SEMITOPOLOGICAL BL-ALGEBRAS AND MV-ALGEBRAS

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Abstract. In this paper, by considering the notion of upsets we construct some topologies on BL-algebras. We show that any BL-algebras with respect to these topologies forms semitopological BL-algebra. Then we obtain some of the topological aspects of these structures such as connectivity and compactness. Moreover, we introduced two kinds of semitopological MV-algebra by using two kinds of definition of MV-algebra and show that they are equivalent.

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1. Introduction

In [11], Hájek proposed his logic BL as a common fragment of all traditional many valued logics (Łukasiewicz, Gödel and Product Logics). In [14], BL has been proved to be complete with respect to a variety of algebras called BL-algebras. BL-algebras are the algebraic structure for Hájek’s Basic Logic. MV-algebras, Gödel algebras and Product algebras are the most known classes of BL-algebras.

In the last 10 years, many mathematicians have studied the properties of BL-algebras endowed with a topology. For example Di Nola and Leustean [9] studied compact representations of BL-algebras, Ciungu [7] investigated some concepts of convergence in the class of perfect BL-algebras, Mi Ko and Kim [15] studied relationships between closure operators and BL-algebras, Haveshki et al. [12] applied filters to construct a topology on BL-algebras. Borzooei et al. [3] defined semitopological and topological BL-algebras, and they stated and proved some theorems that determine the relationships between them. In [4], Borzooei et al. introduced quasi-filter neighborhoods and studied metrizability on (semi)topological BL-algebras. In [2], Borumand Saeid and et al. introduced the set of double complemented elements for any filter F in BL-algebras and obtained some of the properties of it.

In this paper, we generalized this definition and defined the concept of $D_p(F)$, for any upset F of BL-algebra L. In fact, $D(F) = D_1(F)$, for any filter F of L. Then we attempt to construct a topology on L, by using of these subsets. Our aim is to show that L with this topology is a semitopological BL-algebra. With this way, we can construct many semitopological BL-algebras. In the last section, we verify semitopological MV-algebras and find relation between semitopological l-groups and semitopological MV-algebras.

2. Preliminaries

Definition 2.1. [1, 10, 11] (i) A residuated lattice is an algebra $(L, \lor, \land, \circ, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

1. $(L, \lor, \land, 0, 1)$ is a bounded lattice with 1 as the greatest element and 0 as the smallest element,
2. $(L, \circ, 1)$ is a commutative monoid,
3. $a \leq b \rightarrow c$ if and only if $a \circ b \leq c$, for all $a, b, c \in L$.

(ii) A residuated lattice L is called a BL-algebra, if it satisfies the following conditions:

1. $(x \rightarrow y) \lor (y \rightarrow x) = 1$, for all $x, y \in L$.
2. $x \land y = x \circ (x \rightarrow y)$, for all $x, y \in L$. 

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Proposition 2.2. [1, 10, 11, 17] The following properties hold for any BL-algebra:

\( (B1) \) \( x \leq y \Leftrightarrow x \rightarrow y = 1 \),

\( (B2) \) \( 1 \rightarrow x = x, x \circ x' = 0, x \rightarrow 1 = 1, 0 \rightarrow x = 1 \) and \( x \rightarrow (y \rightarrow x) = 1 \),

\( (B3) \) \( x \leq y \rightarrow z \Leftrightarrow y \leq x \rightarrow z \),

\( (B4) \) \( x \rightarrow (y \rightarrow z) = (x \circ y) \rightarrow z = y \rightarrow (x \rightarrow z) \),

\( (B5) \) \( x \leq y \) implies \( z \rightarrow x \leq z \rightarrow y \) and \( y \rightarrow z \leq x \rightarrow z \),

\( (B6) \) \( z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y) \) and \( z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x) \),

\( (B7) \) \( x \circ (y \rightarrow z) \leq x \rightarrow z \),

\( (B8) \) \( x'' = x' \) and \( x \leq x'' \), where \( x' = x \rightarrow 0 \),

\( (B9) \) \( x' \land y' = (x \lor y)' \),

\( (B10) \) \( x \land x' = 1 \) implies \( x \land x' = 0 \),

\( (B11) \) \( x \circ y \leq x \land y \),

\( (B12) \) \( x \leq y \) implies \( x \circ z \leq y \circ z \) and \( x^n \leq y^n \),

\( (B13) \) \( y \rightarrow z \leq x \rightarrow y \rightarrow x \rightarrow z \),

\( (B14) \) \( (x \land y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z) \) and \( (x \land y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z) \),

\( (B15) \) \( x' \lor y' = (x \land y)' \),

\( (B16) \) \( x \lor y = ((x \rightarrow y) \rightarrow y) \land ((y \rightarrow x) \rightarrow x) \),

\( (B17) \) \( x \rightarrow (y \lor z) = (x \rightarrow y) \lor (x \rightarrow z) \) and \( x \rightarrow (y \land z) = (x \rightarrow y) \land (x \rightarrow z) \),

\( (B18) \) \( x \land (y \lor z) = (x \land y) \lor (x \land z) \),

\( (B19) \) \( (x \rightarrow y')'' = x'' \lor y'' \) and \( (x \land y')'' = x'' \lor y'' \),

where \( x^0 = 1, x^1 = x \) and \( x^n = x^{n-1} \circ x \), for any \( 2 \leq n \).

Definition 2.3. Let \((X, \leq)\) be an ordered set. Then we define \(\uparrow: P(X) \rightarrow P(X)\), by \(\uparrow S = \{x \in X | a \leq x, \text{ for some } a \in S\}\), for any subset \(S\) of \(X\). A subset \(F\) of \(X\) is called an upset if \(\uparrow F = F\). We usually use \(U(X)\) to denote the set of all upsets of \(X\). A upset \(F\) is called finite generated if there exists \(n \in \mathbb{N}\) such that \(F = \uparrow \{x_1, x_2, \ldots, x_n\}\), for some \(x_1, x_2, \ldots, x_n \in X\).

Definition 2.4. [11] A non-empty upset \(F\) of \(BL\)-algebra \((L, \lor, \land, \circ, \rightarrow, 0, 1)\) is called a filter, if \(x \circ y \in F\), for any \(x, y \in F\).

Definition 2.5. [17] Let \(D\) be a non-empty subset of \(BL\)-algebra \((L, \lor, \land, \circ, \rightarrow, 0, 1)\) containing \(1\). Then \(D\) is called a deductive system if \(x \in D\) and \(x \rightarrow y \in D\) imply \(y \in D\), for any \(x, y \in L\).

Proposition 2.6. [17] Let \(F\) be a non-empty subset of \(BL\)-algebra \((L, \lor, \land, \circ, \rightarrow, 0, 1)\). Then \(F\) is a deductive system if and only if \(F\) is a filter.

