \leq \begin{cases} 
\lambda, & 1 - n^s \leq ||x||^s, \\
\frac{1}{n - n^s} \lambda||x||^s, & 1 - n^s > ||x||^s; \\
\frac{1}{n - n^{(n+1)s}} \lambda||x||^s, & 1 - n^s < ||x||^s; 
\end{cases} 
\]

(4.16)

for all \( x \in X \) and all \( u \in X \).
5. Counter Examples for Non Stability Cases

In this section, the counter example for non stable cases is discussed.

Now we will provide an example to illustrate that the functional equation \((1.7)\) is not stable for \(s = 1\) in Condition \((ii)\) of Corollary \(4.2\).

**Example 5.1.** Let \(\alpha : \mathbb{R} \to \mathbb{R}\) be a function defined by

\[
\alpha(x) = \begin{cases} 
\mu x, & \text{if } |x| < 1 \\
\mu, & \text{otherwise}
\end{cases}
\]

where \(\mu > 0\) is a constant, and define a function \(f : \mathbb{R} \to \mathbb{R}\) by

\[
N(f(x),u) = \sum_{m=0}^{\infty} \frac{\alpha(n^m x)}{n^m} \text{ for all } x \in \mathbb{R}
\]

Then \(f\) satisfies the functional inequality

\[
N \left( f \left( n x_0 \pm \sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} f(x_0 \pm x_i), u \right) \leq \frac{n \times (n + 1) \mu}{(n - 1)} \sum_{i=0}^{n} ||x_i||
\]

for all \(x_0, x_1, \ldots, x_n \in \mathbb{R}\) and \(u \in \mathbb{R}\). Then there do not exists a additive mapping \(A : \mathbb{R} \to \mathbb{R}\) and a constant \(\delta > 0\) such that

\[
N \left( f(x) - A(x), u \right) \leq \delta ||x|| \text{ for all } x \in \mathbb{R}
\]

**Proof.** Now \(N(f(x),u) \leq \sum_{m=0}^{\infty} \frac{\alpha(n^m x)}{n^m} = \sum_{n=0}^{\infty} \frac{\mu}{n^m} = \frac{n \mu}{n - 1} \).

Therefore we see that \(f\) is bounded. We are going to prove that \(f\) satisfies \((5.1)\).

If \(x_i = 0, i = 0, 1, 2, \ldots n\) then \((5.1)\) is trivial. If \(\sum_{i=0}^{n} ||x_i|| \geq 1\) then the left hand side of \((5.1)\) is less than \(\frac{n(n + 1) \mu}{n - 1}\). Now suppose that \(0 < \sum_{i=0}^{n} ||x_i|| < 1\). Then there exists a positive integer \(k\) such that

\[
\frac{1}{n^k} \leq \sum_{i=0}^{n} ||x_i|| < \frac{1}{n^{k-1}};
\]

so that \(n^{k-1} ||x_i|| < 1, i = 0, 1, 2, \ldots, n\) and consequently

\[
n^{k-1} \left( n x_0 \pm \sum_{i=1}^{n} x_i \right), n^{k-1} \sum_{i=1}^{n} (x_0 \pm x_i) \in (-1, 1).
\]
Therefore for each $m = 0, 1, \ldots, k - 1$, we have
\[
n^m \left( nx_0 \pm \sum_{i=1}^{n} x_i \right), n^m \sum_{i=1}^{n} (x_0 \pm x_i), \in (-1, 1),
\]
and
\[
N \left( \alpha \left( n^m (nx_0 \pm \sum_{i=1}^{n} x_i) \right) - \sum_{i=1}^{n} \alpha(n^m(x_0 \pm x_i)), u \right) = 0
\]
for $m = 0, 1, \ldots, k - 1$. From the definition of $f$ and (5.3), we obtain that
\[
N \left( f \left( nx_0 \pm \sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} f(x_0 \pm x_i), u \right)
\]
\[
\leq \sum_{m=0}^{\infty} \frac{1}{n^m} N \left( \alpha \left( n^m(nx_0 \pm \sum_{i=1}^{n} x_i) \right) - \sum_{i=1}^{n} \alpha(n^m(x_0 \pm x_i)), u \right)
\]
\[
\leq \sum_{m=k}^{\infty} \frac{1}{n^m} \left( n^m (n+1) \mu = \frac{n \times (n+1) \mu}{(n-1)n^k} \leq \frac{n \times (n+1) \mu}{(n-1)} \left( \sum_{i=0}^{n} ||x_i|| \right) \right)
\]
Thus $f$ satisfies (5.1) for all $x_i \in \mathbb{R}, i = 0, 1, 2 \ldots n$ with $0 < \sum_{i=0}^{n} ||x_i|| < 1$.

We claim that the additive functional equation (1.7) is not stable for $s = 1$ in condition $(ii)$ of Corollary 4.2. Suppose on the contrary that there exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\delta > 0$ satisfying (5.2). Since $f$ is bounded and continuous for all $x \in \mathbb{R}$, $A$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 4.1, $A$ must have the form $A(x) = cx$ for any $x$ in $\mathbb{R}$. Thus we obtain that
\[
N \left( f(x), u \right) \leq (\delta + ||c||) ||x||
\]
But we can choose a positive integer $\ell$ with $\ell \mu > \beta + ||c||$.

If $x \in \left(0, \frac{1}{n^{\ell-1}}\right)$, then $n^m x \in (0, 1)$ for all $m = 0, 1, \ldots, \ell - 1$. For this $x$, we get
\[
N \left( f(x), u \right) = \sum_{m=0}^{\infty} \frac{\alpha(n^m x)}{n^m} \geq \sum_{m=0}^{\ell-1} \frac{\mu \cdot n^m x}{n^m} = \ell \mu x > (\delta + ||c||) ||x||
\]
which contradicts (5.4). Therefore the additive functional equation (1.7) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality (4.16).

Now we will provide an example to illustrate that the functional equation (1.7) is not stable for $s = \frac{1}{n+1}$ in Condition $(iii)$ of Corollary 4.2.
Example 5.2. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\alpha(x) = \begin{cases} 
\mu x, & \text{if } |x| < \frac{1}{n+1} \\
\frac{\mu}{n+1}, & \text{otherwise}
\end{cases}$$

where $\mu > 0$ is a constant, and define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$N(f(x), u) = \sum_{n=0}^{\infty} \frac{\alpha(n^m x)}{n^m}$$

for all $x \in \mathbb{R}$.

Then $f$ satisfies the functional inequality

$$N \left( f \left( nx_0 \pm \sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} f(x_0 \pm x_i), u \right) \leq \frac{(n+1)\mu}{(n-1)} \left( \prod_{i=0}^{n} ||x_i||^{2^{n-i}} + \sum_{i=0}^{n} ||x_i|| \right)$$

for all $x_0, x_1, \ldots, x_n \in \mathbb{R}$ and $u \in \mathbb{R}$. Then there do not exists a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$N \left( f(x) - A(x), u \right) \leq \delta ||x|| \quad \text{for all } x \in \mathbb{R}. \quad (5.6)$$

References


[8] M. Arunkumar, Solution and stability of a functional equation originating from arithmetic mean of consecutive Terms of an arithmetic progression, FUNCTIONAL EQUATIONS IN MATHEMATICAL ANALYSIS, Dedicated to the Memory of the 100th Anniversary of S.M.Ulam, SPRINGER.
