A method of calculating intersection of quadratic surfaces in quaternion algebra

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Introduction

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Non-translated quadratic surface in algebra of unit quaternions will be defined later.

Introduction

General features:

- each non-translated quadratic surface is defined by 10 coefficients
- corresponding configuration space is homeomorphic to S^3 with antipodal points identified
- ambient space is 4-dimensional

Quaternion algebra

Quaternion algebra $\mathbb{H}$

An algebra of hyper-complex numbers of form:

$$q = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} + d \quad (1)$$

which base elements $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ have the following multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

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Note: Multiplication in quaternion algebra is non-commutative!

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Subalgebra of unit quaternions

The subalgebra U of $\mathbb{H}$ such that for each $q \in U$:

$$q^* q = 1$$

where $q^* = -a\mathbf{i} - b\mathbf{j} - c\mathbf{k} + d$ is called **subalgebra of unit quaternions**.

Topology: Subalgebra of unit quaternions is a Lie algebra. It is a two-fold universal cover of SO(3) which is topologically homeomorphic to S^3 with antipodal points defined.

Quadratic surface in quaternion algebra

A non-translated quadratic surface in $\mathbb{H}$ can be written as a quadratic form in $\mathbb{R}^4$. Assuming that $q = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} + d \in \mathbb{H}$ and its corresponding vector $\bar{q} = [a, b, c, d]^T \in \mathbb{R}^4$ there is a function $\bar{S} \colon \mathbb{R}^4 \longrightarrow \mathbb{R}$:

$$\bar{S}(\bar{q}) = \bar{q}^T M \bar{q} = 0$$

where

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

which is equivalent to function $S \colon \mathbb{H} \longrightarrow \mathbb{R}$:

$$\begin{aligned} S(q) =& a_{11}a^2 + a_{22}b^2 + a_{33}c^2 + a_{44}d^2 + \\ & 2(a_{12}ab + a_{13}ac + a_{14}ad + a_{23}bc + a_{24}bd + a_{34}cd) = 0 \end{aligned}$$

it will be called shortly $\mathbb{H}$-quadric.

The following theorem allows a person to use a simpler Cartesian intersection routines instead of hyper-complex one.

Theorem

Let $\Gamma(\xi) = [\Gamma_x(\xi), \Gamma_y(\xi), \Gamma_z(\xi), \Gamma_w(\xi)]^T$ *be the intersection of two quadratic surfaces in homogeneous space* $\mathbb{P}^3$ *for* $\xi \in \mathbb{P}^1$.
Intersection of two $\mathbb{H}$*-quadrics is equal to:*

$$\bar{q}(\xi) = \pm \frac{\Gamma(\xi)}{\|\Gamma(\xi)\|} \tag{2}$$

Proof

Assume that in $\mathbb{H}$ there are given two $\mathbb{H}$-quadrics S_1 and S_2 with matrices M_1 and M_2. The intersection is a set of quaternions $\bar{q} = [a, b, c, d]^T$ satisfying:

$$\begin{cases} \bar{q}^T \bar{q} = 1 \\ \bar{q}^T M_1 \bar{q} = 0 \\ \bar{q}^T M_2 \bar{q} = 0 \end{cases}$$

It is impossible that all of a, b, c, d are equal to zero simultaneously because of the first equation. Assume for now that d is non-zero. The second and the third equation can be divided by d^2, resulting in:

$$\begin{cases} \mathbf{t}^T M_1 \mathbf{t} = 0 \\ \mathbf{t}^T M_2 \mathbf{t} = 0 \end{cases} \tag{3}$$

Proof

The newly introduced vector $\mathbf{t}$ is equal to $[\frac{a}{d}, \frac{b}{d}, \frac{c}{d}, 1]^T$. It can be observed that the two equations of (3) form a quadric intersection problem in $\mathbb{R}^3$. Important note: both $\mathbf{t}^T M_1 \mathbf{t}$ and $\mathbf{t}^T M_2 \mathbf{t}$ are not necessarily quadratic forms. Since t_4 is equal to 1, each formula may contain the parameter t in the first order as well as scalars. As a result, a person must consider an intersection of **two general quadrics** in $\mathbb{R}^3$. This problem can be effectively solved. Both quadrics are given in terms of $t_x = \frac{a}{d}, t_y = \frac{b}{d}, t_z = \frac{c}{d}$. Now, we assume that the intersection curve $\Gamma(\xi) = [\Gamma_x(\xi), \Gamma_y(\xi), \Gamma_z(\xi), \Gamma_w(\xi)]^T$ is in homogeneous coordinates:

$$t_x = \frac{\Gamma_x(\xi)}{\Gamma_w(\xi)}, t_y = \frac{\Gamma_y(\xi)}{\Gamma_w(\xi)}, t_z = \frac{\Gamma_z(\xi)}{\Gamma_w(\xi)}, \xi \in \mathbb{P}^1$$