For any filter \(F\) of \(BL\)-algebra \(L\) we can define a relation \(\Xi_F\) on \(L\) by \(x \Xi_F y \Leftrightarrow x \rightarrow y \in F\), for all \(x, y \in L\). Then \(\Xi_F\) is a congruence relation on \(L\) and we use \(x/F\) or \([x]\) to denote \(\{y \in L | x \Xi_F y\}\), for any \(x \in L\). Let \(A/F = \{[x] | x \in L\}\). For all \(x, y \in L\), define \([x] \lor [y] = [x \lor y], [x] \land [y] = [x \land y], [x] \circ [y] = [x \circ y]\) and \([x] \rightarrow [y] = [x \rightarrow y]\). Then \((L/F, \lor, \land, \circ, \rightarrow, 0, 1)\) is a \(BL\)-algebra. It is called quotient \(BL\)-algebra with respect to \(F\) (see [10]). If \(S \subseteq L\), then \(S/F = \{[x] | x \in S\}\).

Definition 2.7. [8, 11, 17, 18] Let \((L, \lor, \land, \circ, \rightarrow, 0, 1)\) and \((L', \lor, \land, \circ, \rightarrow, 0, 1)\) be two \(BL\)-algebras. A map \(f : L \rightarrow L'\) is called a homomorphism if it satisfies the following conditions:

(i) \(f(0) = 0\),

(ii) \(f(x \circ y) = f(x) \circ f(y)\), for any \(x, y \in L\),

(iii) \(f(x \rightarrow y) = f(x) \rightarrow f(y)\), for any \(x, y \in L\).

Let \(f : L \rightarrow L'\) be a homomorphism. Then \(f(x \lor y) = f(x) \lor f(y), f(x \land y) = f(x) \land f(y), f(x') = f(x)\) and \(f(1) = f(1)\). Moreover, if \(F\) and \(G\) are filters of \(L\) and \(L'\), respectively, then \(f^{-1}(G)\) is a filter of \(L\).

If \(f\) is a onto homomorphism, then \(f(F)\) is a filter of \(L'\). We use \(\ker f\) to denote \(\{x \in L | f(x) = 1\}\). It is a filter of \(L\).

Definition 2.8. [2] Let \(F\) be a filter of \(BL\)-algebra \((L, \lor, \land, \circ, \rightarrow, 0, 1)\). The set of double complemented elements, \(D(F)\), is defined by \(D(F) = \{x \in L | x'' \in F\}\).
**Definition 2.9.** [16] Let $\tau$ and $\tau'$ be two topologies on a given set $X$. If $\tau' \subseteq \tau$, then we say that $\tau'$ is finer than $\tau$. Let $(X, U)$ and $(Y, U')$ be two topological spaces. A map $f : X \to Y$ is called a continuous if the inverse image of each open set of $Y$ is open in $X$. A homeomorphism is a continuous function, which is one to one, onto and has a continuous inverse.

**Lemma 2.10.** [16] Let $\beta$ and $\beta'$ be two bases for topologies $\tau$ and $\tau'$, respectively on $X$. Then the following are equivalent:

(i) $\tau'$ is finer than $\tau$,

(ii) For any $x \in X$ and each basis element $B \in \beta$ containing $x$, there is a basis element $B' \in \beta'$ such that $x \in B' \subseteq B$.

**Definition 2.11.** [3] Let $(A, *)$ be an algebra of type 2 and $U$ be a topology on $A$. Then $(A, *, U)$ is called a semitopological algebra.

(i) Right (left) topological algebra, if for all $a \in A$ the map $* : A \to A$ is defined by $x \mapsto a * x$ ($x \mapsto x * a$) is continuous.

(ii) Semitopological algebra if $A$ is a right and left topological algebra.

Note that, if $(A, *)$ is a commutative algebra, then right and left topological algebras are equivalent.

**Definition 2.12.** [3] Let $A$ be a non-empty set and $\{*, i\}_{i \in I}$ be a family of operations of type 2 on $A$ and $U$ be a topology on $A$. Then,

(i) $(A, \{*, i\}_{i \in I}, U)$ is a right (left) topological algebra, if for $i \in I$, $(A, *, U)$ is a right (left) topological algebra.

(ii) $(A, \{*, i\}_{i \in I}, U)$ is a right (left) semitopological algebra, if for $i \in I$, $(A, *, U)$ is a right (left) semitopological algebra.

**Definition 2.13.** A *abelian l-group* is a structure $(G, +, -, 0, \lor, \land)$ such that $(G, +, -, 0)$ is an abelian group, $(G, \lor, \land)$ is a lattice and for any $x, y, a \in G$, $x \leq y$ implies $a + x \leq a + y$.

### 3. Constraction some topologies on BL-algebras

In this section, for any upset $F$ of BL-algebra $L$ and any $y \in L$, we defined the concept $D_y(F)$.

We show that $\{D_y(F)\}$ form a topology on $L$, for any $y \in L$. We denote this topological space by $(L, \tau_y)$. Then some of the properties of this topological space were obtained. We prove that $(L, \{\lor, \land, \land\}, \tau_y)$ is a semitopological BL-algebra, for any $y \in L$.

From now on, in this paper, $(L, \lor, \land, \land, \to, 0, 1)$ or simply $L$ is a BL-algebra, unless otherwise specified.

**Definition 3.1.** Let $y \in L$. For any non-empty upset $F$ of $L$ we define

$$D_y(F) = \{x \in L | y^n \Delta x \in F, \text{ for some } n \in \mathbb{N}\}$$

where $y \Delta x = (y \to x)^n$ and $y^n \Delta x = y \Delta (y^{n-1} \Delta x)$, for any $n \in \{2, 3, 4, \ldots\}$.

**Note 3.2.** By (B4), (B8) and (B19), it can be easily obtained that,

$$y^n \Delta x = \left(y^n \Delta (y^{n-1} \Delta (y^{n-2} \Delta \cdots \Delta (y \Delta x))\cdots)\right) = (y^n)^n \to x''$$

for any $x, y \in L$ and $n \in \mathbb{N}$. Hence, $D_y(F) = \{x \in L | (y^n)^n \to x'' \in F, \text{ for some } n \in \mathbb{N}\}$.

In the next proposition, we want to verify some properties of this operator. Note that, if $F = \emptyset$, then clearly, $D_y(F) = \emptyset$ and so we verify non-empty upsets in the next proposition.

**Proposition 3.3.** Let $x, y \in L$ and $F, G$ be two non-empty upsets of $L$. Then the following hold:

(i) $D_y(F)$ is an upset of $L$,

(ii) $y \in D_y(F)$ and $F \subseteq D_y(F)$,

(iii) $y^n \Delta (y^n \Delta x) = y^{n+m} \Delta x$, for any $m, n \in \mathbb{N}$,

(iv) if $F \subseteq G$, then $D_y(F) \subseteq D_y(G)$,

(v) $D_y(D_y(F)) = D_y(F)$,
(vi) if $F$ is a filter of $L$, then $D_y(F)$ is a filter of $L$, too,
(vii) if $x \leq y$, then $D_y(F) \subseteq D_x(F)$,
(viii) if $\{F_\alpha | \alpha \in I\}$ be a family of filters of $L$, then $D_y(\bigcup\{F_\alpha | \alpha \in I\}) = \bigcup\{D_y(F_\alpha) | \alpha \in I\}$,
(ix) if $\{G_1, G_2, ..., G_n\}$ be a set of filters of $L$, then

$$D_y(\cap\{G_i|i=1,2,...,n\}) = \cap\{D_y(G_i)i=1,2,...,n\}$$

(x) $D_y(D_y(F)) = D_y(D_x(F))$.

