Modelling of Default Risk: Mathematicals Tools*

Monique Jeanblanc
Equipe d'Analyse et Probabilités, Université d'Evry Val d'Essonne
Boulevard des Coquibus, 91025 Evry Cedex, France
jeanbl@maths.univ-evry.fr

Marek Rutkowski
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland
markrut@mini.pw.edu.pl

March 30, 2000

Abstract

Our aim is to give a theoretical study for the modelling of default risk. We provide first a detailed analysis of the relatively simple case when the flow of informations available to an agent reduces to the observations of the random time which models the default event. Subsequently, the case of the general filtration is examined. The focus is on the evaluation of conditional expectations with respect to the filtration generated by a default time with the use of the intensity function, and on various versions of the martingale representation theorem. For a more detailed account of the potential applications of results presented in this work in the context of the modelling of default risk, we refer to the companion work: M. Jeanblanc and M. Rutkowski: *Modelling of Default Risk: An Overview.*

Contents

*Presented at the Workshop *Fixed Income and Credit Risk Modeling and Management,* New York University, Stern School of Business, Statistics and Operations Research Department, May 5, 2000.

1 Introduction

The aim of this paper is to give a theoretical study for the modelling of default risk, taking care to the meaning of the information. In the first part, we provide a detailed analysis of the relatively simple case when the flow of informations available to an agent reduces to the observations of the random time which models the default event. The focus is on the evaluation of conditional expectations with respect to the filtration generated by a default time with the use of the intensity function.

These results are then generalized to the case when an additional information flow - formally represented by some filtration $\mathbb{F}$ - is present. At the intuitive level, $\mathbb{F}$ is generated by prices of some assets, or by other economic factors (e.g., interest rates). Though in typical examples $\mathbb{F}$ is chosen to be the Brownian filtration, most theoretical results obtained do not rely on such a specification of the filtration $\mathbb{F}$. Special attention is paid here to the hypothesis (H), which postulates the invariance of the martingale property with respect to the enlargement of $\mathbb{F}$ by the observations of a default time. This hypothesis prevails in the literature, and means that all the contingent claims (with or without default, in particular the $\mathbb{F}$-measurable ones) are hedgeable. We establish a representation theorem, in order to understand the meaning of complete market in a defaultable world.

We examine non-trivial example of the calculation of the stochastic intensity of a default time (or rather of the dual predictable projection of the associated first jump process). Since in this section the underlying filtration $\mathbb{F}$ is assumed to be generated by a Brownian motion, and it is well known that all stopping time with respect to the Brownian filtration are predictable (so that they do not admit intensity with respect to $\mathbb{F}$), it is natural to examine random times which are not $\mathbb{F}$-stopping times. To be more specific, we study last passage time of a Brownian motion. The last section is devoted to analysis of the minimum of several random times and to the Kusuoka's example.

2 Hazard Function of a Random Time

In this section, the problem of quasi-explicit evaluation of various conditional expectations is studied in a very special case when the only filtration available in calculations is the natural filtration of a random time. In practical terms, we consider here an individual who observes a certain random time τ, but has no access to any other information. More general situations are examined in the next section.

2.1 Conditional Expectations w.r.t. Natural Filtrations

Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$, referred to as the *random time* in what follows. For convenience, we assume that $\mathbf{P}(\tau = 0) = 0$ and $\mathbf{P}(\tau > t) > 0$ for any $t \in \mathbb{R}_+$. We introduce processes $D_t = 1\!\!1_{\{\tau \le t\}}$ and $\tilde{D}_t = 1\!\!1_{\{\tau < t\}}$, for $t \in \mathbb{R}_+$. It is obvious that D ($\tilde{D}$, resp.) is a right-continuous (left-continuous, resp.) process. Moreover, $\tilde{D}$ is the left-continuous version of D. We introduce filtrations $\mathbb{D} = (\mathcal{D}_t)_{t \ge 0}$ and $\tilde{\mathbb{D}} = (\tilde{\mathcal{D}}_t)_{t \ge 0}$ generated by these processes, namely, for any $t \in \mathbb{R}_+$ we set (both filtrations below are assumed to be $(\mathbf{P}, \mathcal{G})$-completed) $\mathcal{D}_t = \sigma(D_u : u \le t)$, and $\tilde{\mathcal{D}}_t = \sigma(\tilde{D}_u : u \le t)$. The σ-field $\mathcal{D}_t$ represents the information generated by the observations of the random time τ up to time t and including t – that is, on the time interval $[0, t]$. The σ-field $\tilde{\mathcal{D}}_t$ represents the information generated by the observations of the random time τ strictly before time t – that is, on the time interval $[0, t[$. Finally, we set $\mathcal{D}_\infty = \sigma(D_u : u \in \mathbb{R}_+)$ and $\tilde{\mathcal{D}}_\infty = \sigma(\tilde{D}_u : u \in \mathbb{R}_+)$. We denote by $\sigma(\eta)$ the σ-field generated by a random variable η.

Let us first examine the basic properties of filtrations $\tilde{\mathbb{D}}$ and $\mathbb{D}$. We have:

(D.1) $\tilde{\mathcal{D}}_t = \sigma(\{\tau < u\} : u \le t,) = \sigma(\{\tau \le u\} : u < t,)$,
(D.2) $\tilde{\mathcal{D}}_t = \sigma(\tau \wedge t) = \sigma(\tau_t)$, where $\tau_t = \tau \wedge t$,
(D.3) $\mathcal{D}_t = \sigma(\{\tau \le u\} : u \le t)$,
(D.4) $\mathcal{D}_t = \sigma(\sigma(\tau) \cap \{\tau \le t\})$,

(D.5) $\mathcal{D}_t = \sigma(\tau \wedge t) \vee (\{\tau > t\}) = \tilde{\mathcal{D}}_t \vee (\{\tau > t\})$,
(D.6) $\tilde{\mathcal{D}}_t \subseteq \mathcal{D}_t$,
(D.7) $\mathcal{D}_t = \mathcal{D}_{t+}$,
(D.8) $\mathcal{D}_t = \tilde{\mathcal{D}}_{t+}$,
(D.9) $\tilde{\mathcal{D}}_\infty = \mathcal{D}_\infty = \sigma(\tau)$,
(D.10) for any $A \in \tilde{\mathcal{D}}_\infty$ we have $A \cap \{\tau < t\} \in \tilde{\mathcal{D}}_t$,
(D.11) for any $A \in \mathcal{D}_\infty$ we have $A \cap \{\tau \le t\} \in \mathcal{D}_t$.
To establish (D.10) or (D.11), it is enough to consider an arbitrary set A of the form $A = \{\tau \le s\}$ for some $s \in \mathbb{R}_+$. Notice also that for some $A \in \tilde{\mathcal{D}}_\infty$ we have $A \cap \{\tau \le t\} \notin \tilde{\mathcal{D}}_t$ (it is enough to take $A = \{\tau \le t\}$). It follows from (D.5) that any $\mathcal{D}_t$-measurable random variable D is equal to a function of $\tau \wedge t$, therefore it is constant on the set $\{\tau > t\}$, and equal to a function of τ on $\{\tau \le t\}$,.

We denote by F the right-continuous cumulative distribution function of τ, i.e., $F(t) = \mathbf{P}(\tau \le t)$. The function F is continuous (that is, $\mathbf{P}(\tau = t) = 0$ for any $t \in \mathbb{R}_+$) if and only if the following condition holds:
(D.12) $\tilde{\mathcal{D}}_t = \mathcal{D}_t$ for any $t \in \mathbb{R}_+$.

In what follows, we shall deal with the filtration $\mathbb{D}$ (the corresponding results for the filtration $\tilde{\mathbb{D}}$ are thus left to the reader as exercises).

In this section, it is assumed throughout that Y is an integrable random variable on the probability space $(\Omega, \mathcal{G}, \mathbf{P})$, that is, $\mathbf{E}\,|Y| < \infty$.

We start with some well known facts established for the first time in Dellacherie [7] or [8] (p.122), and used again in Chou and Meyer [5], Liptser and Shiryaev [21], Elliott [14], Dellacherie and Meyer [9] (p.237) or more recently in Rogers and Williams [24], among others.

Lemma 2.1 *Let Y be a $\mathcal{G}$-measurable random variable. Then*

$$\mathbf{E}\,(Y \,|\, \mathcal{D}_t) = \mathbb{1}_{\{\tau \le t\}} \mathbf{E}\,(Y \,|\, \tau) + \mathbb{1}_{\{\tau > t\}} \frac{\mathbf{E}\,(\mathbb{1}_{\{\tau > t\}} Y)}{\mathbf{P}(\tau > t)}. \tag{2.1}$$

Proof. We shall first check that

$$\mathbf{E}\,(\mathbb{1}_{\{\tau \le t\}} Y \,|\, \mathcal{D}_\infty) = \mathbf{E}\,(\mathbb{1}_{\{\tau \le t\}} Y \,|\, \mathcal{D}_t).$$

In view of (D.11), we have $A \cap \{\tau \le t\} \in \mathcal{D}_t$ for any $A \in \mathcal{D}_\infty$. Consequently

$$\begin{aligned}\int_A \mathbf{E}\,(\mathbb{1}_{\{\tau \le t\}} Y \,|\, \mathcal{D}_\infty)\, d\mathbf{P} &= \int_A \mathbb{1}_{\{\tau \le t\}} Y\, d\mathbf{P} = \int_{A \cap \{\tau \le t\}} Y\, d\mathbf{P} = \int_{A \cap \{\tau \le t\}} \mathbf{E}\,(Y \,|\, \mathcal{D}_t)\, d\mathbf{P} \\ &= \int_A \mathbb{1}_{\{\tau \le t\}} \mathbf{E}\,(Y \,|\, \mathcal{D}_t)\, d\mathbf{P} = \int_A \mathbf{E}\,(\mathbb{1}_{\{\tau \le t\}} Y \,|\, \mathcal{D}_t)\, d\mathbf{P}\end{aligned}$$

since the event $\{\tau \le t\}$ belongs to $\mathcal{D}_t$. We know that the conditional expectation is constant on the set $\tau > t$. Therefore, to establish (2.1), we need to identify the constant c such that

$$\mathbf{E}\,(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{D}_t) = c \mathbb{1}_{\{\tau > t\}},$$

which can be done taking the expectation of both members. □

For easy further reference, we give a special case of the formulae above. For any $t < s$ we have

$$\mathbf{P}(\tau > s \,|\, \mathcal{D}_t) = \mathbb{1}_{\{\tau > t\}} \mathbf{P}(\tau > s \,|\, \tau > t). \tag{2.2}$$

Definition 2.1 The increasing function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula

$$\Gamma(t) = -\ln(1 - F(t)), \quad \forall\, t \in \mathbb{R}_+, \tag{2.3}$$

is called the *hazard function* of τ. It is clear that relationship $F(t) = 1 - e^{-\Gamma(t)}$ is satisfied.

If the cumulative distribution function F is an absolutely continuous function, that is, $F(t) = \int_0^t f(u)\,du$, for some function $f:\mathbb{R}_+ \to \mathbb{R}_+$, then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)\,du}, \quad \forall\, t \in \mathbb{R}_+,$$

where $\gamma(t) = f(t)(1-F(t))^{-1}$. The function γ is called the *intensity function* of τ.

Notice that $\Gamma(t)$ is well defined for any t (since by assumption $F(t) < 1$ for every t) and $\Gamma(\infty) := \lim_{t\to\infty} \Gamma(t) = \infty$ (since $\lim_{t\to\infty}(1-F(t)) = 0$). Also, it is clear that the intensity function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ (if it exists) is a non-negative function, which is Lebesgue integrable on any bounded interval, and such that $\int_0^\infty \gamma(u)\,du = \infty$.

The straightforward consequence of (2.2) appears to be useful.

Lemma 2.2 *The process L given by the formula*

$$L_t := \frac{1-D_t}{1-F(t)} = 1\!\!1_{\{\tau>t\}} e^{\Gamma(t)} = (1-D_t)e^{\Gamma(t)}, \quad \forall\, t \in \mathbb{R}_+, \tag{2.4}$$

is a $\mathbb{D}$-martingale. Equivalently,

$$\mathbf{E}\,(D_s - D_t \,|\, \mathcal{D}_t) = (1-D_t)\frac{F(s)-F(t)}{1-F(t)} = 1\!\!1_{\{\tau>t\}}\,\frac{F(s)-F(t)}{1-F(t)}, \quad \forall\, t \le s, \tag{2.5}$$

Proof. Let us check, for instance, that the process L given by (2.4) is a $\mathbb{D}$-martingale. The first equality in (2.2) can be rewritten as follows

$$\mathbf{E}\,(1 - D_s \,|\, \mathcal{D}_t) = (1-D_t)\,\frac{1-F(s)}{1-F(t)} = 1\!\!1_{\{\tau>t\}}\, e^{\Gamma(t)-\Gamma(s)}.$$

This immediately yields the martingale property of L. □

Using the hazard function Γ, we may rewrite (2.1) as follows

$$\mathbf{E}\,(Y \,|\, \mathcal{D}_t) = 1\!\!1_{\{\tau\le t\}}\mathbf{E}\,(Y \,|\, \tau) + 1\!\!1_{\{\tau>t\}}\, e^{\Gamma(t)}\,\mathbf{E}\,(1\!\!1_{\{\tau>t\}}Y), \tag{2.6}$$

Corollary 2.1 *Assume that Y is $\mathcal{D}_\infty$-measurable, so that $Y = h(\tau)$ for some Borel measurable function $h:\mathbb{R}_+ \to \mathbb{R}$. If the hazard function Γ of τ is continuous then*

$$\mathbf{E}\,(Y \,|\, \mathcal{D}_t) = 1\!\!1_{\{\tau\le t\}}h(\tau) + 1\!\!1_{\{\tau>t\}} \int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}\, d\Gamma(u). \tag{2.7}$$

If τ admits the intensity function γ then

$$\mathbf{E}\,(Y \,|\, \mathcal{D}_t) = 1\!\!1_{\{\tau\le t\}}h(\tau) + 1\!\!1_{\{\tau>t\}} \int_t^\infty h(u)\gamma(u)e^{-\int_t^u \gamma(v)\,dv}\, du.$$

In particular, for any $t \le s$ we have

$$\mathbf{P}(\tau > s \,|\, \mathcal{D}_t) = 1\!\!1_{\{\tau>t\}} e^{-\int_t^s \gamma(v)\,dv}$$

and

$$\mathbf{P}(t < \tau < s \,|\, \mathcal{D}_t) = 1\!\!1_{\{\tau>t\}}\Big(1 - e^{-\int_t^s \gamma(v)\,dv}\Big).$$

2.2 Martingales Associated with a Continuous Hazard Function

We know already that the $\mathbb{D}$-adapted process of finite variation L given by formula (2.4) is a $\mathbb{D}$-martingale (no matter whether Γ is a continuous function or a discontinuous one). In this section, we shall introduce further important examples of martingales associated with the hazard function. We shall work under an additional assumption that the function Γ is continuous, however.

We shall first consider a very special case, when the cumulative distribution function F is an absolutely continuous function, that is, when the random time τ admits the intensity function γ. Our goal is to provide the martingale characterization of γ. To be more specific, we shall check directly that the process

$$M_t := D_t - \int_0^t \gamma(u) \mathbb{1}_{\{u \leq \tau\}} \, du = D_t - \int_0^{t \wedge \tau} \gamma(u) \, du \tag{2.8}$$

follows a $\mathbb{D}$-martingale. To this end, recall that by virtue of (2.5) we have

$$\mathbf{E}\left(D_s - D_t \,|\, \mathcal{D}_t\right) = \mathbb{1}_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)} =: I.$$

On the other hand,

$$\begin{aligned} \mathbf{E}\Big(\int_s^t \gamma(u) \mathbb{1}_{\{u \leq \tau\}} \, du \,\Big|\, \mathcal{D}_t\Big) &= \int_s^t \gamma(u) \mathbf{P}(u \leq \tau | \mathcal{D}_t) \, du \\ &= \mathbb{1}_{\{\tau > t\}} \frac{\int_t^s \gamma(u)(1 - F(u)) \, du}{1 - F(t)} = I. \end{aligned}$$

This shows that the process M defined by (2.8) follows a $\mathbb{D}$-martingale. We have thus established the following well-known result.

