Common Fixed Point Theorems for a pair of Weakly Compatible Mappings in Fuzzy Metric Spaces using Common Limit in the Range Property

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Abstract: The aim of this work is to introduce the new property which is so called “common limit in the range” for four self-mappings and give some examples of mappings which satisfy this property. Moreover, we establish some new existence of a common fixed point theorem for generalized contractive mappings in fuzzy metric spaces by using new property and give some examples. Our results do not require the condition of closeness of range and so our theorems generalize, unify and extend many results in the literature.

Key words: Weakly Compatible Maps, Fuzzy metric space, property (E.A), Common property (E.A), CLRg property, Common limit in range property.

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1. Introduction
The concept of fuzzy sets was introduced by Zadeh [28] in 1965. In 1975, Kramosil and Michalek [14] gave the notion of fuzzy metric spaces which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger
On the other hand, fixed point theory is one of the most famous mathematical theories with applications in several branches of science. Fixed point theory in fuzzy metric spaces has been developed starting with the work of Heilpern [9]. He introduced the concept of fuzzy contraction mappings and proved some fixed point theorems for fuzzy contraction mappings in metric linear spaces, which is a fuzzy extension of the Banach contraction principle. In [6, 7], George and Veeramani introduced and studied the notion of fuzzy metric spaces which constitutes a modification of the one due to Kramosil and Michalek. From now on, by fuzzy metric we mean a fuzzy metric in the sense of George and Veeramani. Many authors have contributed to the development of this theory and apply to fixed point theory, for instance [1,3-5,8,10,11,15,16,18-20,22,24-27].

In 1976, Jungck [12] introduced the notion of commuting mappings. Afterward, Sessa [23] gave the notion of weakly commuting mappings. Jungck [13] defined the notion of compatible mappings to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible but the converse is not true. The concept of property \((E.A)\) in metric space has been recently introduced by Aamri and El Moutawakil [2]. The concept of property \((E.A)\) allows us to replace the completeness requirement of the space with a more natural condition of closeness of the range. In 2009, M. Abbas et al. [1] introduced the notion of common property \((E.A)\).

Recently in 2011, Sintunavarat and Kumam [25] introduced the concept of the common limit in the range property and also established the existence of common fixed point theorems for generalized contractive mappings satisfy this property in fuzzy metric spaces.

The aim of this work is to introduce the new property which is the so called “common limit in the range” for four self-mappings and give some examples of mappings which satisfy this property. Moreover, we establish some new existence of a common fixed point theorem for generalized contractive mappings in fuzzy metric spaces by using new property and give some examples. Ours results do not require the condition of closeness of range and so our theorems generalize, unify and extend many results in the literature.

2. Preliminaries

The concept of triangular norms \((t\text{-norms})\) is originally introduced by Menger [17] in study of statistical metric spaces.
**Definition 2.1** [21] A binary operation \( * : [0,1] \times [0,1] \rightarrow [0,1] \) is continuous \( t \)-norm if \( * \) satisfies the following conditions:

(i) \( * \) is commutative and associative;

(ii) \( * \) is continuous;

(iii) \( a * I = a \) for all \( a \in [0,1] \);

(iv) \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \) for all \( a,b,c,d \in [0,1] \).

Examples of \( t \)-norms are: \( a*b = \min\{a,b\} \), \( a*b = ab \) and \( a*b = \max\{a+b-1,0\} \).

**Definition 2.2** [6] A 3-tuple \( (X, M, *) \) is said to be a fuzzy metric space if \( X \) is an arbitrary set, \( * \) is a continuous \( t \)-norm, and \( M \) is fuzzy sets on \( X \times [0, \infty) \) satisfying the following conditions for all \( x, y, z \in X \) and \( s, t > 0 \)

(i) \( M(x, y, 0) = 0 \);

(ii) \( M(x, y, t) = 1 \) for all \( t > 0 \) if and only if \( x = y \);

(iii) \( M(x, y, t) = M(y, x, t) \);

(iv) \( M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \);

(v) \( M(x, y, .) : [0, \infty) \rightarrow [0, 1] \) is left continuous.

The function \( M(x, y, t) \) denotes the degree of nearness between \( x \) and \( y \) w.r.t. \( t \) respectively.