Proof. (i) Let $u \in \uparrow D_y(F)$. Then there exists $x \in D_y(F)$ such that $x \leq u$ and so $(y''^n) \rightarrow x'' \in F$, for some $n \in N$. By $x \leq u$ and (B5), we get $x'' \leq u''$ and so $(y''^n) \rightarrow x'' \leq (y''^n) \rightarrow u''$. Since $F$ is an upset and $(y''^n) \rightarrow x'' \in F$, then $(y''^n) \rightarrow u'' \in F$ and so $u \in D_y(F)$. Therefore, $D_y(F)$ is an upset.

(ii) Since $1 \in F$ and $(y\Delta y) = 1$, then we get $y \in D_y(F)$. Now, let $x \in F$. Then by (B2) and (B8) we have $x \leq y \rightarrow x \leq (y \rightarrow x)' = y\Delta x$, we conclude that $x \in D_y(F)$. Hence $F \subseteq D_y(F)$.

(iii) It follows from Note 3.2 and (B4).

(iv) Let $F \subseteq G$ and $x \in D_y(F)$. Then there exists $n \in N$ such that $y^n\Delta x \in F \subseteq G$ and so $x \in D_y(G)$.

(v) By (iv), we have $D_y(F) \subseteq D_y(D_y(F))$. Let $x \in D_y(D_y(F))$. Then there exists $m \in N$ such that $y^n\Delta x \in D_y(F)$ and so $y^n\Delta ((y^n\Delta x)) \in F$, for some $n \in N$. Now, by (iii), we get $y^{n+m}\Delta x \in F$ and so $x \in D_y(F)$. Hence $D_y(D_y(F)) = D_y(F)$.

(vi) Let $F$ be a filter of $L$. Then $F$ is a non-empty upset and so by (i), $1 \in D_y(F)$. Let $a, a \rightarrow b \in D_y(F)$. Then there exist $m, n \in N$ such that $(y''^n) \rightarrow a'' \in F$ and $(y''^m) \rightarrow (a \rightarrow b)'' \in F$. Moreover,

$$(y''^n) \rightarrow a'' \rightarrow ((y''^n)^{m+n} \rightarrow b'') = (y''^m) \rightarrow ((y''^n) \rightarrow a'') \rightarrow ((y''^n) \rightarrow b'')$$

by (B4)

$$\geq (y''^m) \rightarrow (a'' \rightarrow b'')$$

by (B6)

$$= (y''^m) \rightarrow (a \rightarrow b)'' \in F$$

by (B19)

Since $F$ is a filter and $(y''^n) \rightarrow a'' \in F$, then we get $(y''^n)^{m+n} \rightarrow b'' \in F$ and so $b \in D_y(F)$. Therefore, $D_y(F)$ is a filter of $L$.

(vii) Let $x \leq y$ and $a \in D_y(F)$. Then there exists $n \in N$ such that $(y''^n) \rightarrow a'' \in F$. By $x \leq y$, (B5) and (B12), we get $x'' \leq y''$ and so $(y''^n) \rightarrow a'' \leq (x''^n) \rightarrow a''$. Since $(y''^n) \rightarrow a'' \in F$ and $F$ is an upset, then $(x''^n) \rightarrow a'' \in F$ and so $a \in D_x(F)$. Therefore, $D_y(F) \subseteq D_x(F)$.

(viii) Let $x \in L$. Then

$$x \in D_y(\bigcup\{F_\alpha | \alpha \in I\}) \iff y^n\Delta x \in \bigcup\{F_\alpha | \alpha \in I\} \text{ for some } n \in N$$

$$\iff y^n\Delta x \in F_\alpha \text{ for some } n \in N \text{ and } \alpha \in I$$

$$\iff x \in D_y(F_\alpha) \text{ for some } n \in N \text{ and } \alpha \in I$$

$$\iff x \in \bigcup\{D_y(F_\alpha) | \alpha \in I\}$$

(ix) By (iv), $D_y(\cap\{G_i|i=1,2,...,n\}) \subseteq \cap\{D_y(G_i) | i=1,2,...,n\}$. Let $x \in \cap\{D_y(G_i) | i=1,2,...,n\}$. Then there exist $m_1, ..., m_n \in N$ such that $(y''^m_i) \rightarrow x'' \in G_i$, for any $i \in \{1,2,...,n\}$. Let $t = \max\{m_1, m_2, ..., m_n\}$. Then by (B2), $(y''^t) \rightarrow x'' \leq (y''^t) \rightarrow x'' \in G_i$, for any $i \in \{1,2,...,n\}$. It follows that $(y''^t) \rightarrow x'' \in \cap\{G_i | i=1,2,...,n\}$, and so $x \in D_y(\cap\{G_i | i=1,2,...,n\})$. Therefore, $D_y(\cap\{G_i | i=1,2,...,n\}) = \cap\{D_y(G_i) | i=1,2,...,n\}$.

(x) It is straightforward by Note 3.2 and (B4).

Corollary 3.4. For any $y \in L$, the map $D_y : U(L) \rightarrow U(L)$ is a closure operator and $D_y = D_{y''}$.

Proof. By Proposition 3.3(i), (ii), (iv) and (v), $D_y$ is a closure operator. Let $F$ be a non-empty upset of $L$. By $y \leq y''$ and Proposition 3.3(vii), $D_{y''}(F) \subseteq D_y(F)$. Let $x \in D_y(F)$. Then there exists $n \in N$ such that $(y''^n) \rightarrow x'' \in F$. By (B8), $y''^n = y''$ and so $x \in D_{y''}(F)$. Therefore, $D_y = D_{y''}$.

Clearly, if $F$ is a filter and $y = 1$, then $D_y(F) = D(F)$. In the next theorem, we want to answer to this question "under what condition $D_y(F) = D(F)$?"

Theorem 3.5. Let $y \in L$ and $F$ be a filter of $L$. Then $D_y(F) = D(F)$ if and only if $y'' \in F$.\[\square\]
Proof. Let \( D_y(F) = D(F) \). Then by Proposition 3.3(ii), we get \( y \in D(F) \) and so \( y'' \in F \). Conversely, suppose that \( y'' \in F \). By Proposition 3.3(vii), \( D(F) \subseteq D_y(F) \). Let \( x \in D_y(F) \). Then there exists \( n \in \mathbb{N} \) such that \( (y'')^n \to x \in F \). Since \( y'' \in F \) and \( F \) is a filter, then \( (y'')^n \in F \) and so \( x'' \in F \). Hence \( x \in D(F) \). Therefore, \( D_y(F) = D(F) \). \( \Box \)

**Proposition 3.6.** Let \( y \in L \) and \( F \) be a filter of \( L \). Then

(i) if \( F \) is a maximal filter of \( L \) and \( y'' \in L \), then \( D_y(F) = F \).