Lemma 2.3 *Assume that $F(t) = 1 - e^{-\int_0^t \gamma(u)\,du}$ for some function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$. Then the process M given by* (2.8) *follows a $\mathbb{D}$-martingale.*

It appears that Lemma 2.3 remains valid when F is merely continuous. Before we state the next result, let us recall the following lemma.

Lemma 2.4 *Let g and h be two right-continuous functions with left-hand limits. If g and h are of finite variation on $[0, t]$ then the following integration by parts formulae are valid*

$$\begin{aligned} g(t)h(t) &= g(0)h(0) + \int_{]0,t]} g(u-) \, dh(u) + \int_{]0,t]} h(u) \, dg(u) \\ &= g(0)h(0) + \int_{]0,t]} g(u) \, dh(u) + \int_{]0,t]} h(u-) \, dg(u) \\ &= g(0)h(0) + \int_{]0,t]} g(u-) \, dh(u) + \int_{]0,t]} h(u-) \, dg(u) + \sum_{u \leq t} \Delta g(u) \, \Delta h(u), \end{aligned}$$

where $\Delta g(u) = g(u) - g(u-)$ and $\Delta h(u) = h(u) - h(u-)$.

In what follows we shall frequently apply this lemma to stochastic processes of finite variation (therefore, the integrals should be understood as pathwise integrals, defined for a.e. ω).

Proposition 2.1 *Assume that Γ is a continuous function. Then the process of finite variation $M = D_t - \Gamma(t \wedge \tau)$ follows a $\mathbb{D}$-martingale. Furthermore, we have*

$$L_t = 1 - \int_{]0,t]} L_{u-} \, dM_u. \tag{2.9}$$

Proof. For sake of brevity, we prefer to make use of Lemma 2.2, rather than to rely on direct calculations. It is clear that M follows a $\mathbb{D}$-adapted integrable process. Using the integration by parts formula for functions of finite variation[1], we obtain

$$L_t = (1-D_t)e^{\Gamma(t)} = 1 + \int_{]0,t]} e^{\Gamma(u)}\big((1-D_u)\,d\Gamma(u) - dD_u\big) \tag{2.10}$$

since Γ is a continuous increasing function. This in turn yields

$$M_t = D_t - \Gamma(t\wedge\tau) = \int_{]0,t]} \big(dD_u - (1-D_u)\,d\Gamma(u)\big) = -\int_{]0,t]} e^{-\Gamma(u)}\,dL_u,$$

and thus M is manifestly a $\mathbb{D}$-martingale. To establish (2.9), notice that (2.10) can be rewritten as follows

$$L_t = 1 + \int_{]0,t]} e^{\Gamma(u)}(1-D_{u-})(d\Gamma(u\wedge\tau) - dD_u) = 1 - \int_{]0,t]} L_{u-}\,dM_u.$$

This ends the proof. □

Proposition 2.2 *Assume that Γ is a continuous function. Then for any Borel measurable function $h:\mathbb{R}_+\to\mathbb{R}$ such that the random variable $h(\tau)$ is integrable, the process*

$$\hat{M}^h_t = 1\!\!1_{\{\tau\le t\}}h(\tau) - \int_0^{t\wedge\tau} h(u)\,d\Gamma(u) \tag{2.11}$$

is a $\mathbb{D}$-martingale.

Proof. We shall establish the assertion through the direct verification of the martingale property of the process $\hat{M}^h$ given by formula (2.11). Therefore, the demonstration given below provides also an alternative proof of Proposition 2.1. First, formula (2.7) in Corollary 2.1 yields

$$I := \mathbf{E}\left(h(\tau)1\!\!1_{\{t<\tau\le s\}}\,|\,\mathcal{D}_t\right) = 1\!\!1_{\{\tau>t\}}e^{\Gamma(t)}\int_t^s h(u)e^{-\Gamma(u)}\,d\Gamma(u).$$

On the other hand, Corollary 2.1 leads to

$$J := \mathbf{E}\Big(\int_{t\wedge\tau}^{s\wedge\tau} h(u)\,d\Gamma(u)\,\Big|\,\mathcal{D}_t\Big) = 1\!\!1_{\{\tau>t\}}e^{\Gamma(t)}\Big(\int_t^s \tilde{h}(u)e^{-\Gamma(u)}\,d\Gamma(u) + e^{-\Gamma(s)}\tilde{h}(s)\Big)$$

where we set $\tilde{h}(s) = \int_t^s h(u)\,d\Gamma(u)$. To conclude the proof, it is enough to observe that the Fubini theorem yields

$$\begin{aligned}\int_t^s e^{-\Gamma(u)}\int_t^u h(v)\,d\Gamma(v)\,d\Gamma(u) \;+\; e^{-\Gamma(s)}\tilde{h}(s) &= \int_t^s h(u)\int_u^s e^{-\Gamma(v)}\,d\Gamma(v)\,d\Gamma(u)\\ &+e^{-\Gamma(s)}\int_t^s h(u)\,d\Gamma(u) = \int_t^s h(u)e^{-\Gamma(u)}\,d\Gamma(u),\end{aligned}$$

as expected. □

Remarks. It is apparent that $\hat{M}^h$ admits the following integral representation (which shows that the martingale property of $\hat{M}^h$ is also a straightforward consequence of Proposition 2.1)

$$\hat{M}^h_t = \int_{]0,t]} h(u)\,dM_u.$$

[1] Or equivalently, a version of Itô's formula for discontinuous semimartingales (see, for instance, [14] or [23]). Notice that since M follows a processes of finite variation, it is a *purely discontinuous martingale,* in the terminology of stochastic analysis. Whenever possible, we shall avoid making use of the results from the modern martingale theory.

Corollary 2.2 *Assume that Γ is a continuous function. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a Borel measurable function such that the random variable $Y = e^{h(\tau)}$ is integrable. Then the process*

$$\tilde{M}_t^h = \exp\big(1\!\!1_{\{\tau \le t\}} h(\tau)\big) - \int_0^{t\wedge\tau} (e^{h(u)} - 1)\, d\Gamma(u) \tag{2.12}$$

is a $\mathbb{D}$-martingale.

Proof. Notice that

$$\exp\big(1\!\!1_{\{\tau \le t\}} h(\tau)\big) - 1 = 1\!\!1_{\{\tau \le t\}} e^{h(\tau)} + 1\!\!1_{\{\tau > t\}} - 1 = 1\!\!1_{\{\tau \le t\}} e^{h(\tau)} - D_t$$

and thus

$$\tilde{M}_t^h = 1\!\!1_{\{\tau \le t\}} e^{h(\tau)} - \int_0^{t\wedge\tau} e^{h(u)}\, d\Gamma(u) - M_t.$$

It is thus enough to apply Proposition 2.2. □

Corollary 2.3 *Assume that Γ is a continuous function. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative Borel measurable function such that the random variable $h(\tau)$ is integrable. Then the process*

$$\bar{M}_t^h = (1 + 1\!\!1_{\{\tau \le t\}} h(\tau)) \exp\Big(- \int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big) \tag{2.13}$$

is a $\mathbb{D}$-martingale.

Proof. Let us denote by U the decreasing continuous process

$$U_t = \exp\Big(- \int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big).$$

It is apparent that

$$1 + 1\!\!1_{\{\tau \le t\}} h(\tau) = 1 + \int_{]0,t]} h(u)\, dD_u =: H_t^h.$$

An application of the integration by parts formula yields

$$d\bar{M}_t^h = d(H_t^h U_t) = U_t h(t)\, dD_t - (1 + 1\!\!1_{\{\tau \le t\}} h(\tau))\, U_t h(t)\, d\Gamma(t\wedge\tau).$$

Consequently,

$$d\bar{M}_t^h = U_t h(t)\, d(D_t - \Gamma(t\wedge\tau)) = U_t h(t)\, dM_t. \qquad \square$$

2.3 Martingale Representation Theorem

The following result is an extension of a well known result, established when F is absolutely continuous (see, for instance, Brémaud [3]).

Proposition 2.3 *Assume that F is a continuous function. Let $M_t^h := \mathbf{E}\,(h(\tau)\,|\,\mathcal{D}_t)$ for some Borel measurable function $h : \mathbb{R} \to \mathbb{R}$ such that the random variable $h(\tau)$ is integrable. Then*

$$M_t^h = M_0^h + \int_{]0,t]} \hat{h}(u)\, dM_u, \tag{2.14}$$

where $M_t = D_t - \Gamma(t\wedge\tau)$, and $\hat{h}$ is given by

$$\hat{h}(t) = h(t) - e^{\Gamma(t)}\, \mathbf{E}\,\big(h(\tau) 1\!\!1_{\{\tau > t\}}\big). \tag{2.15}$$

Proof. By virtue of (2.7), the left-hand side in (2.14) equals

$$I = \mathbf{E}\,(h(\tau)\,|\,\mathcal{D}_t) = D_t h(\tau) + (1 - D_t) g(t),$$

where the function $g : \mathbb{R} \to \mathbb{R}$ equals

$$g(t) := e^{\Gamma(t)} \mathbf{E}\left(\mathbb{1}_{\{\tau > t\}} h(\tau)\right) = e^{\Gamma(t)} \int_t^\infty h(u)\, dF(u). \tag{2.16}$$

On the other hand, the right-hand side in (2.14) can be rewritten as follows

$$\begin{aligned} J &= g(0) + \int_{]0,t]} \hat{h}(u)\, dM_u \\ &= g(0) + \int_{]0,t]} (h(u) - g(u))\, d(D_u - \Gamma(u \wedge \tau)) \\ &= g(0) + D_t(h(\tau) - g(\tau)) + \int_0^{t\wedge\tau} (g(u) - h(u))\, d\Gamma(u) \\ &= g(0) + D_t h(\tau) + (1 - D_t) g(t) - g(t \wedge \tau) + \int_0^{t\wedge\tau} (g(u) - h(u))\, d\Gamma(u) \end{aligned}$$

To check that $I = J$, it is thus enough to show that

$$g(t \wedge \tau) = g(0) + \int_0^{t\wedge\tau} (g(u) - h(u))\, d\Gamma(u),$$

or equivalently, that for every $t \in \mathbb{R}_+$

$$g(t) = g(0) + \int_0^t (g(u) - h(u))\, d\Gamma(u).$$

In other words, we need to verify that

$$e^{\Gamma(t)} \int_t^\infty h(u)\, dF(u) = \int_0^\infty h(u)\, dF(u) + \int_0^t e^{\Gamma(u)} (g(u) - h(u))\, dF(u).$$

To this end, notice that Fubini's theorem yields (recall that $e^{\Gamma(u)} dF(u) = d\Gamma(u)$)

$$\begin{aligned} \int_0^t e^{\Gamma(u)} g(u)\, dF(u) &= \int_0^t e^{2\Gamma(u)} \int_u^\infty h(v)\, dF(v)\, dF(u) \\ &= \int_0^t h(v) \int_0^v e^{\Gamma(u)}\, d\Gamma(u)\, dF(v) + \int_t^\infty h(v) \int_0^t e^{\Gamma(u)}\, d\Gamma(u)\, dF(v) \\ &= \int_0^t h(u)\big(e^{\Gamma(u)} - 1\big)\, dF(u) + \big(e^{\Gamma(t)} - 1\big) \int_t^\infty h(u)\, dF(u). \end{aligned}$$

This completes the proof. □

Notice that representation (2.14) can also be rewritten as follows (cf. formula (3.25))

$$M_t^h = M_0^h + \int_{]0,t]} (h(u) - M_{u-}^h)\, dM_u. \tag{2.17}$$

Remarks. Since any $\mathcal{D}_\infty$-measurable random variable X is of the form $X = h(\tau)$, we deduce from Proposition 2.3 that any $\mathbb{D}$-martingale admits the representation (2.14) and thus is a purely discontinuous martingale (as a martingale of finite variation). Put another way, any continuous $\mathbb{D}$-martingale is constant.

2.4 Change of a Probability Measure

Let $\mathbf{P}^*$ be an arbitrary probability measure on $(\Omega, \mathcal{D}_\infty)$, which is absolutely continuous with respect to $\mathbf{P}$. We denote by η the $\mathcal{D}_\infty$ measurable density of $\mathbf{P}^*$ with respect to $\mathbf{P}$

$$\eta := \frac{d\mathbf{P}^*}{d\mathbf{P}} = h(\tau) \geq 0, \quad \mathbf{P}\text{-a.s.}, \tag{2.18}$$

where $h : \mathbb{R} \to \mathbb{R}_+$ is a Borel measurable function satisfying

$$\mathbf{E}_\mathbf{P}(h(\tau)) = \int_0^\infty h(u)\, dF(u) = 1.$$

We can use Girsanov's theorem. Nevertheless, we prefer here to establish this theorem in our particular setting. We write $\mathbf{E}_\mathbf{P}$ ($\mathbf{E}_{\mathbf{P}^*}$, resp.) to denote the expected value with respect to the probability measure $\mathbf{P}$ ($\mathbf{P}^*$, resp.) Of course, the probability measure $\mathbf{P}^*$ is equivalent to $\mathbf{P}$ if and only if the inequality in (2.18) is strict $\mathbf{P}$-a.s. Furthermore, we shall assume that $\mathbf{P}^*(\tau = 0) = 0$ and $\mathbf{P}^*(\tau > t) > 0$ for any $t \in \mathbb{R}_+$. Actually the first condition is satisfied for any $\mathbf{P}^*$ absolutely continuous with respect to $\mathbf{P}$. For the second condition to hold, it is sufficient and necessary to assume that for every t

$$\mathbf{P}^*(\tau > t) = 1 - F^*(t) = \int_{]t,\infty[} h(u)\, dF(u) > 0,$$

where the c.d.f. F^* of τ under $\mathbf{P}^*$

$$F^*(t) := \mathbf{P}^*(\tau \leq t) = \int_{[0,t]} h(u)\, dF(u). \tag{2.19}$$

Put another way, we assume that (cf. (2.16))

$$g(t) = e^{\Gamma(t)} \mathbf{E}\left(\mathbb{1}_{\{\tau > t\}} h(\tau)\right) = e^{\Gamma(t)} \int_{]t,\infty[} h(u)\, dF(u) = e^{\Gamma(t)}\, \mathbf{P}^*(\tau > t) > 0.$$

We assume throughout that this is the case, so that the hazard function Γ^* of τ with respect to $\mathbf{P}^*$ is well defined. Our goal is to examine relationships between hazard functions Γ^* and Γ. It is easily seen that in general we have

$$\frac{\Gamma^*(t)}{\Gamma(t)} = \frac{\ln\left(\int_{]t,\infty[} h(u)\, dF(u)\right)}{\ln(1 - F(t))} =: g^*(t), \tag{2.20}$$

since by definition $\Gamma^*(t) = -\ln(1 - F^*(t))$.

Assume first that F is an absolutely continuous function, so that the intensity function γ of τ under $\mathbf{P}$ is well defined. Recall that γ is given by the formula

$$\gamma(t) = \frac{f(t)}{1 - F(t)}.$$

On the other hand, the c.d.f. F^* of τ under $\mathbf{P}^*$ now equals

$$F^*(t) := \mathbf{P}^*(\tau \leq t) = \mathbf{E}_\mathbf{P}(\mathbb{1}_{\{\tau \leq t\}} h(\tau)) = \int_0^t h(u) f(u)\, du.$$

so that F^* follows an absolutely continuous function. Therefore, the intensity function γ^* of the random time τ under $\mathbf{P}^*$ exists, and it is given by the formula

$$\gamma^*(t) = \frac{h(t) f(t)}{1 - F^*(t)} = \frac{h(t) f(t)}{1 - \int_0^t h(u) f(u)\, du}.$$

To derive a more straightforward relationship between the intensities γ and γ^*, let us introduce an auxiliary function $h^* : \mathbb{R}_+ \to \mathbb{R}$, given by the formula $h^*(t) = h(t)/g(t)$.