**Remark 2.3** [6] In fuzzy metric spaces \( (X, M, *) \) and \( M(x, y, *) \) are non-decreasing for all \( x, y \in X \).

**Definition 2.4** [6] Let \( (X, M, *) \) be a fuzzy metric space. Then a sequence \( \{x_n\} \) in \( X \) is said to be

(i) convergent to a point \( x \in X \) if, for all \( t > 0 \),

\[
\lim_{n \to \infty} M(x_n, x, t) = 1
\]

(ii) a Cauchy sequence if, for all \( t > 0 \) and \( p > 0 \),
\[
\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1.
\]

**Definition 2.5** [6] A fuzzy metric space \((X, M, \ast)\) is said to be complete if and only if every Cauchy sequence in \(X\) is convergent.

**Example 2.6** [6] Let \(X = \{1/n : n \in \mathbb{N}\} \cup \{0\}\) and let \(*\) be the continuous \(t\)-norm defined by \(a \ast b = ab\) (or \(a \ast b = \min\{a, b\}\)) respectively, for all \(a, b \in [0,1]\). For each \(t > 0\) and \(x, y \in X\), define \((X, M, \ast)\) by

\[
M(x, y, t) = \begin{cases} 
\frac{t}{t + |x - y|}, & t > 0, \\
0, & t = 0
\end{cases}
\]

Clearly, \((X, M, \ast)\) is a complete fuzzy metric space.

**Definition 2.7** [13] A pair of self mappings \((f, g)\) of a fuzzy metric space \((X, M, \ast)\) is said to be compatible if \(\lim_{n \to \infty} M(fg x_n, gfx_n, t) = 1\) whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = z\) for some \(z\) in \(X\).

**Definition 2.8** [13] Two self-mappings \(f\) and \(g\) of a fuzzy metric space \((X, M, \ast)\) are called non-compatible if there exists at least one sequence \(\{x_n\}\) such that \(\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = z\) for some \(z\) in \(X\) but either \(\lim_{n \to \infty} M(fg x_n, gfx_n, t) \neq 1\) or the limit does not exist.

**Definition 2.9** [8] Two self-mappings \(f\) and \(g\) on a non empty set \(X\) are said to be weakly compatible if \(fg x = gfx\) for all \(x\) at which \(fx = gx\).

**Definition 2.10** [2] A pair of self mappings \((f, g)\) on a fuzzy metric space \((X, M, \ast)\) is said to satisfy the property \((E.A)\) if there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} fx_n = z = \lim_{n \to \infty} gx_n\) for some \(z \in X\).

The class of \(E.A\). mappings contains the class of non compatible mappings.
Definition 2.11 [1] The pairs \((A, S)\) and \((B, T)\) on a fuzzy metric space \((X, M, *)\) are said to satisfy the common property \((E.A)\) if there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} A(x_n) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(y_n) = \lim_{n \to \infty} B(y_n) = p \quad \text{for some} \quad p \in X.
\]

Definition 2.12 [25] A pair of self mappings \((f, g)\) on an fuzzy metric space \((X, M, *)\) is said to satisfy the common limit in the range of \(g\) property \((CLRg)\) if there exists a sequence \(\{x_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = gz \quad \text{for some} \quad z \in X.
\]

Inspired by Sintunavarat et al. [25], we introduce the following:

Definition 2.13: The pairs \((A, S)\) and \((B, T)\) on a fuzzy metric space \((X, M, *)\) are said to share the common limit in the range of \(S\) property if there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} A(x_n) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(y_n) = \lim_{n \to \infty} B(y_n) = Sz \quad \text{for some} \quad z \in X.
\]

Example 2.14[8] Let \((X, M, *)\) be a fuzzy metric space with \(X = [-1, 1]\) and for all \(x, y \in X\)
\[
M(x, y, t) = \begin{cases} 
\frac{|x-y|}{t}, & t > 0, \\
0, & t=0.
\end{cases}
\]
Define self mappings \(A, B, S\) and \(T\) on \(X\) by
\[
A(x) = \frac{x}{3}, S(x) = x, T(x) = -x, B(x) = \frac{-x}{3}
\]
for all \(x \in X\). Then with sequences \(\{x_n = \frac{1}{n}\}\) and \(\{y_n = \frac{-1}{n}\}\) in \(X\), one can easily verify that
\[
\lim_{n \to \infty} A(x_n) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(y_n) = \lim_{n \to \infty} B(y_n) = S(0).
\]
This shows that the pairs \((A, S)\) and \((B, T)\) share the common limit in the range of \(S\) property.