Proof

To recover all of a, b, c, d, first d is computed. Summing up the squares of t_x, t_y and t_z one obtains:

$$t_x^2 + t_y^2 + t_z^2 = \frac{a^2}{d^2} + \frac{b^2}{d^2} + \frac{c^2}{d^2} = \frac{a^2 + b^2 + c^2}{d^2} = \frac{1 - d^2}{d^2} = \frac{1}{d^2} - 1$$

so,
$t_x^2 + t_y^2 + t_z^2 + 1 = \frac{1}{d^2}$ and $d^2 = \frac{1}{t_x^2+t_y^2+t_z^2+1}$
by plugging in the intersection curve Γ, one can write:

$$\begin{aligned} d^2 &= \frac{1}{\frac{\Gamma_x(\xi)^2}{\Gamma_w(\xi)^2} + \frac{\Gamma_y(\xi)^2}{\Gamma_w(\xi)^2} + \frac{\Gamma_z(\xi)^2}{\Gamma_w(\xi)^2} + \frac{\Gamma_w(\xi)^2}{\Gamma_w(\xi)^2}} \\ &= \frac{\Gamma_w(\xi)^2}{\Gamma_x(\xi)^2 + \Gamma_y(\xi)^2 + \Gamma_z(\xi)^2 + \Gamma_w(\xi)^2} \end{aligned}$$

A rotation by a quaternion q is identified with a rotation by a quaternion $-q$. Hence, in the above equation a square root can be taken of both sides without a loss of generality.

Proof

Finally one obtains:

$$d = \frac{\Gamma_w(\xi)}{\|\Gamma(\xi)\|}$$

where $\|\Gamma(\xi)\| = \sqrt{\Gamma_x(\xi)^2 + \Gamma_y(\xi)^2 + \Gamma_z(\xi)^2 + \Gamma_w(\xi)^2}$. The remaining quaternion coordinates are:

$$a = dt_x = \frac{\Gamma_w(\xi)}{\|\Gamma(\xi)\|} \frac{\Gamma_x(\xi)}{\Gamma_w(\xi)} = \frac{\Gamma_x(\xi)}{\|\Gamma(\xi)\|}$$

$$b = dt_y = \frac{\Gamma_w(\xi)}{\|\Gamma(\xi)\|} \frac{\Gamma_y(\xi)}{\Gamma_w(\xi)} = \frac{\Gamma_y(\xi)}{\|\Gamma(\xi)\|}$$

$$c = dt_z = \frac{\Gamma_w(\xi)}{\|\Gamma(\xi)\|} \frac{\Gamma_z(\xi)}{\Gamma_w(\xi)} = \frac{\Gamma_z(\xi)}{\|\Gamma(\xi)\|}$$

Proof

The above formulas can be finally rewritten as the $\mathbb{H}$-**quadric intersection parametrization**:

$$q(\xi) = \pm \frac{\Gamma_x(\xi)\mathbf{i} + \Gamma_y(\xi)\mathbf{j} + \Gamma_z(\xi)\mathbf{k} + \Gamma_w(\xi)}{\|\Gamma(\xi)\|}$$

or, equivalently in a vector form:

$$\bar{q}(\xi) = \pm \frac{[\Gamma_x(\xi), \Gamma_y(\xi), \Gamma_z(\xi), \Gamma_w(\xi)]^T}{\|\Gamma(\xi)\|}$$

$$\bar{q}(\xi) = \pm \frac{\Gamma(\xi)}{\|\Gamma(\xi)\|}$$

where $\Gamma := intersect(M_1, M_2)$.

Proof

A note is also needed about initial choice of d as the coordinate by which the remaining coordinates were divided. Because it is not possible that all of a, b, c, d are simultaneously zero, it is possible to non-constructively divide the $\mathbb{H}$ space into four subspaces in which the selected quaternion component is non-zero. In each of these fragments, the proof is repeated, regards to different quaternion component.

$\square$

Conclusions

- intersection of non-translated quadrics on S^3 sphere is not more difficult than intersection of general quadrics in $\mathbb{P}^3$ (nevertheless, it is still a complex problem!)
- an ingredient for other motion planning algorithms (involving 3D rotations)

Thank you