Notice that

$$\gamma^*(t) = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\,du} = \frac{h(t)f(t)}{\int_t^\infty h(u)f(u)\,du} = \frac{h(t)f(t)}{e^{-\Gamma(t)}g(t)} = h^*(t)\,\frac{f(t)}{1-F(t)} = h^*(t)\gamma(t).$$

This means also that $d\Gamma^*(t) = h^*(t)\,d\Gamma(t)$. It appears that the last equality holds true if F is merely a continuous function. Indeed, if F (and thus F^*) is continuous, we get

$$d\Gamma^*(t) = \frac{dF^*(t)}{1-F^*(t)} = \frac{d(1-e^{-\Gamma(t)}g(t))}{e^{-\Gamma(t)}g(t)} = \frac{g(t)d\Gamma(t) - dg(t)}{g(t)} = h^*(t)\,d\Gamma(t).$$

To summarize, if the hazard function Γ is continuous then Γ^* is also continuous and $d\Gamma^*(t) = h^*(t)\,d\Gamma(t)$.

To understand better the origin of the function h^*, let us introduce the following non-negative $\mathbf{P}$-martingale (which is strictly positive when the probability measures $\mathbf{P}^*$ and $\mathbf{P}$ are equivalent)

$$\eta_t := \frac{d\mathbf{P}^*}{d\mathbf{P}}_{\,|\mathcal{D}_t} = \mathbf{E}_{\mathbf{P}}(\eta\,|\,\mathcal{D}_t) = \mathbf{E}_{\mathbf{P}}(h(\tau)\,|\,\mathcal{D}_t), \tag{2.21}$$

so that $\eta_t = M_t^h$. The general formula for η_t reads (cf. (2.6))

$$\eta_t = \mathbb{1}_{\{\tau\le t\}}h(\tau) + \mathbb{1}_{\{\tau>t\}}\,e^{\Gamma(t)}\int_{]t,\infty[} h(u)\,dF(u) = \mathbb{1}_{\{\tau\le t\}}h(\tau) + \mathbb{1}_{\{\tau>t\}}g(t).$$

Assume now that F is a continuous function. Then also (cf. (2.7))

$$\eta_t = \mathbb{1}_{\{\tau\le t\}}h(\tau) + \mathbb{1}_{\{\tau>t\}}\int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}\,d\Gamma(u).$$

On the other hand, using (2.14) and (2.17), we get

$$M_t^h = M_0^h + \int_{]0,t]} M_{u-}^h(h^*(u)-1)\,dM_u$$

where $h^*(u) = h(u)/g(u)$. We conclude that

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-}(h^*(u)-1)\,dM_u. \tag{2.22}$$

It is thus easily seen that

$$\eta_t = \big(1 + \mathbb{1}_{\{\tau\le t\}}v(\tau)\big)\exp\Big(-\int_0^{t\wedge\tau} v(u)\,d\Gamma(u)\Big), \tag{2.23}$$

where we write $v(t) = h^*(t)-1$. Therefore, the martingale property of the process η, which is obvious from (2.21), is also a consequence of Corollary 2.3.

Remarks. In view of (2.22), we have

$$\eta_t = \mathcal{E}_t\Big(\int_0^{\cdot}(h^*(u)-1)\,dM_u\Big),$$

where $\mathcal{E}$ stands for the Doléans exponential. Representation (2.23) for the random variable η_t can thus be obtained from the general formula for the Doléans exponential. Recall that if X is a semimartingale, then the process $Z = \mathcal{E}(X)$ is the unique solution to the SDE

$$Z_t = 1 + \int_{]0,t]} Z_{u-}\,dX_u.$$

It is known that

$$\mathcal{E}_t(X) = \exp\big(X_t - X_0 - \tfrac{1}{2}\langle X^c\rangle_t\big)\prod_{u\le t}(1+\Delta X_u)e^{-\Delta X_u},$$

where X^c is the continuous martingale component of X.

We are in the position to formulate the following result (all statements were already established above).

Proposition 2.4 *Let $\mathbf{P}^*$ be any probability measure on $(\Omega, \mathcal{D}_\infty)$ absolutely continuous with respect to $\mathbf{P}$, so that (2.18) holds for some function h. Assume that $\mathbf{P}^*(\tau > t) > 0$ for every $t \in \mathbb{R}_+$. Then*

$$\frac{d\mathbf{P}^*}{d\mathbf{P}}_{\,|\mathcal{D}_t} = \mathcal{E}_t\Big(\int_0^{\cdot}(h^*(u)-1)\,dM_u\Big), \tag{2.24}$$

where

$$h^*(t) = h(t)/g(t), \quad g(t) = e^{\Gamma(t)}\int_t^\infty h(u)\,dF(u),$$

and $\Gamma^*(t) = g^*(t)\Gamma(t)$ with

$$g^*(t) = \frac{\ln\Big(\int_{]t,\infty[} h(u)\,dF(u)\Big)}{\ln(1-F(t))}. \tag{2.25}$$

If, in addition, the random time τ admits the intensity function γ under $\mathbf{P}$, then the intensity function γ^* of τ under $\mathbf{P}^*$ satisfies $\gamma^*(t) = h^*(t)\gamma(t)$ a.e. on $\mathbb{R}_+$. More generally, if the hazard function Γ of τ under $\mathbf{P}$ is continuous, then the hazard function Γ^* of τ under $\mathbf{P}^*$ is also continuous, and it satisfies $d\Gamma^*(t) = h^*(t)\,d\Gamma(t)$.

Corollary 2.4 *If F is continuous then $M^*_t = D_t - \Gamma^*(t\wedge\tau)$ is a $\mathbb{D}$-martingale under $\mathbf{P}^*$.*

Proof. In view Proposition 2.1, the assertion is an immediate consequence of the continuity of Γ^*. Alternatively, we may check directly that the product $U_t = \eta_t M^*_t = \eta_t(D_t - \Gamma^*(t\wedge\tau))$ follows a $\mathbb{D}$-martingale under $\mathbf{P}$. To this end, observe that the integration by parts formula for functions of finite variation yields

$$\begin{aligned} U_t &= \int_{]0,t]}\eta_{t-}\,dM^*_t + \int_{]0,t]}M^*_t\,d\eta_t \\ &= \int_{]0,t]}\eta_{t-}\,dM^*_t + \int_{]0,t]}M^*_{t-}\,d\eta_t + \sum_{u\le t}\Delta M^*_u\Delta\eta_u \\ &= \int_{]0,t]}\eta_{t-}\,dM^*_t + \int_{]0,t]}M^*_{t-}\,d\eta_t + \mathbb{1}_{\{\tau\le t\}}(\eta_\tau - \eta_{\tau-}). \end{aligned}$$

Using (2.22), we obtain

$$\begin{aligned} U_t &= \int_{]0,t]}\eta_{t-}\,dM^*_t + \int_{]0,t]}M^*_{t-}\,d\eta_t + \eta_{\tau-}\mathbb{1}_{\{\tau\le t\}}(h^*(\tau)-1) \\ &= \int_{]0,t]}\eta_{t-}\,d\big(\Gamma(t\wedge\tau) - \Gamma^*(t\wedge\tau) + \mathbb{1}_{\{\tau\le t\}}(h^*(\tau)-1)\big) + N_t, \end{aligned}$$

where the process N, which equals

$$N_t = \int_{]0,t]}\eta_{t-}\,dM_t + \int_{]0,t]}M^*_{t-}\,d\eta_t$$

is manifestly a $\mathbb{D}$-martingale with respect to $\mathbf{P}$. It remains to show that the process

$$N^*_t := \Gamma(t\wedge\tau) - \Gamma^*(t\wedge\tau) + \mathbb{1}_{\{\tau\le t\}}(h^*(\tau)-1)$$

follows a $\mathbb{D}$-martingale with respect to $\mathbf{P}$. By virtue of Proposition 2.2, the process

$$\mathbb{1}_{\{\tau\le t\}}(h^*(\tau)-1)+\Gamma(t\wedge\tau)-\int_0^{t\wedge\tau}h^*(u)\,d\Gamma(u)$$

is a $\mathbb{D}$-martingale. Therefore, to conclude the proof it is enough to notice that

$$\int_0^{t\wedge\tau}h^*(u)\,d\Gamma(u)-\Gamma^*(t\wedge\tau)=\int_0^{t\wedge\tau}(h^*(u)\,d\Gamma(u)-d\Gamma^*(u))=0,$$

where the last equality is a consequence of the relationship $d\Gamma^*(t)=h^*(t)\,d\Gamma(t)$ established in Proposition 2.4. □

By virtue of Proposition 2.1 if Γ^* is a continuous function then the process $M^*=D_t-\Gamma^*(t\wedge\tau)$ follows a $\mathbb{D}$-martingale under $\mathbf{P}^*$. The next result suggests that this martingale property uniquely characterizes the (continuous) hazard function of a random time. We shall examine this issue in more detail in Section 2.6.

Lemma 2.5 *Suppose that an equivalent probability measure* $\mathbf{P}^*$ *is given by formula* (2.18) *for some function* h. *Let* $\Lambda^*:\mathbb{R}_+\to\mathbb{R}_+$ *be an arbitrary continuous increasing function, with* $\Lambda^*(0)=0$. *If the process* $M_t^*:=D_t-\Lambda^*(t\wedge\tau)$ *follows a* $\mathbb{D}$*-martingale under* $\mathbf{P}^*$, *then* $\Lambda^*(t)=-\ln(1-F^*(t))$ *with* F^* *given by formula* (2.24).

Proof. The Bayes rule implies

$$\mathbf{E}_{\mathbf{P}^*}(M_t^*\,|\,\mathcal{D}_s)=\frac{\mathbf{E}_{\mathbf{P}}(M_t^*\eta\,|\,\mathcal{D}_s)}{\mathbf{E}_{\mathbf{P}}(\eta\,|\,\mathcal{D}_s)}=\eta_s^{-1}\,\mathbf{E}_{\mathbf{P}}(M_t^*\eta_t\,|\,\mathcal{D}_s)$$

and thus

$$\mathbf{E}_{\mathbf{P}^*}(M_t^*\,|\,\mathcal{D}_s)=\frac{\mathbf{E}_{\mathbf{P}}\big((D_t-\Lambda^*(t\wedge\tau))(D_th(\tau)+(1-D_t)g(t))\,\big|\,\mathcal{D}_s\big)}{D_sh(\tau)+(1-D_s)g(s)}\,,$$

or equivalently

$$\mathbf{E}_{\mathbf{P}^*}(M_t^*\,|\,\mathcal{D}_s)=\frac{\mathbf{E}_{\mathbf{P}}\big(D_th(\tau)-D_t\Lambda^*(t\wedge\tau)h(\tau)-(1-D_t)\Lambda^*(t\wedge\tau)g(t)\,\big|\,\mathcal{D}_s\big)}{D_sh(\tau)+(1-D_s)g(s)}\,.$$

This means that

$$\mathbf{E}_{\mathbf{P}^*}(M_t^*\,|\,\mathcal{D}_s)=\frac{J}{D_sh(\tau)+(1-D_s)g(s)}\,,$$

where we write

$$J=\mathbf{E}_{\mathbf{P}}\big(D_th(\tau)-D_t\Lambda^*(t\wedge\tau)h(\tau)-(1-D_t)\Lambda^*(t\wedge\tau)g(t)\,\big|\,\mathcal{D}_s\big).$$

Using (2.1), we obtain

$$J=D_sh(\tau)-D_s\Lambda^*(\tau)h(\tau)-(1-D_s)(1-F(s))^{-1}\,\mathbf{E}_{\mathbf{P}}\big(\mathbb{1}_{\{s<\tau\le t\}}(\Lambda^*(\tau)-1)h(\tau)+\mathbb{1}_{\{\tau>t\}}\Lambda^*(t)g(t)\big)$$

and thus the martingale condition $\mathbf{E}_{\mathbf{P}^*}(M_t^*\,|\,\mathcal{D}_s)=M_s^*$, is equivalent to the following equality

$$(1-D_s)(1-F(s))^{-1}\,\mathbf{E}_{\mathbf{P}}\big(\mathbb{1}_{\{s<\tau\le t\}}(\Lambda^*(\tau)-1)h(\tau)+\mathbb{1}_{\{\tau>t\}}\Lambda^*(t)g(t)\big)=\Lambda^*(s)(1-D_s)g(s).$$

Therefore, for every $s\le t$ we have

$$\mathbf{E}_{\mathbf{P}}\big(\mathbb{1}_{\{s<\tau\le t\}}(\Lambda^*(\tau)-1)h(\tau)+\mathbb{1}_{\{\tau>t\}}\Lambda^*(t)g(t)\big)=\Lambda^*(s)(1-F(s))g(s)$$

so that

$$\int_s^t(\Lambda^*(u)-1)h(u)\,dF(u)+\Lambda^*(t)g(t)(1-F(t))=\Lambda^*(s)\int_s^\infty h(u)\,dF(u),$$

and finally,

$$\int_s^t (\Lambda^*(u) - 1)\, dF^*(u) + \Lambda^*(t)(1 - F^*(t)) = \Lambda^*(s)(1 - F^*(s)).$$

After simple manipulations involving the integration by parts, we get for $s \le t$

$$\int_s^t (1 - F^*(u))\, d\Lambda^*(u) = F^*(t) - F^*(s),$$

and since $\Lambda^*(0) = F^*(0) = 0$, we find that $\Lambda^* = -\ln(1 - F^*(t))$. □

2.5 Applications to the Valuation of Defaultable Claims

Let us fix $T > 0$. We assume that the continuously compounded interest rate r follows a non-negative deterministic function so that the price at time t of a unit default-free zero-coupon bond of maturity T equals

$$B(t,T) = e^{-\int_t^T r(v)\,dv}, \quad \forall\, t \in [0,T].$$

Our goal is to find quasi-explicit expressions for "values" of certain defaultable claims. Let us assume that $Y = \mathbb{1}_{\{\tau \le T\}}\, h(\tau) + \mathbb{1}_{\{\tau > T\}}\, c$, where c is a constant. If Γ is continuous then (2.7) yields

$$\mathbf{E}\,(Y \,|\, \mathcal{D}_t) = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \Big(\int_t^T h(u) e^{\Gamma(t) - \Gamma(u)}\, d\Gamma(u) + c e^{\Gamma(t) - \Gamma(T)} \Big).$$

Similarly, for a fixed $t \le T$ denote by Y_t the random variable (discounted payoff at time t)

$$Y_t = \mathbb{1}_{\{\tau \le T\}}\, h(\tau) e^{-\int_t^\tau r(v)\,dv} + \mathbb{1}_{\{\tau > T\}}\, c e^{-\int_t^T r(v)\,dv}. \tag{2.26}$$

If Γ is an absolutely continuous function, then we get

$$\mathbf{E}\,(Y_t \,|\, \mathcal{D}_t) = \mathbb{1}_{\{\tau \le t\}} h(\tau) e^{\int_\tau^t r(v)\,dv} + \mathbb{1}_{\{\tau > t\}} \Big(\int_t^T h(u)\gamma(u) e^{-\int_t^u \hat{r}(v)\,dv}\, du + c e^{-\int_t^T \hat{r}(v)\,dv} \Big),$$

where $\hat{r}(v) = r(v) + \gamma(v)$.

(a) The case of a defaultable zero-coupon T-maturity bond with zero recovery corresponds to $h = 0$ and $c = 1$ in (2.26). If we denote the "value" at time t of such a bond by $D^0(t,T)$ then we have

$$D^0(t,T) = \mathbb{1}_{\{\tau > t\}} e^{-\int_t^T (r(v) + \gamma(v))\,dv}, \quad \forall\, t \in [0,T].$$

(b) Assume now that $h = \delta$ for some constant $0 < \delta \le 1$ and $c = 1$. Put more explicitly, we consider the random variable $\tilde{Y}_t^\delta$ which equals

$$\tilde{Y}_t^\delta = \mathbb{1}_{\{\tau \le T\}}\, \delta e^{-\int_t^\tau r(v)\,dv} + \mathbb{1}_{\{\tau > T\}}\, e^{-\int_t^T r(v)\,dv}.$$

In this case, for $\tilde{D}^\delta(t,T) := \mathbf{E}\,(\tilde{Y}_t^\delta \,|\, \mathcal{D}_t)$ we get

$$\tilde{D}^\delta(t,T) = \mathbb{1}_{\{\tau \le t\}} \delta e^{\int_t^\tau r(v)\,dv} + \mathbb{1}_{\{\tau > t\}} \Big(\delta \int_t^T h(u)\gamma(u) e^{-\int_t^u \hat{r}(v)\,dv}\, du + c e^{-\int_t^T \hat{r}(v)\,dv} \Big).$$

Notice that $\tilde{D}^\delta(t,T)$ represents the value at time t of a T-maturity defaultable bond which pays a constant payoff δ at time of default, if default takes place before maturity date T.