Definition 2.15 [8] Two finite families of self mappings \(\{A_i\}_{i=1}^m\) and \(\{B_j\}_{j=1}^n\) on a set \(X\) are said to be pairwise commuting if
\[
(i) \ A_i A_j = A_j A_i, \quad i, j \in \{1, 2, 3,...m\},
\]
(ii) $B_i B_j = B_j B_i$, $i, j \in \{1, 2, 3, \ldots, n\}$.

(iii) $A_i B_j = B_j A_i$, $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{1, 2, 3, \ldots, n\}$.

**Definition 2.16** Let $*$ be a continuous $t$ - norm and let $\Psi_6$ be the set of all continuous functions $F : [0,1]^6 \rightarrow [0,1]$ satisfying the following conditions:

(F$_1$) $F$ is non- increasing in the fifth and sixth variables,

(F$_2$) if for some constant $k \in (0,1)$, we have

$$F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) \ast v\left(\frac{t}{2}\right)\right) \geq 1$$

or

$$F\left(u(kt), v(t), u(t), v(t), u(t), u\left(\frac{t}{2}\right) \ast v\left(\frac{t}{2}\right), 1\right) \geq 1$$

for any $t > 0$ and any non decreasing functions $u, v : (0, \infty) \rightarrow [0,1]$, then there exists $h \in (0,1)$ with $u(ht) \geq u(t) \ast v(t)$

(F$_3$) if, for some constant $k \in (0,1)$, we have $F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$ for any fixed $t > 0$ and any non decreasing function $u : (0, \infty) \rightarrow [0,1]$, then $u(kt) \geq u(t)$.

**Example 2.17:** Let $F : [0,1]^6 \rightarrow [0,1]$ be defined by $F(u_1, u_2, u_3, u_4, u_5, u_6) = \frac{u_1}{\min\{u_2, u_3, u_4, u_5, u_6\}}$

Clearly, $F$ in $\Psi_6$.

3. Main Results

**Lemma 3.1:** Let $A$, $B$, $S$ and $T$ be self mappings of a fuzzy metric space $(X, M, *)$ with $a \ast b = \min \{a, b\}$ satisfying the following:

(i) the pair $(A, S)$ (or $(B, T)$) satisfies the common limit in the range of $S$ property;

(ii) for any $x, y \in X$, $F$ in $\Psi_6$ and $t > 0$ such that
(iii) \( A(X) \subseteq T(X) \) (or \( B(X) \subseteq S(X) \)).

Then the pairs \( (A,S) \) and \( (B,T) \) share the common limit in the range of \( S \) property.

**Proof:** Suppose that the pair \( (A,S) \) satisfies the common limit in the range of \( S \) property, then there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} A(x_n) = \lim_{n \to \infty} S(x_n) = z \) for some point \( z \) in \( X \). Then as \( A(X) \subseteq T(X) \), hence for each \( \{x_n\} \), there exist \( \{y_n\} \) in \( X \) such that \( A(x_n) = T(y_n) \). Therefore, \( \lim_{n \to \infty} T(y_n) = z \). Thus in all, we have \( \lim_{n \to \infty} A(x_n) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(y_n) = z \). Now, we assert that \( \lim_{n \to \infty} B(y_n) = z \). By inequality (3.1), we get

\[
F \left( M(A_n, B_n, kt), M(S_n, T_n, t), M(S_n, A_n, t), \right. \\
M(T_n, B_n, t), M(T_n, A_n, t), M(S_n, B_n, t), \\
\left. n \to \infty \right) \geq 1
\]

\[
F \left( M(S_n, \lim_{n \to \infty} B_n, kt), M(S_n, S_n, t), M(S_n, S_n, t), \right. \\
M(S_n, \lim_{n \to \infty} B_n, t), M(S_n, S_n, t), M(S_n, \lim_{n \to \infty} B_n, t), \\
\left. n \to \infty \right) \geq 1
\]

\[
F \left( M(S_n, \lim_{n \to \infty} B_n, kt), 1, 1, \\
M(S_n, \lim_{n \to \infty} B_n, t), 1, M(S_n, \lim_{n \to \infty} B_n, t), \right. \\
\left. n \to \infty \right) \geq 1
\]