(ii) if \( F \) is a prime filter of \( L \), then \( D_y(F) \) is a prime filter of \( L \).

(iii) \( D_y(F) = L \) if and only if \( (y'')^n \) \( \in F \), for some \( n \in \mathbb{N} \).

(iv) if \( M \) is a maximal filter of \( L \), then \( D_y(M) = L \leftrightarrow y \in L - M \).

Proof. (i) Suppose that \( F \) is a maximal filter of \( L \) and \( y'' \in F \). By Proposition 3.3(ii) and (vi) \( D_y(F) \) is a filter and \( F \subseteq D_y(F) \). If \( 0 \in D_y(F) \), then \( (y'')^n \to 0 \in F \), for some \( n \in \mathbb{N} \) and so by (B4), \( (y'')^{n-1} \to (y'' \to 0) = (y'')^{n-1} \to y' \in F \). Since \( y'' \in F \) and \( F \) is a filter then we get \( y' \in F \) and so by (B2), \( 0 = y'' \lor y' \in F \), which is a contradiction. Hence \( D_y(F) \neq L \). Now, by assumption we get \( F = D_y(F) \).

(ii) Let \( F \) be a prime filter of \( L \) and \( a \lor b \in D_y(F) \). Then there exists \( n \in \mathbb{N} \) such that \( (y'')^n \to (a \lor b)'' \in F \). Hence by

\[
(y'')^n \to (a \lor b)'' = (y'')^n \to (a'' \lor b''), \text{ by (B19)}
\]

we get \((y'')^n \lor (a'' \lor b'') \in F \). Since \( F \) is a prime filter, then \( (y'')^n \lor a'' \in F \) or \( (y'')^n \lor b'' \in F \) and so \( a, b \in D_y(F) \). Therefore, \( D_y(F) \) is a prime filter of \( L \).

(iii) \( 0 \in D_y(F) \iff (y'')^n \to 0 \in F \), for some \( n \in \mathbb{N} \iff (y'' \to 0)'' \in F \iff (y'')' \in F \).

(iv) Let \( D_y(M) = L \). If \( y \in M \), then \( y'' \in M \) and so by (i), we have \( L = D_y(M) = M \), which is a contradiction. Hence \( y \in L - M \). Conversely, let \( y \in L - M \). Since \( M \) is a maximal filter, then there exists \( n \in \mathbb{N} \) such that \( (y'')' \in M \). Hence by (iii), \( D_y(F) = L \). \( \Box \)

**Proposition 3.7.** Let \( F \) and \( G \) be two filters of \( L \) such that \( G \subseteq F \). Then \( D_{[y]}(F/G) = D_y(F)/G \).

Proof. Let \( x \in L \). Then

\[
D_{[y]}(F/G) = \{ [x] \in L/G \mid [y]'' \Delta [x] \in F/G, \text{ for some } n \in \mathbb{N} \}
= \{ [x] \in L/G \mid [y]'' \Delta x \in F \}
= \{ [x] \in L/G \mid [y]'' \Delta x \in F \}
= D_y(F)/G
\]

\( \Box \)

**Definition 3.8.** Let \( a \in L \). By Proposition 3.3(viii) and (ix), the set \( \tau_a = \{ D_a(F) \mid F \in U(L) \} \) is a topology on \( L \) and \( (L, \tau_a) \) is a topological space.

**Proposition 3.9.** Let \( f : L \rightarrow L \) be a homomorphism, \( y \in L \) and \( F \) be a filter of \( L \). Then

(i) if \( y \in \text{Im} f \) and \( y = f(a) \), then \( f^{-1}(D_y(F)) = D_a(f^{-1}(F)) \),

(ii) \( D_y(\ker(f)) = f^{-1}(D_{f(y)}(\{1\})) \).

(iii) Consider the topological space \( (L, \tau_y) \). If \( y'' = f(a) \), for some \( a \in L \), then \( f \) is a continuous map from \( (L, \tau_a) \) to \( (L, \tau_y) \).
Proof. (i) Let $y'' = f(a)$. Since $f$ is a homomorphism, then $f^{-1}(F)$ is a filter of $L$ and we get

$$f^{-1}(D_y(F)) = \{ x \in L | f(x) \in D_y(F) \} = \{ x \in L | (y'')^n \rightarrow f(x)' = F, \exists n \in \mathbb{N} \} = \{ x \in L | f((a'')^n) \rightarrow f(x)' = F, \exists n \in \mathbb{N} \} = \{ x \in L | (a'')^n \rightarrow x'' \in f^{-1}(F), \exists n \in \mathbb{N} \} = D_a(f^{-1}(F))$$

(ii)

$$x \in D_y(\ker(f)) \iff (y'')^n \rightarrow x'' \in \ker(f), \text{ for some } n \in \mathbb{N} \iff f((y'')^n) \rightarrow f(x)' = 1 \iff f((y'')^n) \rightarrow f(x)' = 1 \iff f(x) \in D_{f(y)}(\{1\})$$

(iii) Let $F \in U(L)$. It suffices to show that $f^{-1}(D_a(F)) \in \tau_a$. By Corollary 3.4 and (i), we get $f^{-1}(D_a(F)) = f^{-1}(D_y(F)) = D_a(f^{-1}(F))$. Hence $f^{-1}(D_a(F)) \in \tau_a$ and so $f$ is a continuous map. \(\square\)

In the following, we want to verify some of the properties of topological space $(L, \tau_a)$, for any $a \in L$.

**Proposition 3.10.** The set $B_a = \{ D_a(\uparrow x) | x \in L \}$ is a base for the topology $\tau_a$ on $L$.

Proof. Let $X$ be an open subset of $(L, \tau_a)$. Then there exists $F \in U(L)$ such that $X = D_a(F)$. Since $F$ is an upset, then $F = \cup \{ \uparrow x | x \in F \}$ and so by Proposition 3.3(viii), $D_a(F) = \cup \{ D_a(\uparrow a) | a \in F \}$. Hence $B_a$ is a base for the topology $\tau_a$ on $L$. \(\square\)