(c) Let us finally consider the following random variable

$$Y_t^\delta = \big(\mathbb{1}_{\{\tau \le T\}} \delta + \mathbb{1}_{\{\tau > T\}} \big)\, e^{-\int_t^T r(v)\,dv} = B(t,T) \big(\mathbb{1}_{\{\tau \le T\}} \delta + \mathbb{1}_{\{\tau > T\}} \big).$$

Equivalently, we have

$$Y_t^\delta = \mathbb{1}_{\{\tau\le T\}}\delta e^{-\int_\tau^T r(v)\,dv} e^{-\int_t^\tau r(v)\,dv} + \mathbb{1}_{\{\tau>T\}} e^{-\int_t^T r(v)\,dv},$$

and the last expression leads to $h(\tau) = \delta e^{-\int_\tau^T r(v)\,dv}$, $c = 1$, in formula (2.26). The above specification of Y_t^δ corresponds to a defaultable zero-coupon T-maturity bond with fractional recovery of par. This means that the bond pays δ at maturity T if default occurs before maturity (otherwise, it pays the face value 1). For the value $D^\delta(t,T) := \mathbf{E}\,(Y_t^\delta \,|\, \mathcal{D}_t)$ of such a bond we get

$$D^\delta(t,T) = \mathbb{1}_{\{\tau\le t\}}\delta B(t,T) + \mathbb{1}_{\{\tau>t\}}\delta B(t,T)\big(1 - e^{-\int_t^T \gamma(v)\,dv}\big) + \mathbb{1}_{\{\tau>t\}} e^{-\int_t^T \hat{r}(v)\,dv}. \tag{2.27}$$

2.6 Martingale Characterization of the Hazard Function

In view of Lemma 2.5, the natural question which arises in this context reads: does the $\mathbb{D}$-martingale property of the process $D_t - \Gamma(t\wedge\tau)$ uniquely characterize the hazard function of τ? To examine this problem, it is useful to notice that the process $A_t := \Gamma(t\wedge\tau)$ satisfies: (i) A is an increasing, continuous, $\mathbb{D}$-adapted process, and (ii) $D - A$ is a $\mathbb{D}$-martingale. It is thus clear that A is a dual predictable projection (or $\mathbb{D}$-*compensator*) of the increasing, right-continuous, $\mathbb{D}$-adapted process D. We shall see that the answer to the question formulated above is indeed positive when the hazard function Γ is a continuous function.

It is useful to observe that when Γ is discontinuous, equality (2.10) takes the following form

$$L_t = L_0 + \int_{]0,t]} (1 - D_u)\, de^{\Gamma(u)} - \int_{]0,t]} e^{\Gamma(u-)}\, dD_u,$$

that is (we write $\Delta\Gamma(s) = \Gamma(s) - \Gamma(s-)$),

$$L_t = 1 + \int_{]0,t]} e^{\Gamma(u-)}\big((1-D_u)\,d\Gamma(u) - dD_u\big) + \sum_{s\le t,\, s<\tau} \big(e^{\Gamma(s)} - e^{\Gamma(s-)} - e^{\Gamma(s-)}\,\Delta\Gamma(s)\big).$$

Let us stress that both A and Γ exist for arbitrary random time τ, and are unique. We find it convenient to introduce the notion of a martingale hazard function of a random time.

Definition 2.2 An increasing function $\Lambda : \mathbb{R}_+ \to \mathbb{R}$ such that $\Lambda(0) = 0$ is called a *martingale hazard function* of a random time τ with respect to the natural filtration $\mathbb{D}$ if and only if the process $D_t - \Lambda(t\wedge\tau)$ is a $\mathbb{D}$-martingale.

The function Λ can also be seen as a $\mathbb{F}^0$-adapted right-continuous stochastic process, where $\mathbb{F}^0$ is the trivial filtration, $\mathcal{F}_t^0 = \mathcal{F}_0 = \{\emptyset, \Omega\}$ for every $t\in\mathbb{R}_+$. We shall sometimes find it useful to refer to the martingale hazard function as the $(\mathbb{F}^0,\mathbb{D})$-*martingale hazard process* of a random time τ. The reason for this convention will become clear in Section 4, where the notion of a martingale hazard process with respect to a non-trivial filtration $\mathbb{F}$ is examined.

Proposition 2.5 (i) *The (unique) martingale hazard function of τ with respect to $\mathbb{D}$ is the right-continuous increasing function Λ given by the formula*

$$\Lambda(t) = \int_{]0,t\,]} \frac{dF(u)}{1 - F(u-)} = \int_{]0,t\,]} \frac{d\mathbf{P}(\tau\le u)}{1 - \mathbf{P}(\tau<u)}, \quad \forall\, t\in\mathbb{R}_+. \tag{2.28}$$

(ii) *The martingale hazard function Λ is continuous if and only if the c.d.f. F is continuous. In this case, Λ satisfies $\Lambda(t) = -\ln\big(1 - F(t)\big)$ (equivalently, $F(t) = 1 - e^{-\Lambda(t)}$).*

(iii) *The martingale hazard function* Λ *coincides with the hazard function* Γ *if and only if* F *is a continuous function. In general*

$$e^{-\Gamma(t)} = e^{-\Lambda^c(t)} \prod_{0 \le u \le t} (1 - \Delta\Lambda(u)), \tag{2.29}$$

where $\Lambda^c(t) = \Lambda(t) - \sum_{0 \le u \le t} \Delta\Lambda(u)$, *and* $\Delta\Lambda(u) = \Lambda(u) - \Lambda(u-)$.
(iv) *If* F *is an absolutely continuous function then*

$$\Lambda(t) = \Gamma(t) = \int_0^t f(u)(1 - F(u))^{-1} \, du. \tag{2.30}$$

Proof. We shall first focus on Λ. The definition of Λ implies that $\mathbf{E}\,(D_t) = \mathbf{E}\,(\Lambda(t \wedge \tau))$, that is (recall that $F(0) = 0$),

$$F(t) = \int_{]0,t\,]} \Lambda(u) \, dF(u) + \Lambda(t)(1 - F(t)) \tag{2.31}$$

The first equality shows that Λ is necessarily a right-continuous function. If Λ_1 and Λ_2 are right-continuous functions which satisfy (2.31) then for every $t \in \mathbb{R}_+$

$$\int_{]0,t\,]} (\Lambda_1(u) - \Lambda_2(u)) \, dF(u) + (\Lambda_1(t) - \Lambda_2(t))(1 - F(t)) = 0.$$

This shows, by standard contraction arguments, that the martingale hazard function Λ, if it exists, is unique (a similar argument applies to $\tilde{\Lambda}$).

To establish (i), it is enough to check that for any $t \le s$ we have (cf. (2.5))

$$\mathbf{E}\,(D_s - D_t \,|\, \mathcal{D}_t) = 1\!\!1_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)} = 1\!\!1_{\{\tau > t\}} \mathbf{E}\,(Y_\tau \,|\, \mathcal{D}_t),$$

where the process Y is defined as

$$Y_u = \int_{]t \wedge u, s \wedge u]} \frac{dF(v)}{1 - F(v-)}.$$

The random variable $\mathbf{E}\,(Y_\tau \,|\, \mathcal{D}_t)$ is equal to a constant on the set $\tau > t$

$$1\!\!1_{\{\tau > t\}} \mathbf{E}\,\Big(\int_{]\tau \wedge s, \tau \wedge t]} \frac{dF(u)}{1 - F(u-)} \,\Big|\, \mathcal{D}_s \Big) = 1\!\!1_{\{\tau > t\}} \, g(t),$$

where

$$\begin{aligned} g(t) &= \frac{1}{1 - F(t)} \mathbf{E}\,\Big(1\!\!1_{\{t < \tau\}} \int_{]\tau \wedge t, \tau \wedge s]} \frac{dF(u)}{1 - F(u-)} \Big) \\ &= \frac{1 - F(s)}{1 - F(t)} \int_{]t,s]} \frac{dF(u)}{1 - F(u-)} - \frac{1}{1 - F(t)} \int_{]t,s]} dF(v) \int_{]s,v]} \frac{dF(u)}{1 - F(u-)}. \end{aligned}$$

Applying Fubini's theorem, we conclude that $g(t) = \dfrac{F(s) - F(t)}{1 - F(s)}$. This completes the proof of (i). Statements (ii)-(iv) are almost immediate consequences of (i) and the definition of a hazard function. Let us only observe that at any point of discontinuity of F we have

$$\Delta\Lambda(t) := \Lambda(t) - \Lambda(t-) = \frac{F(t) - F(t-)}{1 - F(t-)}.$$

On the other hand, for the hazard function Γ we obtain

$$e^{-\Delta\Gamma(t)} = e^{-(\Gamma(t) - \Gamma(t-))} = \frac{1 - F(t)}{1 - F(t-)} = 1 - \Delta\Lambda(t). \tag{2.32}$$

This shows that the martingale hazard function Λ and the hazard function Γ cannot coincide when F is discontinuous. In view of (2.32), relationship (2.29) is also easy to establish. □

As was already mentioned, the notion of a martingale hazard function is closely related to the $\mathbb{D}$-compensator of τ (or rather, the $\mathbb{D}$-compensator of the associated jump process D). Let us first recall the definition of a compensator of an increasing process. In our context, it can be stated as follows.

Definition 2.3 A process A is called a $\mathbb{D}$-*compensator* of the jump process D if and only if the following hold: (i) A is a $\mathbb{D}$-predictable right-continuous increasing process, with $A_0 = 0$, (ii) the process $H - A$ is a $\mathbb{D}$-martingale.

Using the well-known result on the existence and uniqueness of the Doob-Meyer decomposition with respect to the filtration $\mathbb{D}$ which satisfies the 'usual conditions,' it is easy to check that a process A is a $\mathbb{D}$-compensator of the jump process D if and only if $A_t = \Lambda(t \wedge \tau)$, where Λ is the martingale hazard function of τ. Therefore, we have the following result.

Lemma 2.6 *The unique $\mathbb{D}$-compensator A of a random time τ is given by the formula*

$$A_t = \int_{]0, t\wedge\tau\,]} \frac{dF(u)}{1 - F(u-)} = \Lambda(t \wedge \tau), \quad \forall\, t \in \mathbb{R}_+. \tag{2.33}$$

Proof. In view of the definition of the martingale hazard function and Proposition 2.5, it is enough to check that the process $A_t = \Lambda(t \wedge \tau)$, is $\mathbb{D}$-predictable. But this is obvious, since $t \to t \wedge \tau$ is a continuous $\mathbb{D}$-adapted process, so that it is $\mathbb{D}$-predictable. □

Combining part (ii) in Proposition 2.5 with Lemma 2.6 we get immediately the following corollary.

Corollary 2.5 *The hazard function Γ of τ is related to the $\mathbb{D}$-compensator A of the jump process D through the formula $A_t = \Gamma(t \wedge \tau)$ if and only if the cumulative distribution function F of τ is continuous.*

3 Hazard Process of a Random Time

In this section, previously introduced concepts are extended to the case when a larger flow of information – formally represented by a filtration $\mathbb{G}$ – is available. Generally speaking, our goal is to establish useful formulae for the conditional expectation of the form $\mathbf{E}\,(Y \,|\, \mathcal{G}_t)$.

3.1 Hazard Process Γ

As before, we denote by τ a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$, such that $\mathbf{P}(\tau = 0) = 0$ and $\mathbf{P}(\tau > t) > 0$ for any $t \in \mathbb{R}_+$. We introduce a right-continuous process D by setting $D_t = \mathbb{1}_{\{\tau \le t\}}$, and we denote by $\mathbb{D}$ the associated filtration: $\mathcal{D}_t = \sigma(D_u : u \le t)$. Let $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$ be an arbitrary filtration[2] on $(\Omega, \mathcal{G}, \mathbf{P})$. For each t, the σ-field $\mathcal{G}_t$ is assumed to represent all observations available at time t. We shall consider the following case:

We assume that we are given an auxiliary filtration[3] $\mathbb{F}$ such that $\mathbb{G} = \mathbb{D} \vee \mathbb{F}$, that is, $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t$ for every t. Then, for every t, we have $\mathcal{D}_t \subset \mathcal{G}_t$, that is, the random time τ is observed at any time t, and τ is a $\mathbb{G}$-stopping time.

[2]Recall that all filtrations are assumed to be $(\mathbf{P}, \mathcal{G})$-completed. We assume also that the enlarged filtration $\mathbb{G}$ satisfies the 'usual conditions.'

[3]In most applications, $\mathbb{F}$ is the natural filtration of a certain stochastic process.

The process D is obviously $\mathbb{G}$-adapted, but not necessarily $\mathbb{F}$-adapted. In other words, τ is a $\mathbb{G}$-stopping time, but not necessarily a $\mathbb{F}$-stopping time.

For any $t \in \mathbb{R}_+$, we write $F_t = \mathbf{P}(\tau \leq t \,|\, \mathcal{F}_t)$, so that $1 - F_t = \mathbf{P}(\tau > t \,|\, \mathcal{F}_t)$. It is easily seen that F ($1 - F$, resp.) is a bounded, non-negative $\mathbb{F}$-submartingale ($\mathbb{F}$-supermartingale, resp.) We may thus deal with the right-continuous modification of F. The next definition is a straightforward generalization of Definition 2.1.

Definition 3.1 Assume that $F_t < 1$ for every $t \in \mathbb{R}_+$. The $\mathbb{F}$-*hazard process* of τ, denoted by Γ, is defined through the formula $1 - F_t = e^{-\Gamma_t}$, or equivalently, $\Gamma_t = -\ln(1 - F_t)$ for every $t \in \mathbb{R}_+$.

In Section 3.1, it is assumed throughout that the inequality $F_t < 1$ holds for every t, so that the $\mathbb{F}$-hazard process Γ is well defined. It should be stressed that the case when τ is a $\mathbb{F}$-stopping time, that is, the case when $\mathbb{F} = \mathbb{G}$, is not dealt with in this section (we postpone the study of the case of a $\mathbb{F}$-stopping time to Section 4.5).

Remarks. For given filtrations $\mathbb{D} \subset \mathbb{G}$, the equality $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t$ does not specify uniquely an auxiliary filtration $\mathbb{F}$. For instance, when $\mathcal{G}_t = \mathcal{D}_t$, we may take $\mathbb{F} = \mathbb{F}^0$ (as in the previous section), but also $\mathbb{F} = \mathbb{D}$ (or any other sub-filtration of $\mathbb{D}$ for that matter). At the intuitive level, given the random time τ and the associated filtration $\mathbb{D}$, we are interested in a filtration $\hat{\mathbb{F}}$ which represents an additional flow of informations.[4] To be a bit more specific, one could search for the 'minimal' filtration $\hat{\mathbb{F}}$ for which $\mathbb{G} = \mathbb{D} \vee \hat{\mathbb{F}}$, where the concept of minimality needs to be examined. The ultimate goal could be to provide a representation of the conditional expectation $\mathbf{E}(Y \,|\, \mathcal{G}_t)$ in terms of the random time τ and the hazard process of τ with respect to $\hat{\mathbb{F}}$.

3.1.1 Evaluation of Conditional Expectation with respect to $\mathbb{G}$

We start with the following two lemmas.

Lemma 3.1 *We have $\mathcal{G}_t \subset \mathcal{G}^*_t$, where*

$$\mathcal{G}^*_t := \big\{ A \in \mathcal{G} \,|\, \exists\, B \in \mathcal{F}_t \;\; A \cap \{\tau > t\} = B \cap \{\tau > t\} \big\}.$$

Proof. Observe that $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t = \sigma(\mathcal{D}_t, \mathcal{F}_t) = \sigma(\{\tau \leq u\}, u \leq t, \mathcal{F}_t)$. Also, it is easily seen that the class $\mathcal{G}^*_t$ is a sub-σ-field of $\mathcal{G}$. Therefore, it is enough to check that if either $A = \{\tau \leq u\}$ for some $u \leq t$ or $A \in \mathcal{F}_t$, then there exists an event $B \in \mathcal{F}_t$ such that $A \cap \{\tau > t\} = B \cap \{\tau > t\}$. Indeed, in the former case we may take $B = \emptyset$, in the latter $B = A$. □

Lemma 3.2 *For any $\mathcal{G}$-measurable random variable Y we have, for any $t \in \mathbb{R}_+$,*

$$\mathbf{E}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbf{E}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{F}_t)}{\mathbf{P}(\tau > t \,|\, \mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbf{E}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{F}_t). \tag{3.1}$$

Proof. This follows from the observation (cf. Lemma 3.1) that any $\mathcal{G}_t$-measurable random variable coincides on the set $\{\tau > t\}$ with some $\mathcal{F}_t$-measurable random variable. Therefore,

$$\mathbf{E}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} Z$$

where Z is a $\mathcal{F}_t$-measurable random variable, and taking expectation with respect to $\mathcal{F}_t$, we get

$$\mathbf{E}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{F}_t) = P(\tau > t \,|\, \mathcal{F}_t)\, Z. \qquad \square$$

In the next result, we take $t < s \leq \infty$.