On the other hand, since

\[
M(S_n, \lim_{n \to \infty} B_n, t) \geq M \left( S_n, \lim_{n \to \infty} B_n, \frac{t}{2} \right) = M \left( S_n, \lim_{n \to \infty} B_n, \frac{t}{2} \right) * 1
\]

and as \( F \) is non-increasing in the sixth variable, we have, for any \( t > 0 \),

\[
F \left( M(S_n, \lim_{n \to \infty} B_n, kt), 1, 1, \\
M(S_n, \lim_{n \to \infty} B_n, t), 1, M(S_n, \lim_{n \to \infty} B_n, t), \right. \\
\left. n \to \infty \right) \geq 1
\]
which by using (F₂) gives \( \lim_{n \to \infty} B(y_n) = Sz \). Then the pairs \((A,S)\) and \((B,T)\) share the common limit in the range of \(S\) property.

**Theorem 3.2:** Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X,M,*\)) satisfying inequality (3.1). Suppose that

(i) the pair \((A,S)\) (or \((B,T)\)) satisfies the common limit in the range of \(S\) property;

(ii) \(A(X) \subseteq T(X)\) ( or \(B(X) \subseteq S(X)\)).

Then the pairs \((A,S)\) and \((B,T)\) each have a point of coincidence. Moreover, \(A, B, S\) and \(T\) have a unique common fixed point provided that both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

**Proof:** In view of Lemma 3.1, the pairs \((A, S)\) and \((B, T)\) share the common limit in the range of \(S\) property, that is there exists two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} A(x_n) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(y_n) = \lim_{n \to \infty} B(y_n) = Sz \text{ for some } z \text{ in } X.
\]

Firstly, we assert that \(Az = Sz\). By (3.1), we have

\[
F\left(\frac{M(Az,By_n,kt)}{M(Sz,Ty_n,t),M(Sz,Az,t)},\frac{M(Sz,Ty_n,t),M(Ty_n,Az,t)}{M(Sz,By_n,t)}\right) \geq 1
\]

\[
F\left(\frac{M(Az,Sz,kt)}{M(Sz,Sz,t),M(Sz,Az,t)},\frac{M(Sz,Sz,t),M(Sz,Az,t)}{M(Sz,Sz,t)}\right) \geq 1
\]

\[
F\left(\frac{M(Az,Sz,kt),1}{M(Sz,Az,t),1},\frac{M(Sz,Az,t)}{1}\right) \geq 1
\]

On the other hand, since

\[
M(Sz,Az,t) \geq M\left(Sz,Az,\frac{t}{2}\right) = M\left(Sz,Az,\frac{t}{2}\right) \ast 1
\]

and as \(F\) is non-increasing in the fifth variable, we have, for any \(t > 0\),
whic

which, by using (F_2) gives, \(Az = Sz\).

Since \(A(X) \subseteq T(X)\), there exists \(v\) in \(X\) such that \(Az = Tv\).

Secondly, we assert that \(Bv = Tv\). By (3.1), we get

\[
F \left( M(Az, Bv, kt), M(Sz, Tv, t), M(Sz, Az, t), M(Tv, Bv, t), M(Tv, Az, t), M(Sz, Bv, t) \right) \geq 1
\]

\[
F \left( M(Tv, Bv, t), M(Tv, Tv, t), M(Tv, Bv, t), M(Tv, Bv, t) \right) \geq 1
\]

\[
F \left( M(Tv, Bv, kt), 1, 1, M(Tv, Bv, t), 1, M(Tv, Bv, t) \right) \geq 1
\]

On the other hand, since

\[
M(Tv, Bv, t) \geq M \left(Tv, Bv, \frac{t}{2}\right) = M \left(Tv, Bv, \frac{t}{2}\right)^* 1
\]

and as \(F\) is non-increasing in the sixth variable, we have, for any \(t > 0\),

\[
F \left( M(Tv, Bv, kt), 1, 1, \frac{t}{2}, M(Tv, Bv, \frac{t}{2})^* 1 \right) \geq 1
\]

which, by using (F_2) gives \(Tv = Bv = Az = Sz\).