**Proposition 3.11.** Let $a, b \in L$ such that $a \leq b$. Then the topology $\tau_b$ is finer than topology $\tau_a$.

Proof. Let $z \in L$ and $D_a(\uparrow x)$ be an element of $B_a$ containing $z$. Then there exists $n \in \mathbb{N}$ such that $(a'')^n \rightarrow z'' \in \uparrow x$. By Proposition 3.3(ii), $z \in D_b(\uparrow z)$. Since $a \leq b$, then by Proposition 3.3(vii), $D_b(\uparrow z) \subseteq D_a(\uparrow z)$. Now, we show that $D_a(\uparrow z) \subseteq D_b(\uparrow x)$. Let $u \in D_a(\uparrow z)$. Then there exists $m \in \mathbb{N}$ such that $z \leq (a'')^m \rightarrow u''$ and so by (B5) and (B19), $z'' \leq ((a'')^m \rightarrow (a'')^m \rightarrow u'') = (a'')^m \rightarrow u''$. Hence by (B5), we get $(a'')^n \rightarrow z'' \leq (a'')^n \rightarrow ((a'')^m \rightarrow u'') = (a'')^m \rightarrow u''$ and so $x \leq (a'')^m \rightarrow u''$. It follows that $u \in D_b(\uparrow x)$. Thus $D_a(\uparrow z) \subseteq D_a(\uparrow x)$ and so $D_b(\uparrow z) \subseteq D_a(\uparrow x)$ by Lemma 2.10, $\tau_b$ is finer than topology $\tau_a$. \(\square\)

**Theorem 3.12.** Let $F$ be a non-empty subset of $L$ and $a \in L$. Then $F$ is a compact subset of $(L, \tau_a)$ if and only if $F \subseteq D_a(\uparrow \{ x_1, x_2, ..., x_n \})$, for some $x_1, x_2, ..., x_n \in F$.

Proof. Suppose that $F \subseteq D_a(\uparrow \{ x_1, x_2, ..., x_n \})$, for some $x_1, x_2, ..., x_n \in F$. Let $\{ D_a(F_i) | i \in I \}$ be a family of open subsets of $L$ whose union contains $F$. Then for any $j \in \{ 1, 2, ..., n \}$, there exists $i_j \in I$ such that $x_{i_j} \in D_a(F_{i_j})$ and so by Proposition 3.3(i) and (iv), $D_a(x_{i_j}) \subseteq D_a(F_{i_j})$, for any $j \in \{ 1, 2, ..., n \}$. Now, by Proposition 3.3(viii) we obtain

$$F \subseteq D_a(\uparrow \{ x_1, x_2, ..., x_n \}) = D_a(x_{i_1}) \cup D_a(x_{i_2}) \cup ... \cup D_a(x_{i_n}) \subseteq D_a(F_{i_1}) \cup D_a(F_{i_2}) \cup ... \cup D_a(F_{i_n})$$

Hence $F$ is compact. Conversely, let $F$ be a compact subset of $L$. Since $F \subseteq \cup \{ \uparrow x | x \in F \}$, then by Proposition 3.3(ii), $F \subseteq \cup \{ D_a(\uparrow x) | x \in F \}$. Hence $\{ D_a(\uparrow x) | x \in F \}$ is a family of open subsets of $L$ whose union contains $F$. By assumption, there are $x_1, x_2, ..., x_n \in F$ such that

$$F \subseteq D_a(\uparrow x_1) \cup D_a(\uparrow x_2) \cup ... \cup D_a(\uparrow x_n)$$

It follows from Proposition 3.3(viii) that $F \subseteq D_a(\uparrow \{ x_1, x_2, ..., x_n \})$. \(\square\)
Let $F$ be a filter of $L$, $L/F$ be the quotient $BL$ algebra with respect to $F$, $\pi : L \rightarrow L/F$ be the canonical epimorphism and $\tau$ be a topology on $L$. Then we define a topology on $L/F$ whose open subsets are $\pi^{-1}(U)$, for any $U \in \tau$. This topology on $L/F$ is called the quotient topology induced by $\pi F$. It is well known that it is the largest topology on $L/F$ making $\pi F$ continuous. Suppose that $\tau$ is the quotient topology on $L/F$. If $V \in \tau$ then there exists $U \in \tau$ such that $\pi F(U) = V$ (see [4]).

**Theorem 3.13.** Let $a, b \in L$ and $F,G$ be two filters of $L$. Consider the topological space $(L, \tau_b)$. Then there exists a homeomorphism from $\frac{L \times L}{\mathcal{D}_a(F \times G)}$ to $\frac{L}{\mathcal{D}_a(F)} \times \frac{L}{\mathcal{D}_a(G)}$.

**Proof.** Define $\varphi : L \times L \rightarrow \frac{L \times L}{\mathcal{D}_a(F \times G)}$, by $\varphi((x,y)) = (x/D_a(F), y/D_a(G))$. Clearly, $\varphi$ is an onto homomorphism. Let $(x,y) \in L \times L$. Then

$$(x,y) \in \text{Ker}(\varphi) \iff x/D_a(F) = 1/D_a(F) \text{ and } y/D_a(G) = 1/D_a(G) \iff x \in D_a(F) \text{ and } y \in D_a(G)$$

Hence $\text{ker}(F) = D_a(F) \cap D_a(G)$. It can be easily obtained that $D_a(F) \cap D_a(G) = D_a(F \times G)$. It follows from the first isomorphism theorem that the map $\overline{\varphi} : \frac{L \times L}{\mathcal{D}_a(F \times G)} \rightarrow \frac{L}{\mathcal{F}_a(F)} \times \frac{L}{\mathcal{F}_a(G)}$ is defined by $\overline{\varphi}(x,y)/D_a(F \times G) = (x/D_a(F), y/D_a(G))$ is an isomorphism. Suppose that $X$ be an open subset of $\frac{L}{\mathcal{F}_a(F)} \times \frac{L}{\mathcal{F}_a(G)}$. Then there exist open subset $U, V \in \tau_b$ such that $X = U_D/F \times V_D/G$. Clearly, $\overline{\varphi}^{-1}(X) = U_D/V_D$, which is an open subset of $\frac{L \times L}{\mathcal{D}_a(F \times G)}$. Hence $\overline{\varphi}$ is a continuous map. By the similar way, we can show that $\overline{\varphi}^{-1}$ is continuous. Therefore, $\overline{\varphi}$ is a homeomorphism. \(\square\)

**Proposition 3.14.** Let $F$ be a filter of $L$ and $a \in L$. Then

(i) for any $x \in L$, $D_a(\uparrow x)/F = D_a(F)(\uparrow x/F)$,

(ii) the topologies $\tau_a/F$ and $\tilde{\tau}_a$ on $L/F$ are the same, where $\tilde{\tau}_a$ is the quotient topology on $L/F$.