[4] Let us emphasize that, typically, filtrations $\mathbb{D}$ and $\mathbb{F}$ are not independent.

Proposition 3.1 *i) Assume that F follows a process of finite variation (not necessarily continuous). Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a (bounded) Borel measurable function. Then*

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}h(\tau)\;|\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\,\mathbf{E}\Big(\int_{]t,s]} h(u)\,dF_u\,\Big|\,\mathcal{F}_t\Big). \tag{3.2}$$

ii) In the general case, let Z be a (bounded) $\mathbb{F}$-predictable process. Then for any $t \le s$

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}Z_\tau\;|\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\,\mathbf{E}\Big(\int_{]t,s]} Z_u\,dF_u\,\Big|\,\mathcal{F}_t\Big). \tag{3.3}$$

Proof. Let us first prove (3.2). In view of (3.1), it is enough to check that the following equality holds

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}\,h(\tau)\,|\,\mathcal{F}_t) = \mathbf{E}\Big(\int_t^s h(u)\,dF_u\,\Big|\,\mathcal{F}_t\Big).$$

We consider first a stepwise function $h(u) = \sum_{i=0}^n h_i 1\!\!1_{]t_i,t_{i+1}]}(u)$, where, w.l.o.g., $t_0 = t < \cdots < t_{n+1} = s$. Then we have

$$\begin{aligned}
\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}h(\tau)\,|\,\mathcal{F}_t) &= \sum_{i=0}^n \mathbf{E}\,\big(\mathbf{E}\,(h_i 1\!\!1_{]t_i,t_{i+1}]}(\tau)\,|\,\mathcal{F}_{t_{i+1}})\,\big|\,\mathcal{F}_t\big)\\
&= \mathbf{E}\Big(\sum_{i=0}^n h_i(F_{t_{i+1}} - F_{t_i})\,\Big|\,\mathcal{F}_t\Big)\\
&= \mathbf{E}\Big(\sum_{i=0}^n \int_{t_i}^{t_{i+1}} h(u)\,dF_u\,\Big|\,\mathcal{F}_t\Big)\\
&= \mathbf{E}\Big(\int_t^s h(u)\,dF_u\,\Big|\,\mathcal{F}_t\Big).
\end{aligned}$$

In the second step we approximate an arbitrary (bounded) measurable function by a sequence of stepwise functions.

The proof of (3.3) is similar. We start by assuming that Z is a stepwise $\mathbb{F}$-predictable process, so that (we are interested only in values of Z for $u > t$)

$$Z_u = \sum_{i=0}^n Z_{t_i} 1\!\!1_{]t_i,t_{i+1}]}(u),$$

where $t_0 = t < \cdots < t_{n+1} = s$. In view of (3.5), for any i we have

$$\mathbf{E}\,\big(\,1\!\!1_{\{t_i<\tau\le t_{i+1}\}}\,Z_\tau\,|\,\mathcal{G}_t\big) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\,\mathbf{E}\,\big(1\!\!1_{\{t_i<\tau\le t_{i+1}\}}Z_{t_i}\;|\,\mathcal{F}_t\big).$$

Then we proceed along the similar lines as in the first part of the proof. □

Let us remark that Proposition 3.1 remains valid if $\mathbb{F} = \mathbb{G}$, that is, when τ is a $\mathbb{F}$-stopping time. However, in this case, it does not provide an interesting result. The left-hand member is $\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}Z_\tau\,|\,\mathcal{F}_t)$. Since $F_t = 1\!\!1_{\{\tau\le t\}}$, the random variable Γ_t is equal to 0 on the set $\{\tau > t\}$, and the right-hand member is obviously equal to $\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}Z_\tau\;|\,\mathcal{F}_t)$.

Corollary 3.1 *Let Y be a $\mathcal{G}$-measurable random variable. Then for any $t \le s$*

$$\mathbf{E}\,(1\!\!1_{\{\tau>s\}}\,Y\,|\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,(1\!\!1_{\{\tau>s\}}\,e^{\Gamma_t}Y\,|\,\mathcal{F}_t), \tag{3.4}$$

and

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}\,Y\,|\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}\,e^{\Gamma_t}Y\,|\,\mathcal{F}_t). \tag{3.5}$$

Furthermore, for any $\mathcal{F}_s$-measurable random variable Y we have

$$\mathbf{E}\,(1\!\!1_{\{\tau>s\}}\,Y\,|\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,\big(Ye^{\Gamma_t-\Gamma_s}\,|\,\mathcal{F}_t\big). \tag{3.6}$$

If F (and thus also Γ) follows a continuous increasing process then

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}h(\tau)\ |\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,\Big(\int_t^s h(u)e^{\Gamma_t-\Gamma_u}\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big) \tag{3.7}$$

and, for a $\mathbb{F}$ predictable process Z

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}}Z_\tau\ |\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,\Big(\int_t^s Z_u e^{\Gamma_t-\Gamma_u}\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big). \tag{3.8}$$

Proof. In view of (3.1), to show that (3.4) holds it is enough to observe that $1\!\!1_{\{\tau>s\}} = 1\!\!1_{\{\tau>t\}}1\!\!1_{\{\tau>s\}}$. Equalities (3.5)-(3.6) are straightforward consequences of (3.4). The formulae (3.7)-(3.8) are immediate consequences of formulae (3.2)-(3.3). For instance

$$\mathbf{E}\,\Big(\int_t^s h(u)e^{-\Gamma_u}\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big) = \mathbf{E}\,\Big(\int_t^s h(u)\,dF_u\,\Big|\,\mathcal{F}_t\Big)$$

since $dF_u = e^{-\Gamma_u}\,d\Gamma_u$. □

In some instances the following version of Lemma 3.2 will prove useful.

Lemma 3.3 *For any $\mathcal{G}$-measurable random variable Y, and any sub-σ-field $\mathcal{F}$ of $\mathcal{G}$, we have*

$$\mathbf{E}\,(1\!\!1_{\{\tau>t\}}Y\,|\,\mathcal{D}_t\vee\mathcal{F}) = 1\!\!1_{\{\tau>t\}}\,\frac{\mathbf{E}\,(1\!\!1_{\{\tau>t\}}Y\,|\,\mathcal{F})}{\mathbf{P}(\tau>t\,|\,\mathcal{F})}. \tag{3.9}$$

For any $t\le s$

$$\mathbf{P}(\tau\ge s\,|\,\mathcal{D}_t\vee\mathcal{F}) = 1\!\!1_{\{\tau>t\}}\,\frac{\mathbf{P}(\tau\ge s\,|\,\mathcal{F})}{\mathbf{P}(\tau>t\,|\,\mathcal{F})}. \tag{3.10}$$

Proof. The proof of the lemma is essentially the same as the proof of Lemma 3.2. It should be stressed that $\mathcal{F}$ is here an arbitrary sub-σ-field of $\mathcal{G}$. □

3.1.2 Evaluation of the Conditional Expectation $\mathbf{E}\,(Y\,|\,\mathcal{G}_t)$

Lemma 3.4 *Let Y be a bounded random variable.*

$$1\!\!1_{\{\tau\le t\}}\mathbf{E}\,(Y\,|\,\mathcal{G}_t) = 1\!\!1_{\{\tau\le t\}}\mathbf{E}\,(Y\,|\,\mathcal{D}_\infty\vee\mathcal{F}_t) = \mathbf{E}\,(1\!\!1_{\{\tau\le t\}}Y\,|\,\mathcal{D}_\infty\vee\mathcal{F}_t). \tag{3.11}$$

Proof. First, $\mathcal{G}_t\subset\mathcal{D}_t\vee\mathcal{F}_t\subset\mathcal{D}_\infty\vee\mathcal{F}_t$. Furthermore for any $A\in\mathcal{D}_\infty\vee\mathcal{F}_t$ we have

$$\int_A \mathbf{E}\,(1\!\!1_B Y\,|\,\mathcal{D}_\infty\vee\mathcal{F}_t)\,d\mathbf{P} = \int_A 1\!\!1_B Y\,d\mathbf{P} = \int_{A\cap B} Y\,d\mathbf{P} = \int_{A\cap B}\mathbf{E}\,(Y\,|\,\mathcal{G}_t)\,d\mathbf{P} = \int_A 1\!\!1_B\mathbf{E}\,(Y\,|\,\mathcal{G}_t)\,d\mathbf{P},$$

where we write $B=\{\tau\le t\}$. Notice that the random variable $1\!\!1_B\mathbf{E}\,(Y\,|\,\mathcal{G}_t)$ is manifestly $\mathcal{D}_t\vee\mathcal{G}_t$-measurable, and thus it is also $\mathcal{D}_\infty\vee\mathcal{F}_t$-measurable. We conclude that (3.11) holds. □

By combining (3.11) with (3.1), we obtain the following result (notice that formula (3.12) is a straightforward generalization of equality (2.1)).

Lemma 3.5 *For any $\mathcal{G}$-measurable random variable Y we have*

$$\mathbf{E}\,(Y\,|\,\mathcal{G}_t) = 1\!\!1_{\{\tau\le t\}}\,\mathbf{E}\,(Y\,|\,\mathcal{D}_\infty\vee\mathcal{F}_t) + 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,(1\!\!1_{\{\tau>t\}}\,e^{\Gamma_t}Y\,|\,\mathcal{F}_t). \tag{3.12}$$

In some applications, the following version of formula (3.3) appears to be useful.

Proposition 3.2 *Assume that C is a $\mathbb{F}$-predictable increasing process such that C_T is an integrable random variable. If F follows a process of finite variation then for every $t \le T$*

$$\mathbf{E}\Big(\int_{]t,T]}(1-D_u)\,dC_u\,\Big|\,\mathcal{G}_t\Big) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\mathbf{E}\Big(\int_{]t,T]}(1-F_u)\,dC_u\,\Big|\,\mathcal{F}_t\Big), \tag{3.13}$$

or equivalently,

$$\mathbf{E}\Big(\int_{]t,T]}(1-D_u)\,dC_u\,\Big|\,\mathcal{G}_t\Big) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\mathbf{E}\Big(\int_{]t,T]}e^{-\Gamma_u}\,dC_u\,\Big|\,\mathcal{F}_t\Big). \tag{3.14}$$

Proof. For a fixed $t \le T$ we introduce an auxiliary process $\tilde{C}$ by setting $\tilde{C}_u = C_u - C_t$ for $u \in [t,T]$. Then

$$\begin{aligned} J_t &:= \mathbf{E}\Big(\int_{]t,T]}(1-D_u)\,dC_u\,\Big|\,\mathcal{G}_t\Big) \\ &= \mathbf{E}\Big(\int_{]t,T]}1\!\!1_{\{\tau>u\}}\,d\tilde{C}_u\,\Big|\,\mathcal{G}_t\Big) \\ &= \mathbf{E}\Big(\tilde{C}_{\tau-}1\!\!1_{\{t<\tau\le T\}} + \tilde{C}_T 1\!\!1_{\{\tau>T\}}\,\Big|\,\mathcal{G}_t\Big) \\ &= 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\mathbf{E}\Big(\int_{]t,T]}\tilde{C}_{u-}\,dF_u + \tilde{C}_T(1-F_T)\,\Big|\,\mathcal{F}_t\Big), \end{aligned}$$

where the last equality follows from (3.3) (it is clear that the process $\tilde{C}_{u-}$ is $\mathbb{F}$-predictable). If we set $G_t = 1 - F_t$, then

$$\begin{aligned} \tilde{J}_t &:= \mathbf{E}\Big(\int_{]t,T]}\tilde{C}_{u-}\,dF_u + \tilde{C}_T(1-F_T)\,\Big|\,\mathcal{F}_t\Big) \\ &= \mathbf{E}\Big(-\int_{]t,T]}\tilde{C}_{u-}\,dG_u + \tilde{C}_T G_T\,\Big|\,\mathcal{F}_t\Big) \\ &= \mathbf{E}\Big(\int_{]t,T]}G_u\,d\tilde{C}_u + \tilde{C}_t G_t\,\Big|\,\mathcal{F}_t\Big) \\ &= \mathbf{E}\Big(\int_{]t,T]}(1-F_u)\,dC_u\,\Big|\,\mathcal{F}_t\Big), \end{aligned}$$

where we have used the integration by parts formula for processes of finite variation

$$\tilde{C}_T G_T = \tilde{C}_t G_t + \int_{]t,T]}\tilde{C}_{u-}\,dG_u + \int_{]t,T]}G_u\,d\tilde{C}_u,$$

and the obvious equality $\tilde{C}_t = 0$. This proves (3.13). Formula (3.14) is an immediate consequence of (3.13). □

3.1.3 Semimartingale Representation of the Stopped Process

In the next result we assume that a $\mathbb{F}$-predictable process $(m_t, t \ge 0)$ follows a $\mathbb{F}$-martingale. We are interested in the semimartingale decomposition with respect to the enlarged filtration $\mathbb{G}$ of the stopped process $\tilde{m}_t = m_{t\wedge\tau}$ (we shall sometimes use the standard notation m^τ for the process m stopped at τ).

Lemma 3.6 *Assume that a $\mathbb{F}$-predictable process m follows a $\mathbb{F}$-martingale.*
(i) *If F is a continuous, increasing process then the stopped process $\tilde{m}_t = m_{t\wedge\tau}$ is a $\mathbb{G}$-martingale.*
(ii) *If F is a continuous submartingale then the process*

$$M_t^m = \tilde{m}_t + \int_0^{t\wedge\tau} (1-F_u)^{-1}\, d\langle m,F\rangle_u = m_{t\wedge\tau} + \int_0^{t\wedge\tau} (1-F_u)^{-1}\, d\langle m,F\rangle_u \tag{3.15}$$

is a $\mathbb{G}$-martingale.
(iii) *If m is a continuous process then the process M^m given by* (3.15) *follows a $\mathbb{G}$-martingale.*

Proof. To establish the first statement, we fix $s>0$ and we define a $\mathbb{F}$-adapted process $\bar{m}$ by setting $\bar{m}_t = m_{t\wedge s}$ for any $t\in\mathbb{R}_+$. It is clear that for any $t\le s$ we have $\mathbf{E}\,(\tilde{m}_s \,|\, \mathcal{G}_t) = \mathbf{E}\,(\bar{m}_\tau \,|\, \mathcal{G}_t)$. Furthermore

$$\begin{aligned}
\mathbf{E}\,(\bar{m}_\tau \,|\, \mathcal{G}_t) &= 1\!\!1_{\{\tau\le t\}}\bar{m}_\tau + e^{\Gamma_t} 1\!\!1_{\{\tau>t\}} \mathbf{E}\,\Big(\int_t^\infty \bar{m}_u\, dF_u \,\Big|\, \mathcal{F}_t\Big)\\
&= 1\!\!1_{\{\tau\le t\}} m_{\tau\wedge s} + e^{\Gamma_t} 1\!\!1_{\{\tau>t\}} \mathbf{E}\,\Big(\int_t^\infty m_{u\wedge s}\, dF_u \,\Big|\, \mathcal{F}_t\Big)\\
&= 1\!\!1_{\{\tau\le t\}} m_\tau + e^{\Gamma_t} 1\!\!1_{\{\tau>t\}} J_t,
\end{aligned}$$

where

$$J_t = \mathbf{E}\,\Big(\int_t^\infty m_{u\wedge s}\, dF_u \,\Big|\, \mathcal{F}_t\Big) = -\mathbf{E}\,\Big(\int_t^s m_u\, dG_u + \int_s^\infty m_s\, dG_u \,\Big|\, \mathcal{F}_t\Big),$$

where we write $G_t := e^{-\Gamma_t} = 1 - F_t$. Since G is a $\mathbb{F}$-adapted, bounded, continuous, decreasing process, Itô's integration by parts formula combined with the $\mathbb{F}$-martingale property of m yields

$$J_t = \mathbf{E}\,\Big(G_t m_t - G_s m_s + \int_t^s G_{u-}\, dm_u - m_s(G_\infty - G_s) \,\Big|\, \mathcal{F}_t\Big) = G_t m_t = e^{-\Gamma_t} m_t,$$

where we have also used the equality $G_\infty = e^{-\Gamma_\infty} = 0$. Consequently, we get

$$\mathbf{E}\,(\tilde{m}_s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau\le t\}} m_\tau + 1\!\!1_{\{\tau>t\}} m_t = m_{\tau\wedge t} = \tilde{m}_t,$$

the desired result. This ends the proof of (i).