Since the pairs \((A, S)\) and \((B, T)\) are weakly compatible and \(Az = Sz\) and \(Tv = Bv\), we therefore have

\[
ASz = SAz = AAz = SSSz, \quad BTv = TBv = TTv = BBv.
\]
Finally, we assert that \( AAz = Az \). Again by (3.1), we have

\[
F \left( M(AAz, Bv, kt), M(SAz, Tv, t), M(SAz, AAz, t), M(Tv, Bv, t), M(Tv, AAz, t), M(SAz, Bv, t) \right) \geq 1
\]

\[
F \left( M(AAz, Bv, kt), M(AAz, Bv, t), M(AAz, AAz, t), M(Bv, Bv, t), M(Bv, AAz, t), M(AAz, Bv, t) \right) \geq 1
\]

\[
F \left( M(AAz, Az, kt), M(AAz, Az, t), M(AAz, Az, t), 1, M(Az, AAz, t), M(AAz, Az, t) \right) \geq 1
\]

which, by using \( F_3 \), we have \( AAz = Az = SAz \) and so \( Az \) is a common fixed point of \( A \) and \( S \). Similarly, one can easily prove that \( BBv = Bv = TBv \), that is \( Bv \) is common fixed point of \( B \) and \( T \). As \( Az = Bv \), we therefore have that \( Az \) is common fixed point of \( A, S, B \) and \( T \). The uniqueness of the common fixed point is an easy consequence of inequality (3.1).

By choosing \( A, B, S \) and \( T \) suitably, one can derive corollaries involving two or three mappings.

**Corollary 3.3:** Let \( A \) and \( S \) be self mappings of a fuzzy metric space \((X, M, *)\) satisfying:

(i) the pair \((A, S)\) satisfies the common limit in the range of \( S \) property;

(ii) \( A(X) \subset S(X) \)

(iii) \[
F \left( M(Ax, Ay, kt), M(Sx, Sy, t), M(Sx, Ax, t), M(Sy, Ay, t), M(Sx, Ax, t), M(Sy, Ay, t) \right) \geq 1.
\]

Then \( A \) and \( S \) have a point of coincidence. Moreover, \( A \) and \( S \) have a unique common fixed point provided that \( A \) and \( S \) are weakly compatible.

**Proof:** Taking \( B = A \) and \( T = S \) in Theorem 3.2, the result follows

**Corollary 3.4:** Let \( A, B, S \) and \( T \) be self mappings of a fuzzy metric space \((X, M, *)\) satisfying inequality (3.1). Suppose that

(i) the pairs \((A, S)\) and \((B, T)\) satisfy the common limit in the range of \( S \) property.

(ii) \( A(X) \subset T(X) \) (or \( B(X) \subset S(X) \)).
Then the pairs \((A, S)\) and \((B, T)\) have a point of coincidence each. Moreover, \(A, B, S\) and \(T\) have a unique common fixed point provided that both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

**Proof:** Proof easily follows on same lines of Theorem 3.2, using lemma 3.1.

**Theorem 3.5:** Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X, M, \ast)\) satisfying inequality (3.1). Suppose that

(i) the pair \((A,S)\) (or \((B,T)\)) satisfies property \((E.A.)\) and \(S(X)\) is a closed subspace of \(X\);

(ii) \(A(X) \subseteq T(X)\) (or \(B(X) \subseteq S(X)\)).

Then the pairs \((A, S)\) and \((B, T)\) each have a point of coincidence. Moreover, \(A, B, S\) and \(T\) have a unique common fixed point provided that both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

**Proof:** Suppose the pair \((A, S)\) satisfies property \((E.A.)\). Then there exists a sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} A(x_n) = \lim_{n \to \infty} S(x_n) = p \quad \text{for some} \quad p \in X.
\]

It follows from \(S(X)\) being a closed subspace of \(X\) that \(p = Sz\) for some \(z\) in \(X\) and then the pair \((A, S)\) satisfies the common limit in the range of \(S\) property. By Theorem 3.2, we get \(A, B, S\) and \(T\) have a unique common fixed point.