**Proof.** (i) Let $x \in L$. Then

$$D_a(F)(\uparrow x/F) = \{ u/F \in L/F | x/F \leq ((a/F)^{*})^n \rightarrow (u/F)^{*} \} = \{ u/F \in L/F | x \rightarrow [(a/F)^{*}]^n \rightarrow u/F \}$$

and $D_a(\uparrow x)/F = \{ v/F | v \in D_a(\uparrow x) \}$. Let $u/F \in D_a(\uparrow x)/F$ and so there exists $v \in D_a$ such that $u/F = v/F$ and so $x \rightarrow ((a/F)^{*})^n \rightarrow v/F = 1$, for some $n \in \mathbb{N}$. Since $u \equiv v$, then we get $x \rightarrow ((a/F)^{*})^n \rightarrow u/F = 1$ and so $x \rightarrow ((a/F)^{*})^n \rightarrow u/F \in F$. Hence $u/F \in D_a(F)(\uparrow x/F)$. Now, let $u/F \in D_a(\uparrow x)/F$. Then there exists $n \in \mathbb{N}$ such that $x \rightarrow [(a/F)^{*}]^n \rightarrow u/F \in F$. Let $f = x \rightarrow ((a/F)^{*})^n \rightarrow u/F$. Then by (B4) we obtain $x \rightarrow [(a/F)^{*}]^n \rightarrow [f \rightarrow u/F] = f \rightarrow [x \rightarrow [(a/F)^{*}]^n \rightarrow u/F] = 1$ and so by (B8) and (B19),

$$1 = x \rightarrow ((a/F)^{*})^n \rightarrow [f \rightarrow u/F]$$

Hence $f \rightarrow u \in D_a(\uparrow x)$. Since $f \in F$, then $f/F = 1/F$ and so

$$(f/F) = f/F \rightarrow u/F = f/F \rightarrow u/F = 1/F \rightarrow u/F = (1/u)/F = u/F$$

That is $(f/F) \equiv u/F$. Therefore, $u/F \in D_a(\uparrow x)/F$.

(ii) By Proposition 3.10, $\{ D_a(F)(\uparrow x/F) | x/F \in L/F \}$ is a base for $\tau_a/F$ on $L/F$. Moreover,

$$\{ D_a(\uparrow x)/F | x \in L \}$$

is a base for $\tau_a$. Now, the proof of (ii) is straightforward by (i). \(\square\)

**Proposition 3.15.** The topological space $(L, \tau_n)$ is connected.

**Proof.** Let $A$ be a non-empty subset of $L$ which is both closed and open. Then there exists an upset $F$ of $L$ such that $A = D_a(F)$. If $0 \in A$ and so by Proposition 3.3(i), $A = L$. Let $0 \in L - A$. Since $A$ is closed, then $L - A$ is open and so by Proposition 3.3(i), $L - A$ is an upset. Now, $0 \in L - A$ implies that $L - A = L$. Hence $A = \emptyset$, which is a contradiction. It follows that $\{ L, \emptyset \}$ is the set of all subset of $L$ which are both closed and open. Therefore, $L$ is connected. \(\square\)
**Definition 3.16.** [3] Let \( U \) be a topology on BL-algebra \( L \). If \((L, \{\ast\}, U)\), where \( \{\ast\} \subseteq \{\lor, \land, \circ, \to\} \), be a (semi)topological algebra, then \((L, \{\ast\}, U)\) is a (semi)topological BL-algebra. If \( \{\ast\} = \{\lor, \land, \circ, \to\} \), then we consider \((L, U)\) instate of \((L, \{\lor, \land, \circ, \to\}, U)\), for simplicity.

**Theorem 3.17.** Let \( a \in L \). Then

(i) \((L, \{\lor, \land, \circ\}, \tau_a)\) is a semitopological BL-algebra.

(ii) \((L, \{\to\}, \tau_a)\) is a right semitopological BL-algebra.

**Proof.** (i) Let \( b \) be an arbitrary element of \( L \).

(1) Consider the map \( f_b : L \to L \) is defined by \( f_b(x) = b \circ x \), for any \( x \in L \). Since by Proposition 3.10, \( D_a(\uparrow x) \) is a base for \( \tau_a \), then it suffices to show that \( f_b^{-1}(D_a(\uparrow x)) \in \tau_a \), for any \( x \in L \).

Let \( A = \{u \in L | x \leq (a'')^n \to (b \circ u)'', \text{ for some } n \in \mathbb{N}\} \), \( u \in A \) and \( u \leq v \), for some \( v \in L \). Then there exists \( n \in \mathbb{N} \) such that \( x \leq (a'')^n \to (b \circ u)''. \) By (B5) and (B12), \((a'')^n \to (b \circ u)''' \leq (a''')^n \to (b \circ v)'''\) and so \( v \in A \). Hence \( A \subseteq U(L) \). Now, we show that \( D_a(A) = A \). Then we get \( A \in \tau_a \) and so \( f_b \) is a continuous map. Let \( u \in D_a(A) \). Then there exists \( m \in \mathbb{N} \) such that \((a''')^m \to (u'') \in A \) and so \( x \leq (a'')^n \to (b \circ ((a''')^m \to (u'')))'' \), for some \( n \in \mathbb{N} \).

\[
[(a'')^n \to (b \circ ((a''')^m \to (u'')))'''] = [(a'')^{n+m} \to (b \circ u)'''] = [(a'')^{n+m} \to (b \circ u)'''] \geq [(b \circ ((a''')^m \to (u''))'''] \to [(a''')^m \to (b \circ u)'''], \text{ by (B6)}
\]

\[
= [b \circ ((a''')^m \to (u''))'''] \to [a'' \to (b \circ u)]''', \text{ by (B4) and (B19)}
\]

\[
= ([b \circ ((a''')^m \to (u''))]'' \to [a'' \to (b \circ u)])''', \text{ by (B4)}
\]

\[
\geq [b \to (u'') \to (b \circ u)]''', \text{ by (B6)}
\]

\[
= b''' \to (u'' \to (b \circ u)''), \text{ by (B8) and (B19)}
\]

\[
= [b \to (u \to (b \circ u))]''', \text{ by (B19)}
\]

\[
= [(b \circ u) \to (b \circ u)]'''] = 1
\]

Hence \( x \leq (a''')^m \to (b \circ ((a''')^m \to (u'')))'' \leq (a''')^m \to (b \circ u)'' \) and so \( u \in A \). Therefore, by Proposition 3.3(ii), \( D_a(A) = A \). That is \( f_b \) is a continuous map.

(2) Let \( g_b : L \to L \) be a map was defined by \( g_b(x) = b \lor x \), for any \( x \in L \).

\[
g_b^{-1}(D_a(\uparrow x)) = \{u \in L | g_b(u) \in D_a(\uparrow x)\}
\]

\[
= \{u \in L | (a''')^n \to (b \lor u)''' \in \uparrow x, \text{ for some } n \in \mathbb{N}\}
\]

\[
= \{u \in L | x \leq (a''')^n \to (b \lor u)''', \text{ for some } n \in \mathbb{N}\}
\]

Let \( B = \{u \in L | x \leq (a''')^n \to (b \lor u)''', \text{ for some } n \in \mathbb{N}\} \). Then similar to (1), we can show that \( B \) is an upset of \( L \). Let \( u \in D_a(B) \). Then there exists \( m \in \mathbb{N} \) such that \((a''')^m \to u'' \in B \) and so
we conclude that $x \leq (a'')^n \to (b \lor ((a'')^m \to u''))''$, for some $n \in \mathbb{N}$. Hence by

$$[(a'')^n \to (b \lor ((a'')^m \to u''))]' \to [(a'')^{n+m} \to (b \lor u'')]$$

$$\geq [(b \lor ((a'')^m \to (u'')))' \to [(a'')^m \to (b \lor u'')]', \text{ by (B6)}$$

$$= [b \lor ((a'')^m \to (u''))]'' \to [a^m \to (b \lor u'')]''', \text{ by (B4) and (B19)}$$

$$= ([b \lor ((a'')^m \to (u''))] \lor [(a'')^m \to (u'')])'' \to [a^m \to (b \lor u'')]'$$

$$\geq (1 \lor [(a'')^n \to (u'')]) \lor [(a'')^m \to (b \lor u'')]$$

$$= u'' \to (b'' \lor u''), \text{ by (B8) and (B19)}$$

$$= 1$$

we conclude that $x \leq (a'')^{n+m} \to (b \lor u'')''$ and so $u \in A$. By Proposition 3.3(ii), $D_a(A) = A$. Therefore, $g_b$ is a continuous map.