For the second statement, we denote, as before, $G_t = e^{-\Gamma_t} = 1 - F_t$ (G is now a continuous bounded $\mathbb{F}$-supermartingale). For a fixed $s>0$, we introduce an auxiliary process $\bar{m}$

$$\bar{m}_t := m_{t\wedge s} - \int_0^{t\wedge s} G_u^{-1}\, d\langle m,F\rangle_u = m_{t\wedge s} + m^*_{t\wedge s},$$

where we set $m_t^* = \int_0^t G_u^{-1}\, d\langle m,F\rangle_u$. Obviously, for any $t\le s$ we have $\mathbf{E}\,(M_s^m \,|\, \mathcal{G}_t) = \mathbf{E}\,(\bar{m}_\tau \,|\, \mathcal{G}_t)$. We need to show that $\mathbf{E}\,(\bar{m}_\tau \,|\, \mathcal{G}_t) = M_t^m$. To this end, notice that

$$\mathbf{E}\,(\bar{m}_\tau \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau\le t\}} M_\tau + e^{\Gamma_t} 1\!\!1_{\{\tau>t\}} (J_t^1 + J_t^2),$$

where

$$J_t^1 := \mathbf{E}\,\Big(\int_t^\infty m_{u\wedge s}\, dF_u \,\Big|\, \mathcal{F}_t\Big), \quad J_t^2 := \mathbf{E}\,\Big(\int_t^\infty m^*_{u\wedge s}\, dF_u \,\Big|\, \mathcal{F}_t\Big).$$

For J_t^1, using Itô's formula and the martingale property of m, we get (notice that $G_\infty = 1 - F_\infty = 0$)

$$\begin{aligned}
J_t^1 &= -\mathbf{E}\,\Big(\int_t^s m_u\, dG_u + \int_s^\infty m_s\, dG_u \,\Big|\, \mathcal{F}_t\Big)\\
&= \mathbf{E}\,\Big(G_t m_t - G_s m_s + \int_t^s G_u\, dm_u + \int_t^s d\langle G,m\rangle_u - m_s(G_\infty - G_s) \,\Big|\, \mathcal{F}_t\Big)\\
&= G_t m_t + \langle m,G\rangle_s - \langle m,G\rangle_t = e^{-\Gamma_t} m_t + \langle m,F\rangle_t - \mathbf{E}\,(\langle m,F\rangle_s \,|\, \mathcal{F}_t).
\end{aligned}$$

On the other hand, since m^* is a continuous process of finite variation, another application of Itô's formula yields

$$\begin{aligned} J_t^2 &= -\mathbf{E}\Big(\int_t^s m_u^*\,dG_u + \int_s^\infty m_s^*\,dG_u \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbf{E}\Big(G_t m_t^* - G_s m_s^* + \int_t^s G_u\,dm_u^* - m_s^*(G_\infty - G_s)\,\Big|\,\mathcal{F}_t\Big) \\ &= e^{-\Gamma_t} m_t^* - \langle m, F\rangle_t + \mathbf{E}\left(\langle m, F\rangle_s \,|\, \mathcal{F}_t\right), \end{aligned}$$

where the last equality is a consequence of the definition of m^*. Upon simplification, for any $t \leq s$ we obtain

$$\mathbf{E}\left(\bar{m}_\tau \,|\, \mathcal{G}_t\right) = 1\!\!1_{\{\tau \leq t\}} M_\tau^m + 1\!\!1_{\{\tau > t\}}(m_t + m_t^*) = M_t^m.$$

This completes the proof of part (ii). The proof of part (iii) is similar. Let us stress that we no longer assume that F necessarily follows a continuous process, but since m is assumed to be a continuous process, we have $[m, F] = \langle m, F\rangle$. □

3.1.4 Martingales Associated with the Hazard Process Γ

The next result is a generalization of Lemma 2.2. Let us stress that the case when $\mathbb{F} = \mathbb{G}$ is not covered by Lemma 3.7.

Lemma 3.7 *The process L given by the formula*

$$L_t := 1\!\!1_{\{\tau > t\}} e^{\Gamma_t} = (1 - D_t)e^{\Gamma_t} = \frac{1 - D_t}{1 - F_t} \tag{3.16}$$

follows a $\mathbb{G}$-martingale. Moreover, for any $\mathbb{F}$-martingale m, the product Lm is a $\mathbb{G}$-martingale. If, in addition, m follows also a $\mathbb{G}$-martingale then the quadratic variation $[L, m]$, which equals

$$[L, m]_t := L_t m_t - L_0 m_0 - \int_{]0,t]} L_{s-}\,dm_s - \int_{]0,t]} m_{s-}\,dL_s \tag{3.17}$$

is a $\mathbb{G}$-martingale.

Proof. It is enough to check that for any $t \leq s$

$$\mathbf{E}\left(1\!\!1_{\{\tau > s\}} e^{\Gamma_s} \,|\, \mathcal{G}_t\right) = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}.$$

In view of (3.4) this can be rewritten as follows

$$1\!\!1_{\{\tau > t\}} e^{\Gamma_t}\,\mathbf{E}\left(1\!\!1_{\{\tau > s\}} e^{\Gamma_s} \,|\, \mathcal{F}_t\right) = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}.$$

To complete the proof of the first statement, it is enough to observe that

$$\mathbf{E}\left(1\!\!1_{\{\tau > s\}} e^{\Gamma_s} \,|\, \mathcal{F}_t\right) = \mathbf{E}\left(e^{\Gamma_s}\mathbf{E}\left(1\!\!1_{\{\tau > s\}} \,|\, \mathcal{F}_s\right) \,|\, \mathcal{F}_t\right) = 1.$$

For the second part of the lemma, notice that for $t \leq s$ in view of (3.1) we have

$$\begin{aligned} \mathbf{E}\left(L_s m_s \,|\, \mathcal{G}_t\right) &= \mathbf{E}\left(1\!\!1_{\{\tau > s\}} L_s m_s \,|\, \mathcal{G}_t\right) = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}\,\mathbf{E}\left(1\!\!1_{\{\tau > s\}} e^{\Gamma_s} m_s \,|\, \mathcal{F}_t\right) \\ &= 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}\,\mathbf{E}\left(m_s e^{\Gamma_s}\mathbf{E}\left(1\!\!1_{\{\tau > s\}} \,|\, \mathcal{F}_s\right) \,|\, \mathcal{F}_t\right) = (1 - D_t)e^{\Gamma_t} m_t = L_t m_t \end{aligned}$$

so that Lm is a $\mathbb{G}$-martingale. The last statement is obvious. □

Notice that under the assumptions of Lemma 3.7, if Γ is an increasing process, so that L is a process of finite variation, then we have $[L, m]_t = \sum_{u \leq t} \Delta L_u\,\Delta m_u$. In this case, (3.17) can be rewritten as follows

$$L_t m_t = L_0 m_0 + \int_{]0,t]} L_{s-}\,dm_s + \int_{]0,t]} m_s\,dL_s. \tag{3.18}$$

In the next result we deal with the continuous case; more precisely, we assume that Γ is a continuous increasing process. The following result is a counterpart of Propositions 2.1-2.2.

Proposition 3.3 *Assume that the $\mathbb{F}$-hazard process Γ of τ follows an increasing continuous process. Then:*
(i) *the process $M_t = D_t - \Gamma_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale, more specifically,*

$$M_t = -\int_{]0,t]} e^{-\Gamma_u}\, dL_u. \tag{3.19}$$

Furthermore, $L_t = \mathcal{E}_t(-M)$, that is, L satisfies

$$L_t = 1 - \int_{]0,t]} L_{u-}\, dM_u. \tag{3.20}$$

(ii) *If a $\mathbb{F}$-martingale m is also a $\mathbb{G}$-martingale then the product Mm is a $\mathbb{G}$-martingale – that is, M and m are mutually orthogonal $\mathbb{G}$-martingales.*
(iii) *If a $\mathbb{F}$-predictable process m is a $\mathbb{F}$-martingale then the product $M\tilde{m}$ is a $\mathbb{G}$-martingale, where $\tilde{m}$ is the stopped process, that is, $\tilde{m}_t = m_{t\wedge\tau}$ for every t.*

Proof. Proof of (i). The martingale property of M and equalities (3.19)-(3.20) can be shown using the same arguments as in the proof of Proposition 2.1, i.e., Itô's formula combined with Lemma 3.7.

Proof of (ii). We need to show that $\mathbf{E}\,(M_s m_s - M_t m_t \,|\, \mathcal{G}_t) = 0$ for any $t \le s$, or equivalently, that

$$\mathbf{E}\,(m_s(M_s - M_t) \,|\, \mathcal{G}_t) + \mathbf{E}\,(M_t(m_s - m_t) \,|\, \mathcal{G}_t) = 0.$$

Since m is a $\mathbb{G}$-martingale, it is clear that $\mathbf{E}\,(M_t(m_s - m_t) \,|\, \mathcal{G}_t) = M_t \mathbf{E}\,(m_s - m_t \,|\, \mathcal{G}_t) = 0$. On the other hand

$$\mathbf{E}\,(m_s(M_s - M_t) \,|\, \mathcal{G}_t) = -\mathbf{E}\,(m_s \int_{]t,s]} e^{-\Gamma_u}\, dL_u \,|\, \mathcal{G}_t).$$

To show that the last term above vanishes, let us consider a sequence of stepwise processes $\Gamma^n_u = \sum_{i=0}^n \Gamma_{t_i} \mathbb{1}_{[t_i,t_{i+1}[}(u)$, where $t = t_0 < \cdots < t_{n+1} = s$. Then

$$\begin{aligned}
\mathbf{E}\,(m_s \int_{]t,s]} e^{-\Gamma^n_u}\, dL_u \,|\, \mathcal{G}_t) &= \mathbf{E}\Big(m_s \sum_{i=0}^n e^{-\Gamma_{t_i}}(L_{t_{i+1}} - L_{t_i}) \,\Big|\, \mathcal{G}_t\Big) \\
&= \sum_{i=0}^n \mathbf{E}\left(\mathbf{E}\left(e^{-\Gamma_{t_i}} m_s (L_{t_{i+1}} - L_{t_i}) \,\middle|\, \mathcal{G}_{t_{i+1}}\right) \middle|\, \mathcal{G}_t\right) \\
&= \sum_{i=0}^n \mathbf{E}\left(\mathbf{E}\left(e^{-\Gamma_{t_i}} m_{t_{i+1}} (L_{t_{i+1}} - L_{t_i}) \,\middle|\, \mathcal{G}_{t_i}\right) \middle|\, \mathcal{G}_t\right) \\
&= \sum_{i=0}^n \mathbf{E}\left(\mathbf{E}\left(e^{-\Gamma_{t_i}} m_{t_i} (L_{t_i} - L_{t_i}) \,\middle|\, \mathcal{G}_{t_{i+1}}\right) \middle|\, \mathcal{G}_t\right) = 0,
\end{aligned}$$

where we have used the martingale property with respect to $\mathbb{G}$ of m and Lm (since by assumption m is a $\mathbb{F}$-martingale, the second property follows from Lemma 3.7).

The proof of part (ii) can be shortened by the direct use of the Itô calculus. Indeed, by virtue of Lemma 3.7 L and m are orthogonal $\mathbb{G}$-martingales, so that the proces $[L,m]$ is also a $\mathbb{G}$-martingale. Consequently, applying Itô's formula and (3.19), we get

$$\begin{aligned}
M_t m_t &= M_0 m_0 + \int_{]0,t]} M_{u-}\, dm_u + \int_{]0,t]} m_{u-}\, dM_u + [M,m]_t \\
&= M_0 m_0 + \int_{]0,t]} M_{u-}\, dm_u - \int_{]0,t]} m_{u-} e^{-\Gamma_u}\, dL_u - \int_{]0,t]} e^{-\Gamma_u}\, d[L,m]_u
\end{aligned}$$

where L, m and $[L,m]$ follow $\mathbb{G}$-martingales.

Proof of (iii). If m is a predictable $\mathbb{F}$-martingale then by virtue of part (i) in Lemma 3.6 the stopped process $\tilde{m} = m_{t\wedge\tau}$ is a $\mathbb{G}$-martingale. In view of Lemma 3.7, the product Lm is a $\mathbb{G}$-martingale, and thus also the stopped process $(Lm)^\tau = \tilde{L}\tilde{m}$, where $\tilde{L}_t = L_{t\wedge\tau}$ also follows a $\mathbb{G}$-martingale. Since

$$[\tilde{L}, \tilde{m}]_t := \tilde{L}_t \tilde{m}_t - \tilde{L}_0 \tilde{m}_0 - \int_{]0,t]} \tilde{L}_{s-}\, d\tilde{m}_s - \int_{]0,t]} \tilde{m}_{s-}\, d\tilde{L}_s,$$

we conclude that the quadratic variation $[\tilde{L}, \tilde{m}]$ is a $\mathbb{G}$-martingale.

Consequently, the product $M\tilde{m}$ is a $\mathbb{G}$-martingale, since

$$\begin{aligned} M_t \tilde{m}_t &= M_0 \tilde{m}_0 + \int_{]0,t]} M_{u-}\, d\tilde{m}_u + \int_{]0,t]} \tilde{m}_{u-}\, dM_u + [M, \tilde{m}]_t \\ &= M_0 m_0 + \int_{]0,t]} M_{u-}\, d\tilde{m}_u - \int_{]0,t]} m_{u-} e^{-\Gamma_u}\, d\tilde{L}_u - \int_{]0,t]} e^{-\Gamma_u}\, d[L, \tilde{m}]_u \\ &= M_0 m_0 + \int_{]0,t]} M_{u-}\, d\tilde{m}_u - \int_{]0,t]} m_{u-} e^{-\Gamma_u}\, d\tilde{L}_u - \int_{]0,t]} e^{-\Gamma_u}\, d[\tilde{L}, \tilde{m}]_u \end{aligned}$$

where we have used the well-known equality $[L, \tilde{m}] = [L, m]^\tau = [\tilde{L}, \tilde{m}]$. □

Corollary 3.2 *For any bounded $\mathbb{F}$-predictable process Z the following processes are $\mathbb{G}$-martingales:*

$$V_t^1 = Z_\tau \mathbb{1}_{\{\tau \le t\}} - \int_0^{t\wedge\tau} Z_u\, d\Gamma_u = \int_0^t Z_u\, dM_u, \tag{3.21}$$

and

$$V_t^2 = \exp\left(\mathbb{1}_{\{\tau \le t\}} Z_\tau\right) - \int_0^{t\wedge\tau} (e^{Z_u} - 1)\, d\Gamma_u. \tag{3.22}$$

Proof. It is apparent that V^1 is a $\mathbb{G}$-martingale. The martingale property of V^2 is an easy consequence of the martingale property of V^1. □

Remarks. If the continuous process Γ is not of finite variation, formula (2.10) becomes

$$L_t = (1 - D_t) e^{\Gamma_t} = 1 + \int_0^t e^{\Gamma_u} \big((1 - D_u)\, (d\Gamma_u + (1/2)\langle\Gamma\rangle_u) - dD_u \big)$$

and it is no longer true that M is a $\mathbb{G}$-martingale (but, of course, the process $D_t - \Gamma_{t\wedge\tau} - (1/2)\langle\Gamma\rangle_{t\wedge\tau}$ is a $\mathbb{G}$-martingale).