**Corollary 3.6:** Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X, M, \ast)\) satisfying inequality (3.1). Suppose that

(i) the pairs \((A, S)\) and \((B, T)\) satisfy the common property \((E.A.)\) and \(S(X)\) is a closed subspace of \(X\);

(ii) \(A(X) \subseteq T(X)\) (or \(B(X) \subseteq S(X)\)).

Then the pairs \((A, S)\) and \((B, T)\) each have a point of coincidence. Moreover, \(A, B, S\) and \(T\) have a unique common fixed point provided that both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

**Proof:** Since the pairs \((A,S)\) and \((B,T)\) satisfy the common property \((E.A.)\), there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} A(x_n) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} T(y_n) = \lim_{n \to \infty} B(y_n) = p \quad \text{for}
\]
some $p$ in $X$. It follows from $S(X)$ being a closed subspace of $X$ that $p = Sz$ for some $z$ in $X$ and then the pairs $(A, S)$ and $(B, T)$ share the common limit in the range of $S$ property. By Theorem 3.2, we see that $A$, $B$, $S$ and $T$ have a unique common fixed point.

Since the pair of non compatible mappings imply to the pair satisfying $E.A.$ property, we get the following corollary.

**Corollary 3.7:** Let $A$, $B$, $S$ and $T$ be self mappings of a fuzzy metric space $(X,M,*)$ satisfying inequality (3.1). Suppose that

(i) the pair $(A,S)$ (or $(B,T)$) is non compatible mappings and $S(X)$ is a closed subspace of $X$;

(ii) $A(X) \subset T(X)$ ( or $B(X) \subset S(X)$).

Then the pairs $(A, S)$ and $(B, T)$ each have a point of coincidence. Moreover, $A$, $B$, $S$ and $T$ have a unique common fixed point provided that both the pairs $(A, S)$ and $(B, T)$ are weakly compatible.

**Proof:** Since the pair of mappings $(A, S)$ are non-compatible mappings, we get $A$ and $S$ satisfy the $E.A.$ property. Therefore, by Theorem 3.5, we get $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.

**Remark:** Although the property $(E.A.)$ (common property $(E.A.)$) is an essential tool to claim the existence of common fixed points of some mappings, this property however requires the condition of closedness of $S(X)$. Note that Theorem 3.2 weakens the condition of closed subspaces of $S(X)$. Therefore it is most interesting to use the common limit in the range of $S$ property as another auxiliary tool to claim the existence of a common fixed point. However, all the main results in this paper are some of the choices for claim that the existence of common fixed point in fuzzy metric spaces. Our result may be the motivation to other authors for extending and improving these results to suitable tools for these problems.

As an application of Theorem 3.2, we prove a common fixed point theorem for four finite families of mappings on fuzzy metric spaces. While proving our result, we utilize Definition 2.13 which is a natural extension of commutativity condition to two finite families.
Theorem 3.8: Let \( \{ A_1, A_2, \ldots, A_m \} \), \( \{ B_1, B_2, \ldots, B_n \} \), \( \{ S_1, S_2, \ldots, S_p \} \) and \( \{ T_1, T_2, \ldots, T_q \} \) be four finite families of self mappings of a fuzzy metric space \((X, M, \ast)\) such that \( A = A_1 A_2 \ldots A_m \), \( B = B_1 B_2 \ldots B_n \), \( S = S_1 S_2 \ldots S_p \) and \( T = T_1 T_2 \ldots T_q \) satisfying the conditions (3.1) and

(i) the pair \((A, S)\) (or \((B, T)\)) satisfies the common limit in the range of \( S \) property;

(ii) \( A(X) \subset T(X) \) (or \( B(X) \subset S(X) \)).

Then

(a) the pairs \((A, S)\) and \((B, T)\) each have a point of coincidence.