(3) Let $h_b : L \to L$ be a map was defined by $h_b(x) = b \land x$, for any $x \in L$. Then

$$h_b^{-1}(D_a(\uparrow x)) = \{u \in L | x \leq (a'')^n \to (b \lor u'')', \text{ for some } n \in \mathbb{N}\}$$

Similar to (1), we can show that $h_b^{-1}(D_a(\uparrow x))$ is an upset. Let $u \in D_a(h_b^{-1}(D_a(\uparrow x)))$. Then there exists $m \in \mathbb{N}$ such that $(a'')^m \to u'' \in h_b^{-1}(D_a(\uparrow x))$ and so $x \leq (a'')^n \to (b \land ((a'')^m \to u''))''$, for some $n \in \mathbb{N}$.

$$[(a'')^n \to (b \land ((a'')^m \to (u''))')]'' \to [(a'')^{n+m} \to (b \land u'')]$$

$$\geq [b \land ((a'')^m \to (u''))]'' \to [(a'')^m \to (b \land u'')]', \text{ by (B6)}$$

$$= [b'' \land ((a'')^m \to (u''))]'' \to [(a'')^m \to (b \land u'')]', \text{ by (B19)}$$

$$= [b'' \lor [(a'')^m \to (b \land u'')] \lor [(a'')^m \to (u'')]] \lor [(a'')^m \to (b \land u'')]$$

$$\geq [b'' \lor [(a'')^m \to (b \land u'')] \lor [u'' \to (b \land u'')]] \lor [(a'')^m \to (b \land u'')]$$

$$= [b'' \lor [(a'')^m \land u''] \lor (b \land u'')] \lor [(a'')^m \to (b \land u'')]$$

$$= 1, \text{ by (B19)}$$

Hence $x \leq (a'')^n \to (b \land ((a''^m \to (u''))'))'' \leq (a'')^{n+m} \to (b \land u'')'$ and so $u \in h_b^{-1}(D_a(\uparrow x))$. It follows that, $h_b$ is a continuous map.

From (1), (2) and (3), we get that $(L, \{\lor, \land, \circ\}, \tau_a)$ is a semitopological $BL$-algebra.

(ii) We show that the map $f_b : L \to L$ is defined by $f_b(x) = b \to x$ is a continuous map, for any $b \in L$. Let $b \in L$ and $D_a(\uparrow x)$ be an arbitrary element of $B_a$. Then

$$f_b^{-1}(D_a(\uparrow x)) = \{u \in L | b \to u \in D_a(\uparrow x)\} = \{u \in L | x \leq (a'')^n \to (b \to u'')', \text{ for some } n \in \mathbb{N}\}$$

Let $A = \{u \in L | x \leq (a'')^n \to (b \to u'')', \text{ for some } n \in \mathbb{N}\}$. If $u \in A$ and $u \leq v$, for some $v \in L$, then there exists $n \in \mathbb{N}$ such that $x \leq (a'')^n \to (b \to u'')'$. Since $u \leq v$, then by (B5), $(a'')^n \to (b \to u'')'' \leq (a'')^n \to (b \to u'')''$ and so $v \in A$. Now, we show that $D_a(A) = A$. Let $u \in D_a(A)$, then there exists $m, n \in \mathbb{N}$ such that $(a'')^m \to (u'') \in A$ and $x \leq (a'')^m \to [(a'')^n \to (u'')]'$ by (B4) and (B8), we get

$$x \leq (a'')^m \to [(a''^n \to (u''))] = (a'')^{n+m} \to (u'')$$

That is $u \in A$ and so by Proposition 3.3(ii), $D_a(A) = A$. Therefore, $A \in \tau_a$ and so $(L, \{\to\}, \tau_a)$ is a right semitopological $BL$-algebra. \qed
4. Semitopological MV-algebra

In this section, we want to define the concept of semitopological MV-algebra and semitopological abelian l-group and obtain relation between these structures.

There are two equivalent definitions for MV-algebra.

Definition 4.1. [6] An MV-algebra is an algebra \((M, \oplus, *, 0)\) of type \((2, 1, 0)\) satisfying the following condition: for any \(x, y \in M\),

\[(M1) \quad (M, \oplus, 0) \text{ is a commutative monoid,}\]
\[(M2) \quad x^{**} = x,\]
\[(M3) \quad 1 \oplus x = 1, \text{ where } 1 = 0^*,\]
\[(M4) \quad x \oplus x^* = 1,\]
\[(M5) \quad (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x,\]

Definition 4.2. [11] A BL-algebra \((L, \lor, \land, \ominus, \rightarrow, 0, 1)\) is an MV-algebra if \(x'' = x\), for any \(x \in L\).

Theorem 4.3. [11] (i) Let \((L, \lor, \land, \ominus, \rightarrow, 0, 1)\) be an MV-algebra. Define \(x \oplus y = (x^r \otimes y^r)^r\) and \(x^r = x^r\), for any \(x, y \in L\), then \((L, \oplus, *, 0)\) is an MV-algebra as Definition 4.1.

(ii) Let \((M, \oplus, *, 0)\) be an MV-algebra. Define the maps \(\lor, \land, \ominus, \rightarrow: M \rightarrow M\), by \(x \lor y = (a^* \ominus b)^* \ominus b\), \(x \land y = (x^r \otimes y^r)^r\), \(x \ominus y = (x^r \otimes y^r)^r\) and \(x \rightarrow y = x^r \oplus y\), for any \(x, y \in L\). Then \((M, \lor, \land, \ominus, \rightarrow, 0, 1)\) is an MV-algebra as Definition 4.2.

Definition 4.4. (i) A semitopological BL-algebra \(L\) is called a semitopological MV-algebra if \(x'' = x\), for any \(x \in L\).

(ii) Let \((G, +, \lor, \land, \ominus, 0)\) be an abelian l-group and \((G, U)\) be a topological space. Then \(G\) is called a semitopological abelian l-group if \((G, \lor, \land, \ominus, 0)\) is a topological algebra.

Proposition 4.5. Let \((L, \land, \lor, \ominus, \rightarrow, \oplus, *, 0)\) be a semitopological MV-algebra and \(\oplus, *\) be the operation in Theorem 4.3(i). Then

(i) for any \(x \in L\), the map \(f_x: L \rightarrow L\), is defined by \(f_x(x) = x \oplus y\) is continuous.