Lemma 3.8 *The martingale $\rho_t = \mathbf{E}\,(\mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t)$ has the dynamics*

$$d\rho_t = -\rho_{t-} dM_t + L_{t-} dm_t + d[L, m]_t, \tag{3.23}$$

where $m_t = \mathbf{E}\,(e^{-\Gamma_T} \,|\, \mathcal{F}_t)$.

Proof. It suffices to write

$$\rho_t = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} \mathbf{E}\,(e^{\Gamma_T} \,|\, \mathcal{F}_t) = L_t m_t.$$

Decomposition (3.23) is useful in the study of the dynamics of a defaultable zero-coupon bond (see Blanchet-Scalliet and Jeanblanc [2]). □

3.1.5 $\mathbb{F}$-intensity of a Random Time

Let us consider the classic case of an absolutely continuous $\mathbb{F}$-hazard process Γ (of course, Γ is here a continuous increasing process). More precisely, we assume that $\Gamma_t = \int_0^t \gamma_u \, du$ for some $\mathbb{F}$-progressively measurable nonnegative process γ, referred to as the $\mathbb{F}$-*intensity* of a random time τ. By virtue of Proposition 3.3, the process M given by the formula

$$M_t = D_t - \int_0^{t\wedge\tau} \gamma_u \, du = D_t - \int_0^t 1\!\!1_{\{\tau>u\}} \gamma_u \, du = D_t - \int_0^t 1\!\!1_{\{\tau\geq u\}} \gamma_u \, du \tag{3.24}$$

follows a $\mathbb{G}$-martingale. This property is frequently used in the financial literature as a definition of a 'stochastic intensity' of a random time. The intuitive meaning of the $\mathbb{F}$-intensity γ as the 'intensity of survival given the flow of informations $\mathbb{F}$' becomes clear from the following corollary.

Corollary 3.3 *If the $\mathbb{F}$-hazard process Γ of τ is absolutely continuous then for any $t \leq s$*

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \, \mathbf{E}\left(e^{-\int_t^s \gamma_u \, du} \,\middle|\, \mathcal{F}_t\right)$$

and

$$\mathbf{P}(t < \tau \leq s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \, \mathbf{E}\left(1 - e^{-\int_t^s \gamma_u \, du} \,\middle|\, \mathcal{F}_t\right).$$

Remarks. Since obviously the $\mathbb{F}$-hazard function Γ is not well defined when τ is a $\mathbb{F}$-stopping time (that is, when $\mathbb{D} \subset \mathbb{F}$ so that $\mathbb{G} = \mathbb{F}$), Corollaries 3.1-3.3 cannot be directly applied in such a case. However, it appears that for a certain class of a $\mathbb{G}$-stopping times we can find an increasing $\mathbb{G}$-predictable process Λ such that for any $t \leq s$

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \, \mathbf{E}\left(e^{\Lambda_t - \Lambda_s} \,\middle|\, \mathcal{G}_t\right) = 1\!\!1_{\{\tau>t\}} \, \mathbf{E}\left(e^{-\int_t^s \lambda_u \, du} \,\middle|\, \mathcal{G}_t\right),$$

where the second equality holds provided that the process Λ is absolutely continuous, that is, $\Lambda_t = \int_0^t \lambda_u \, du$ for some process λ. It seems natural to conjecture that the *martingale hazard process* Λ, which is formally introduced in Section 4, always represents the $\hat{\mathbb{F}}$-hazard process of τ for some filtration $\hat{\mathbb{F}}$ such that τ is not a $\hat{\mathbb{F}}$-stopping time. The notion of a martingale hazard process should thus be seen as a technical tool which allows us to find the 'natural' hazard process of τ.

3.2 Martingale Representation Theorems

3.2.1 General Case

We shall first consider a general setup: in particular, we do not assume that the filtration $\mathbb{F}$ supports only continuous martingales. We shall assume, however, that the process F is continuous and increasing. Recall that if $F_t = \mathbf{P}(\tau \leq t \,|\, \mathcal{F}_t)$ follows an increasing continuous process then the associated hazard process Γ is also an increasing continuous process, since $\Gamma_t = -\ln(1 - F_t)$. Recall also that in this case the process $M = D_t - \Gamma_{t\wedge\tau}$ is a $\mathbb{G}$-martingale. We reproduce here the results of Blanchet-Scalliet and Jeanblanc [2].

Proposition 3.4 *Assume that the $\mathbb{F}$-hazard process Γ of τ follows an increasing continuous process. Let Z be a $\mathbb{F}$-predictable process such that the random variable Z_τ is integrable. Then the $\mathbb{G}$-martingale $M_t^Z := E(Z_\tau | \mathcal{G}_t)$ admits the following decomposition*

$$M_t^Z = m_0 + \int_{]0,t]} e^{\Gamma_u} d\tilde{m}_u + \int_{]0,t]} (Z_u - z_u) \, dM_u, \tag{3.25}$$

where $\tilde{m}_t = m_{t\wedge\tau}$, *and* m *is a* $\mathbb{F}$*-martingale, namely,*

$$m_t = \mathbf{E}\Big(\int_0^\infty Z_u e^{-\Gamma_u} d\Gamma_u \,\Big|\, \mathcal{F}_t\Big) = \mathbf{E}\Big(\int_0^\infty Z_u \, dF_u \,\Big|\, \mathcal{F}_t\Big) \tag{3.26}$$

so that in particular $m_0 = M_0^Z$. *Moreover*

$$z_t = \mathbf{E}\Big(\int_t^\infty Z_u e^{\Gamma_t - \Gamma_u} d\Gamma_u \,\Big|\, \mathcal{F}_t\Big) = e^{\Gamma_t}\mathbf{E}\Big(\int_t^\infty Z_u \, dF_u \,\Big|\, \mathcal{F}_t\Big). \tag{3.27}$$

Proof. By virtue of Proposition 3.1, we have

$$M_t^Z = \mathbf{E}\,(Z_\tau \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau\le t\}} Z_\tau + 1\!\!1_{\{\tau>t\}} \mathbf{E}\Big(\int_t^\infty Z_u e^{\Gamma_t-\Gamma_u} d\Gamma_u \,\Big|\, \mathcal{F}_t\Big) = 1\!\!1_{\{\tau\le t\}} Z_\tau + 1\!\!1_{\{\tau>t\}} z_t.$$

It follows from our assumptions that e^{Γ_t} is a continuous increasing process. Hence the Itô integration by parts formula yields (notice that m is a $\mathbb{G}$-semimartingale)

$$\begin{aligned} z_t &= e^{\Gamma_t} m_t - e^{\Gamma_t}\int_0^t Z_u e^{-\Gamma_u} d\Gamma_u \\ &= m_0 + \int_{]0,t]} e^{\Gamma_u} dm_u + \int_0^t m_u e^{\Gamma_u} d\Gamma_u - \int_0^t Z_u d\Gamma_u - \int_0^t e^{\Gamma_u}\int_0^u Z_v e^{-\Gamma_v} d\Gamma_v \, d\Gamma_u \\ &= m_0 + \int_{]0,t]} e^{\Gamma_u} dm_u + \int_0^t (z_u - Z_u)\, d\Gamma_u. \end{aligned}$$

Furthermore, since z is a right-continuous process with left-hand side limits, we have

$$1\!\!1_{\{\tau>t\}} z_t = m_0 + \int_{]0,t\wedge\tau]} dz_u - 1\!\!1_{\{\tau\le t\}} z_\tau.$$

We conclude that

$$M_t^Z = m_0 + \int_{]0,t\wedge\tau]} e^{\Gamma_u}\, dm_u + \int_0^{t\wedge\tau} (z_u - Z_u)\, d\Gamma_u + 1\!\!1_{\{\tau\le t\}}(Z_\tau - z_\tau),$$

and this immediately yields (3.25). □

Corollary 3.4 *Under the assumptions of Proposition* 3.4 *if, in addition,* $\Delta z_\tau = 0$ (*or equivalently,* $\Delta m_\tau = 0$) *then*

$$M_t^Z = m_0 + \int_{]0,t]} e^{\Gamma_u} d\tilde{m}_u + \int_{]0,t]} (Z_u - M_{u-}^Z)\, dM_u = m_0 + \tilde{M}_t^Z + \hat{M}_t^Z, \tag{3.28}$$

where $\tilde{M}^Z$ *and* $\hat{M}^Z$ *are mutually orthogonal* $\mathbb{G}$*-martingales, i.e.,* $\tilde{M}^Z\hat{M}^Z$ *is a* $\mathbb{G}$*-martingale.*

Proof. Since obviously $z_u = M_u^Z$ for $u < \tau$, we get also $z_\tau = z_{\tau-} = M_{\tau-}^Z$. Thus

$$1\!\!1_{\{\tau>t\}} z_t = m_0 + \int_{]0,t\wedge\tau]} dz_u - 1\!\!1_{\{\tau\le t\}} M_{\tau-}^Z.$$

Since Γ is a continuous process, we conclude that (3.28) is valid. For the second statement, notice that by virtue of part (iii) in Proposition 3.3 the quadratic variation $[M, \tilde{m}]$ is a $\mathbb{G}$-martingale. Consequently, the process[5]

$$[\tilde{M}^Z, \hat{M}^Z]_t = \int_{]0,t]} e^{\Gamma_u}(Z_u - M_{u-}^Z)\, d[M,\tilde{m}]_u$$

[5] In fact is it easily seen that under the present assumptions we have $[M, \tilde{m}] = 0$.

follows a $\mathbb{G}$-martingale, and thus also the product $\tilde{M}^Z \hat{M}^Z$ is a $\mathbb{G}$-martingale. □

If the filtration $\mathbb{F}$ supports only continuous martingales (for instance, if $\mathbb{F}$ is the natural filtration of some Brownian motion under $\mathbf{P}$) then clearly $\Delta m_\tau = 0$. In this case equality (3.28) represents the decomposition of the $\mathbb{G}$-martingale M^Z into a continuous $\mathbb{G}$-martingale $\tilde{M}^Z$ and a discontinuous $\mathbb{G}$-martingale $\hat{M}^Z$. Since $\tilde{M}^Z$ and $\hat{M}^Z$ are mutually orthogonal $\mathbb{G}$-martingales, formula (3.25) gives in fact the canonical decomposition of the $\mathbb{G}$-martingale M^Z into a continuous martingale part $\tilde{M}^Z$ and a purely discontinuous martingale part $\hat{M}^Z$.

Corollary 3.5 *Assume that Γ is an increasing continuous process. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a Borel measurable function such that $h(\tau)$ is integrable. Then $M_t^h := E(h(\tau)|\,\mathcal{G}_t)$ admits the following decomposition*

$$M_t^h = m_0 + \int_{]0,t]} e^{\Gamma_u} d\tilde{m}_u + \int_{]0,t]} (h(u) - g_u)\, dM_u, \tag{3.29}$$

where $\tilde{m}_t = m_{t\wedge\tau}$ and m is a $\mathbb{F}$-martingale

$$m_t = \mathbf{E}\Big(\int_0^\infty h(u)e^{-\Gamma_u} d\Gamma_u \,\Big|\, \mathcal{F}_t\Big) = \mathbf{E}\Big(\int_0^\infty h(u)\, dF_u \,\Big|\, \mathcal{F}_t\Big) \tag{3.30}$$

so that $m_0 = M_0^h$, and

$$g_t = e^{\Gamma_t}\, \mathbf{E}\Big(\int_t^\infty h(u)e^{-\Gamma_u} d\Gamma_u \,\Big|\, \mathcal{F}_t\Big) = e^{\Gamma_t}\, \mathbf{E}\Big(\int_t^\infty h(u)\, dF_u \,\Big|\, \mathcal{F}_t\Big). \tag{3.31}$$

The following corollary to Proposition 3.4 was already established in Section 2.3.

Corollary 3.6 *Assume that the filtration $\mathbb{F}$ is trivial and Γ follows an increasing continuous function. Let $M_t^h := \mathbf{E}\,(h(\tau)\,|\,\mathcal{D}_t)$ for some bounded Borel measurable function $h : \mathbb{R} \to \mathbb{R}$. Then*

$$M_t^h = M_0^h + \int_{]0,t]} (h(u) - g(u))\, dM_u,$$

where the function $g : \mathbb{R}_+ \to \mathbb{R}$ equals

$$g(t) = e^{\Gamma(t)}\, \mathbf{E}\left(h(\tau)\mathbb{1}_{\{\tau>t\}}\right) = e^{\Gamma(t)} \int_t^\infty h(u)\, dF(u).$$

3.2.2 Case of a Brownian Filtration

In this section, we consider the case of the Brownian filtration, that is, we assume that $\mathbb{F} = \mathbb{F}^W$ for some Brownian motion W. We postulate that the Brownian motion W remains a martingale (and thus a Brownian motion) with respect to the enlarged filtration $\mathbb{G}$. (See section 4.1.1) Let us fix $T > 0$. In the next result, we do not assume that F is an increasing process (in other words, we do not assume that the $\mathbb{G}$-martingale L is of finite variation). We reproduce here a result of Kusuoka [19].

Proposition 3.5 *Let X be a $\mathcal{G}_T$-measurable integrable random variable and let $M_t^X := E(X|\,\mathcal{G}_t)$ for $t \in [0,T]$. Then the $\mathbb{G}$-martingale M^X admits the following decomposition*

$$M_t^X = M_0^X + \int_0^t \xi_u^X\, dW_u + \int_{]0,t]} \tilde{\zeta}_u^X\, dL_u = M_0^X + \tilde{L}_t^X + \hat{L}_t^X, \tag{3.32}$$

for $t \in [0,T]$, where ξ^X and $\tilde{\zeta}^X$ are $\mathbb{G}$-predictable stochastic processes. Moreover, the $\mathbb{G}$-martingales $\tilde{L}_t^X$ and $\hat{L}_t^X$ are mutually orthogonal.

Proof. Since $\mathcal{G}_T = \mathcal{D}_T \vee \mathcal{F}_T$, it is clear that it is enough to consider a random variable X of the form $X = (1 - D_s)Y$ for some fixed $s \le T$, where Y is a $\mathcal{F}_T$-measurable random variable. Notice that

$$X = (1 - D_s)Y = (1 - D_s)e^{\Gamma_s}\tilde{Y} = L_s\tilde{Y},$$

where $\tilde{Y} = e^{-\Gamma_s}Y$ is a $\mathcal{F}_T$-measurable integrable random variable. We introduce the $\mathbb{F}$-martingale

$$U_t = \mathbf{E}\,(\tilde{Y}\,|\,\mathcal{F}_t) = \mathbf{E}\,(\tilde{Y}) + \int_0^t \xi_u\,dW_u, \quad \forall\, t \in [0,T], \tag{3.33}$$

where the second equality is a consequence of the martingale representation property of the Brownian motion W (so that ξ is a $\mathbb{F}$-predictable stochastic process). Since by assumption W is also a $\mathbb{G}$-martingale it is clear the process U is not only a $\mathbb{F}$-martingale, but also a (continuous) $\mathbb{G}$-martingale. Invoking Lemma 3.7, we find that the quadratic variation $[L,U]$ follows a $\mathbb{G}$-martingale. Thus we obtain (L^c stands for the continuous martingale component of L)

$$[L,U]_t = \langle L^c, U\rangle_t + \sum_{u\le t} \Delta L_u\,\Delta U_u = \langle L^c, U\rangle_t.$$

It turns out that $[L,U] = \langle L^c, U\rangle = 0$ (recall that any continuous martingale of finite variation is constant). Clearly $U_T = \tilde{Y}$ so that $X = L_sU_T$. Itô's integration by parts formula yields

$$\begin{aligned} X &= L_0U_0 + \int_0^T L_{t-}\,dU_t + \int_{]0,s]} U_{t-}\,dL_t + [L,U]_s \\ &= \mathbf{E}\,(\tilde{Y}) + \int_0^T \xi_t L_{t-}\,dW_t + \int_{]0,T]} U_t \mathbb{1}_{[0,s]}(t)\,dL_t. \end{aligned}$$

We thus get (3.32) with $\xi^X_t = \xi_t L_{t-}$ and $\tilde{\zeta}^X_t = U_t\mathbb{1}_{[0,s]}(t)$. It remains to show that the $\mathbb{G}$-martingales $\tilde{L}^X$ and $\hat{L}^X$ are mutually orthogonal. As a consequence of Lemma 3.7, we get $[L,W] = \langle L, W\rangle = 0$. The $\mathbb{G}$-martingale property of the product $\tilde{L}^X\hat{L}^X$ thus easily follows from Itô's lemma. □

Corollary 3.7 *Under the assumptions of Proposition* 3.5 *if, in addition, the hazard process* Γ *is an increasing continuous process then*

$$M^X_t = M^X_0 + \int_0^t \xi^X_u\,dW_u + \int_{]0,t]} \zeta^X_u\,dM_u = M^X_0 + \tilde{M}^X_t + \hat{M}^X_t, \tag{3.34}$$

where ξ^X *and* ζ^X *are* $\mathbb{G}$*-predictable stochastic processes. Moreover, the continuous* $\mathbb{G}$*-martingale* $\tilde{M}^X$ *and the purely discontinuous* $\mathbb{G}$*-martingale* $\hat{M}^X$ *are mutually orthogonal.*

Proof. It is enough to observe that part (i) in Proposition 3.3 yields

$$\int_{]0,t]} U_{u-}\,dL_u = -\int_{]0,t]} U_u L_{u-} dM_u$$

so that it is enough to set $\zeta^X_t = -U_tL_{t-}$. The mutual orthogonality of $\mathbb{G}$-martingales $\tilde{M}^X$ and M^X is also obvious. □

In the next result, we no longer assume that W is a martingale with respect to $\mathbb{G}$. By virtue of Lemma 3.6, if F is an increasing continuous process then the stopped process $W_{t\wedge\tau}$ is a $\mathbb{G}$-martingale. Otherwise, the process

$$\hat{W}_t = W_{t\wedge\tau} + \int_0^{t\wedge\tau} (1 - F_u)^{-1}\,d\langle W, F\rangle_u \tag{3.35}$$

is a $\mathbb{G}$-martingale.