(b) \( A_i, S_k, B_r \) and \( T_t \) have a unique common fixed point provided that the pairs of families \( \{A_i, \{S_k\}\} \) and \( \{B_r, \{T_t\}\} \) commute pairwise for all \( i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, p, \)

\[ r = 1, 2, \ldots, n, \quad t = 1, 2, \ldots, q. \]

Proof: As the pairs of families \( \{A_i, \{S_k\}\} \) and \( \{B_r, \{T_t\}\} \) commute pairwise, we first show that \( AS = SA \) since

\[
AS = (A_1 A_2 \ldots A_m)(S_1 S_2 \ldots S_p) = (A_1 A_2 \ldots A_{m-1})(A_m S_1 S_2 \ldots S_p)
\]

\[
= (A_1 A_2 \ldots A_{m-1})(S_1 S_2 \ldots S_p A_m) = (A_1 A_2 \ldots A_{m-2})(A_{m-1} S_1 S_2 \ldots S_p A_m)
\]

\[
= (A_1 A_2 \ldots A_{m-2})(S_1 S_2 \ldots S_p A_{m-1}A_m) = \ldots = A_1(S_1 S_2 \ldots S_p A_2 \ldots A_m)
\]

\[
= (S_1 S_2 \ldots S_p)(A_1 A_2 \ldots A_m) = SA.
\]

Similarly one can prove that \( BT = TB \) and hence that the pair \((A, S)\) is obviously compatible and \((B, T)\) is weakly compatible. Now using Theorem 3.2, we conclude that \( A, S, B \) and \( T \) have a unique common fixed point in \( X \), say \( z \).

Now, one needs to prove that \( z \) remains the fixed point of all the component mappings.

For this consider
\[ A(A_i z) = ((A_1 A_2 \ldots A_m)A_i)z = (A_1 A_2 \ldots A_{m-1})A_i z \]

\[ = (A_1 A_2 \ldots A_{m-1})(A_i A_{m})z = (A_1 A_2 \ldots A_{m-2})A_i z \]

\[ = (A_1 A_2 \ldots A_{m-2})(A_i A_{m-1} A_m)z = \ldots = A_i A_{m} z = A_i z. \]

Similarly, one can prove that

\[ A(S_k z) = S_k (A z) = S_k z, \quad S(S_k z) = S_k z, \]

\[ S(A_i z) = A_i z, \quad B(B_r z) = B_r z, \]

\[ B(T_t z) = T_t z, \quad T(T_t z) = T_t z \]

and

\[ T(B_r z) = B_r (T_t z) = B_r z. \]

This shows that (for all \( i, r, k \) and \( t \)) \( A_i z \) and \( S_k z \) are other fixed points of the pair \((A, S)\) whereas \( B_r z \) and \( T_t z \) are other fixed points of the pair \((B, T)\). As \( A, B, S \) and \( T \) have a unique common fixed point, so, we get

\[ z = A_i z = S_k z = B_r z = T_t z, \quad \text{for all } i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, p, \]

\[ r = 1, 2, \ldots, n, \quad t = 1, 2, \ldots, q. \]

This shows that \( z \) is a unique common fixed point of \( \{ A_i \}_{i=1}^m, \{ S_k \}_{k=1}^p, \{ B_r \}_{r=1}^n \) and \( \{ T_t \}_{t=1}^q \).

We conclude this paper with an example that demonstrates the validity of the hypotheses of Theorem 3.2.
Example 3.9: Let $(X, M, *)$ be a fuzzy metric space where $X = [0, 2)$ and a $t$-norm $*$ be defined by $a * b = \min\{a, b\}$ for all $a, b$ in $[0, 1]$ and $M$ be a fuzzy set on $X^2 \times (0, \infty)$ defined by

$$M(x, y, t) = \left[\exp\left(\frac{|x-y|}{t}\right)\right]^{-1}$$

for all $x, y$ in $X$ and $t > 0$.

Let $F : [0, 1]^6 \rightarrow R$ be defined by $F(u_1, u_2, u_3, u_4, u_5, u_6) = \frac{u_4}{\min\{u_2, u_3, u_4, u_5, u_6\}}$

Clearly, $F$ in $\Psi_6$. Define $A$, $B$, $S$ and $T$ by

$$Ax = Bx = 1, \quad S(x) = \begin{cases} 
1 & x \in \mathcal{Q} \\
2 & x \not\in \mathcal{Q}
\end{cases}$$

and $T(x) = \begin{cases} 
1 & x \in \mathcal{Q} \\
\frac{1}{3} & x \not\in \mathcal{Q}
\end{cases}$.

Clearly, the pair $(A, S)$ satisfies the common limit in the range of $S$ property and $A(X) \subseteq T(X)$.

$A$, $B$, $S$ and $T$ have a unique common fixed point $x = 1$.

References