(ii) \(*: L \rightarrow L\) is a continuous map.

(iii) the MV-algebra \((L, \ominus, *, 0)\) is a semitopological algebra.

Proof. Let \(V \in U\) and \(x \in L\). Then \(f^{-1}_x(V) = \{y \in L | x \oplus y \in V\} = \{y \in L | (x^r \otimes y^r)^r \in V\}\). Since for any \(a \in L\), \(a^r = a\), then

\[
f^{-1}_x(V) = \{y \in L | x^r \otimes y^r \in V\}
= \{y \in L | y^r \in \left( f^{-1}_x(V) \right)^r \}
= \{y \in L | y \in \left( f^{-1}_x(V) \right)^r \}
= \left( f^{-1}_x(V) \right)^r
\]

By assumption, \((L, \land, \lor, \ominus, \rightarrow, \oplus, *, 0)\) is a semitopological MV-algebra. Thus \(V' \in U\) and \(f_x\) is a continuous map and so \(f^{-1}_x(V') = f^{-1}_x(V') \in U\) and \(\left( f^{-1}_x(V') \right)^r \in U\). Hence \(f_x: L \rightarrow L\) is continuous.

(ii) Let \(\alpha: L \rightarrow L\) defined by \(\alpha(x) = x \rightarrow 0\), for any \(x \in L\). Clearly, \(\alpha = *\). Since \((L, \land, \lor, \ominus, \rightarrow, \oplus, *, 0)\) is a semitopological MV-algebra, then \(\alpha\) is a continuous map.

(iii) By Theorem 4.3, we know that \((L, \oplus, *, 0)\) is an MV-algebra. The proof of this part follows from (i) and (ii). □

D. Mundici in [13] shown that if \((G, +, \lor, \land, \ominus, 0)\) is an abelian l-group and \(0 \leq u \in G\), then \(([0, u], \oplus, *, 0)\) is an MV-algebra, where \(\ominus\) and \(*\) were defined by

\[
x \oplus y = (x + y) \land u, \quad x^* = u - x, \text{ for any } x, y \in [0, u]
\]

He denoted this MV-algebra by \(\Gamma(G, u)\). More generally, Mundici proved that every MV-algebra is isomorphic to \(\Gamma(G, u)\), for some abelian l-group \(G\) and some positive element \(u \in G\).
By considering the Definition 4.1, we can propose another definition of semitopological $MV$-algebra in the following form:

**Definition 4.6.** Let $U$ be a topology on $MV$-algebra $(M, \oplus, *, 0)$. If $(M, \{\oplus\}, U)$ is a semitopological algebra and $*: M \to M$ is a continuous map, then $(M, U, \oplus, *, 0)$ is called a *semitopological $MV$-algebra*. For simply we write $(M, U, \oplus, *, 0)$ is a semitopological $MV$-algebra.

**Theorem 4.7.** Let $(M, U, \oplus, *, 0)$ be a semitopological $MV$-algebra and $\forall, \land, \lor, \to: M \to M$ be the operations were defined in Theorem 4.3(ii). Then $(M, \forall, \land, \lor, \to, U)$ is a semitopological $MV$-algebra, which was introduced in Definition 4.4(i).

**Proof.** By Theorem 4.3, $(M, \forall, \land, \lor, \to, 0, 1)$ is an $MV$-algebra. Let $x \in M$. Define $f_x: M \to M$ by $f_x(g) = x \circ g$. Since $(M, U, \oplus, *, 0)$ is a semitopological $MV$-algebra, then the operation $*$ and the map $g_\lambda: M \to M$ was defined by $g_\lambda(x) = x \circ y$ is a continuous map, for any $x \in M$. It can be easily obtained that $f_x(y) = (g_{x*}(y))$. Hence $f_x$ is a composite of two continuous maps and so it is continuous. On the other hand, by $(M1)$, we get $a' = a \to 0 = a^* + 0 = a^*$, for any $a \in M$ and so $'$ is a. Now, by Theorem 3.14 [3], we get $(M, \forall, \land, \lor, \to, U)$ is a semitopological $MV$-algebra.

**Corollary 4.8.** Definition 4.4(i) and 4.6 are equivalent.

**Proof.** It follows from Proposition 4.5 and Theorem 4.7.

**Theorem 4.9.** Let $(G, U)$ be a semitopological abelian $l$-group such that the operation $\land: G \to G$ is continuous map and $0 \leq u$, for some $u \in G$. Then $(\{0, u\}, \tau, \lor, *, 0)$ is a semitopological $MV$-algebra, where $(\{0, u\}, \tau)$ is a subspace topology of $(G, U)$.

**Proof.** Let $x \in \{0, u\}$. First, we show that the maps $f_x: [0, u] \to [0, u]$ was defined by $f_x(y) = x \lor y$, for any $y \in [0, u]$ is continuous. Suppose that $V \in \tau$. Then there exists $W \in U$ such that $V = W \cap [0, u]$. Define $\alpha: G \to G$, by $\alpha(y) = x \lor y$ and $\beta: G \to G$ by $\beta(y) = y \land u$, for any $y \in G$. Since $(G, U)$ is a semitopological abelian $l$-group, then $\alpha$ and $\beta$ are continuous and so $\beta\alpha\circ\mu$ is continuous. Then by

$$f_x^{-1}(V) = \{y \in [0, u]| x \lor y \in V\}$$

$$= \{y \in [0, u]| (x \lor y) \land u \in V\}$$

$$= \{y \in [0, u]| (\beta(\alpha(y)) \in W \cap [0, u])\}$$

$$= \{y \in [0, u]| (\beta(\alpha(y)) \in (\beta\alpha)^{-1}(W) \cap [0, u])\}$$

$$= [0, u] \cap (\beta\alpha)^{-1}(W) \cap [0, u]$$

we conclude that $f_x^{-1}(V) = \beta(\alpha(0)) \cap [0, u]$. Since $\beta\alpha\circ\mu$ is continuous and $W \in U$, then $(\beta\alpha)^{-1}(W) \in U$ and so $f_x^{-1}(V) \in \tau$. Now, we show that $*: [0, u] \to [0, u]$ is continuous. Define $\lambda: G \to G$ by $\lambda(y) = u - y$ and $\mu: G \to G$ by $\mu(y) = -y$, for any $y \in G$. Then by assumption $\lambda$ and $\mu$ are continuous. Since $y^* = \lambda(\mu(y))$, for any $y \in [0, u]$, then we have

$$\{y \in [0, u]| y^* \in V\} = \{y \in [0, u]| \lambda(\mu(y)) \in W \cap [0, u]\}$$

$$= [0, u] \cap (\lambda\mu)^{-1}(W) \cap [0, u]$$

Hence $\{y \in [0, u]| y^* \in V\} \in \tau$ and so $*$ is a continuous map. Therefore, $(\{0, u\}, \tau, \lor, *, 0)$ is a semitopological $MV$-algebra.

**References**