Let Z be a $\mathbb{F}$-predictable process Z, such that the random variable Z_τ is integrable, we define (cf. (3.26))

$$m_t = \mathbf{E}\Big(\int_0^\infty Z_u e^{-\Gamma_u} d\Gamma_u \,\Big|\, \mathcal{F}_t\Big).$$

It is clear that m admits the integral representation

$$m_t = m_0 + \int_0^t \xi_u^Z \, dW_u$$

for some $\mathbb{F}$-predictable process ξ^Z. The next result is an immediate corollary to Proposition 3.4.

Corollary 3.8 *Assume that the $\mathbb{F}$-hazard process Γ of τ follows an increasing continuous process. Then the $\mathbb{G}$-martingale $M_t^Z := E(Z_\tau | \mathcal{G}_t)$ admits the following decomposition into two mutually orthogonal $\mathbb{G}$-martingales*

$$M_t^Z = m_0 + \int_0^t e^{\Gamma_u} \xi_u^Z \, dW_{u\wedge\tau} + \int_{]0,t]} (Z_u - M_{u-}^Z) \, dM_u. \tag{3.36}$$

Notice that on the random set $[0,\tau]$ the process M_{t-}^Z equals to some $\mathbb{F}$-predictable process. Therefore equality (3.36) means also that

$$M_t^Z = m_0 + \int_0^t \xi_u \, dW_{u\wedge\tau} + \int_{]0,t\wedge\tau]} \zeta_u \, dM_u$$

for some $\mathbb{F}$-predictable processes ξ and ζ.

3.3 Change of a Probability Measure

We shall now deal with an equivalent change of a probability measure. We make the following standing assumptions: (a) $\mathbb{F}$ is generated by a Brownian motion, (b) any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale under $\mathbf{P}$, (c) the hazard process Γ of τ is an increasing continuous process. ¿From Proposition 3.3 we know that the process $M_t = D_t - \Gamma_{t\wedge\tau}$ is a $\mathbb{G}$-martingale. We fix $T > 0$. For a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{G}_T)$ we introduce the $\mathbb{G}$-martingale η_t, $t \leq T$, by setting

$$\eta_t := \frac{d\mathbf{P}^*}{d\mathbf{P}}_{\,|\mathcal{G}_t} = \mathbf{E_P}(X \,|\, \mathcal{G}_t), \quad \mathbf{P}\text{-a.s.}, \tag{3.37}$$

where X is a $\mathcal{G}_T$-measurable integrable random variable, such that $\mathbf{P}(X > 0) = 1$. In view of Corollary 3.7 the Radon-Nikodým density process η admits the following representation

$$\eta_t = 1 + \int_0^t \xi_u \, dW_u + \int_{]0,t]} \zeta_u \, dM_u$$

where ξ and ζ are $\mathbb{G}$-predictable stochastic processes. Since η is a strictly positive process, we get

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} \big(\beta_u \, dW_u + \kappa_u \, dM_u\big) \tag{3.38}$$

where β and κ are $\mathbb{G}$-predictable processes, with $\kappa > -1$.

Proposition 3.6 *Let $\mathbf{P}^*$ be a probability measure on $(\Omega, \mathcal{G}_T)$ equivalent to $\mathbf{P}$. If the Radon-Nikodým density of $\mathbf{P}^*$ with respect to $\mathbf{P}$ is given by (3.37) with η satisfying (3.38), then the process*

$$W_t^* = W_t - \int_0^t \beta_u \, du, \quad \forall\, t \in [0,T], \tag{3.39}$$

follows a Brownian motion with respect to $\mathbb{G}$ under $\mathbf{P}^*$, and the process

$$M_t^* := M_t - \int_{]0,t\wedge\tau]} \kappa_u \, d\Gamma_u = D_t - \int_{]0,t\wedge\tau]} (1+\kappa_u)\, d\Gamma_u, \quad \forall\, t \in [0,T], \tag{3.40}$$

is a $\mathbb{G}$-martingale orthogonal to W^*.

Proof. Notice first that for $t \le T$ we have

$$\begin{aligned} d(\eta_t W_t^*) &= W_t^* \, d\eta_t + \eta_{t-}\, dW_t^* + d[W^*, \eta]_t \\ &= W_t^* \, d\eta_t + \eta_{t-}\, dW_t - \eta_{t-}\beta_t \, dt + \eta_{t-}\beta_t \, d[W,W]_t \\ &= W_t^* \, d\eta_t + \eta_{t-}\, dW_t. \end{aligned}$$

This shows that W^* is a $\mathbb{G}$-martingale under $\mathbf{P}^*$. Since the quadratic variation of W^* under $\mathbf{P}^*$ equals $[W^*, W^*]_t = t$ and W^* is continuous, by virtue of Lévy's theorem it is clear that W^* follows a Brownian motion under $\mathbf{P}^*$. Similarly, for $t \le T$

$$\begin{aligned} d(\eta_t M_t^*) &= M_t^* \, d\eta_t + \eta_{t-}\, dM_t^* + d[M^*, \eta]_t \\ &= M_t^* \, d\eta_t + \eta_{t-}\, dM_t - \eta_{t-}\kappa_t \, d\Gamma_{t\wedge\tau} + \eta_{t-}\kappa_t \, dD_t \\ &= M_t^* \, d\eta_t + \eta_{t-}(1+\kappa_t)\, dM_t. \end{aligned}$$

We conclude that M^* is a $\mathbb{G}$-martingale under $\mathbf{P}^*$. To conclude it is enough to observe that W^* is a continuous process and M^* follows a process of finite variation. □

Corollary 3.9 *Let Y be a $\mathbb{G}$-martingale with respect to $\mathbf{P}^*$. Then Y admits the following decomposition*

$$Y_t = Y_0 + \int_0^t \xi_u^* \, dW_u^* + \int_{]0,t]} \zeta_u^* \, dM_u^*, \tag{3.41}$$

where ξ^* and ζ^* are $\mathbb{G}$-predictable stochastic processes.

Proof. Consider the process $\tilde{Y}$ given by the formula

$$\tilde{Y}_t = \int_{]0,t]} \eta_{u-}^{-1} \, d(\eta_u Y_u) - \int_{]0,t]} \eta_{u-}^{-1} Y_{u-}\, d\eta_u.$$

It is clear that $\tilde{Y}$ is a $\mathbb{G}$-martingale under $\mathbf{P}$. Notice also that Itô's formula yields

$$\eta_{u-}^{-1} \, d(\eta_u Y_u) = dY_u + \eta_{u-}^{-1} Y_{u-}\, d\eta_u + \eta_{u-}^{-1}\, d[Y,\eta]_u,$$

and thus

$$Y_t = Y_0 + \tilde{Y}_t - \int_{]0,t]} \eta_{u-}^{-1}\, d[Y,\eta]_u. \tag{3.42}$$

From Corollary 3.7 we know that

$$\tilde{Y}_t = Y_0 + \int_0^t \tilde{\xi}_u \, dW_u + \int_{]0,t]} \tilde{\zeta}_u \, dM_u \tag{3.43}$$

for some $\mathbb{G}$-predictable processes $\tilde{\xi}$ and $\tilde{\zeta}$. Therefore

$$\begin{aligned} dY_t &= \tilde{\xi}_t \, dW_t + \tilde{\zeta}_t \, dM_t - \eta_{t-}^{-1}\, d[Y,\eta]_t \\ &= \tilde{\xi}_t \, dW_t^* + \tilde{\zeta}_t (1+\kappa_t)^{-1}\, dM_t^* \end{aligned}$$

since (3.38) combined with (3.42)-(3.43) yield

$$\eta_{t-}^{-1}\, d[Y,\eta]_t = \tilde{\xi}_t \beta_t \, dt + \tilde{\zeta}_t \kappa_t (1+\kappa_t)^{-1}\, dD_t.$$

To derive the last equality we observe, in particular, that in view of (3.42) we have (we take into account continuity of Γ)

$$\Delta[Y,\eta]_t = \eta_{t-}\tilde{\zeta}_t \kappa_t \, dD_t - \kappa_t \Delta[Y,\eta]_t.$$

We conclude that Y satisfies (3.41) with $\xi^* = \tilde{\xi}$ and $\zeta^* = \tilde{\zeta}(1+\kappa)^{-1}$. □

3.4 Applications to the Valuation of Defaultable Claims

Let us fix $t \le T$, and let δ be a constant. As a simple consequence of (3.4), for arbitrary $\mathcal{G}$-measurable random variable Y we obtain

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le T\}}\,\delta + 1\!\!1_{\{\tau>T\}}\,Y \,|\, \mathcal{G}_t) = \delta\,\mathbf{P}(t<\tau\le T\,|\,\mathcal{G}_t) + 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,(1\!\!1_{\{\tau>T\}}\,Y\,|\,\mathcal{G}_t),$$

so that

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le T\}}\,\delta + 1\!\!1_{\{\tau>T\}}\,Y \,|\, \mathcal{G}_t) = \delta 1\!\!1_{\{\tau>t\}}\mathbf{E}\,\big(1-e^{\Gamma_t-\Gamma_T}\,|\,\mathcal{F}_t\big) + 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,(1\!\!1_{\{\tau>T\}}\,e^{\Gamma_t}Y\,|\,\mathcal{F}_t).$$

If Y is a $\mathcal{F}_T$-measurable random variable, we may rewrite the last formula as follows

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le T\}}\,\delta + 1\!\!1_{\{\tau>T\}}\,Y \,|\, \mathcal{G}_t) = \delta 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,\big(1-e^{\Gamma_t-\Gamma_T}\,|\,\mathcal{F}_t\big) + 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,(e^{\Gamma_t-\Gamma_T}\,Y\,|\,\mathcal{F}_t).$$

If, instead of constant δ, we take a random payoff $h(\tau)$, then we get

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le T\}}\,h(\tau) + 1\!\!1_{\{\tau>T\}}\,Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,\Big(\int_t^T h(u)e^{\Gamma_t-\Gamma_u}\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big) + 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,(e^{\Gamma_t-\Gamma_T}Y\,|\,\mathcal{F}_t),$$

provided that Γ is continuous and Y is $\mathcal{F}_T$-measurable. Finally, under the same assumptions, if Z is a $\mathbb{F}$-predictable process, then

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le T\}}\,Z_\tau + 1\!\!1_{\{\tau>T\}}\,Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,\Big(\int_t^T Z_u e^{\Gamma_t-\Gamma_u}\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big) + 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,(e^{\Gamma_t-\Gamma_T}Y\,|\,\mathcal{F}_t).$$

Let us introduce the savings account process B (B is assumed to be a $\mathbb{F}$-adapted continuous process of finite variation). For any $t \le T$, we set $B(t,T) = B_t\,\mathbf{E}\,(B_T^{-1}\,|\,\mathcal{F}_t)$, and we refer to $B(t,T)$ as the price of a default-free zero-coupon bond which pays 1 at time T. First, for the conditional expectation

$$V_t^Y := B_t\,\mathbf{E}\,(1\!\!1_{\{\tau>T\}}\,B_T^{-1}Y\,|\,\mathcal{G}_t)$$

we obtain

$$V_t^Y = 1\!\!1_{\{\tau>t\}}B_t\,\mathbf{E}\,(e^{\Gamma_t-\Gamma_T}B_T^{-1}Y\,|\,\mathcal{F}_t) = 1\!\!1_{\{\tau>t\}}\hat{B}_t\,\mathbf{E}\,(\hat{B}_T^{-1}Y\,|\,\mathcal{F}_t),$$

where we set $\hat{B}_t = B_t e^{\Gamma_t}$. The next goal is to evaluate the conditional expectation

$$V_t^h := B_t\,\mathbf{E}\,(1\!\!1_{\{t<\tau\le T\}}\,B_\tau^{-1}h(\tau)\,|\,\mathcal{G}_t)$$

and

$$V_t^Z := B_t\,\mathbf{E}\,(1\!\!1_{\{t<\tau\le T\}}\,B_\tau^{-1}Z_\tau\,|\,\mathcal{G}_t).$$

If Γ is continuous, we get

$$V_t^D = 1\!\!1_{\{\tau>t\}}\,\hat{B}_t\,\mathbf{E}\,\Big(\int_t^T \hat{B}_u^{-1}h(u)\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big)$$

and

$$V_t^Z = 1\!\!1_{\{\tau>t\}}\,\hat{B}_t\,\mathbf{E}\,\Big(\int_t^T \hat{B}_u^{-1}Z_u\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big).$$

For instance, if $h=\delta$ for some constant δ and $Y=1$, we obtain

$$V_t := B_t\,\mathbf{E}\,(1\!\!1_{\{t<\tau\le T\}}\,B_\tau^{-1}\delta + 1\!\!1_{\{\tau>T\}}\,B_T^{-1}\,|\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\hat{B}_t\,\Big(\delta\int_t^T \hat{B}_u^{-1}d\Gamma_u + \hat{B}_T^{-1}\,\Big|\,\mathcal{F}_t\Big). \tag{3.44}$$

If we introduce the modified bond price $\hat{B}(t,T) = \hat{B}_t\,\mathbf{E}\,(\hat{B}_T^{-1}\,|\,\mathcal{F}_t)$, then

$$V_t = 1\!\!1_{\{\tau>t\}}\delta\hat{B}_t\,\Big(\int_t^T \hat{B}_u^{-1}d\Gamma_u\,\Big|\,\mathcal{F}_t\Big) + 1\!\!1_{\{\tau>t\}}\hat{B}(t,T).$$

4 Martingale Hazard Process

It should be stressed that the case of $\mathbb{D} \subset \mathbb{F}$ (i.e., the case when $\mathbb{F} = \mathbb{G}$) is not excluded. Put another way, the case when τ is a $\mathbb{F}$-stopping time is also covered by the foregoing results. The concept of a $(\mathbb{F}, \mathbb{G})$-martingale hazard process is a direct counterpart of the notion of the $(\mathbb{F}^0, \mathbb{D})$-martingale hazard process (that is, the martingale hazard function of τ).

4.1 Martingale Hazard Process Λ

Definition 4.2 A $\mathbb{F}$-predictable right-continuous increasing process Λ is called a $(\mathbb{F}, \mathbb{G})$-*martingale hazard process* of a random time τ (or, in short, a $\mathbb{F}$-martingale hazard process) if and only if the process $\widetilde{M}_t := D_t - \Lambda_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale. In addition, $\Lambda_0 = 0$.

Our first goal will be to examine a special case when the $\mathbb{F}$-martingale hazard process Λ can be expressed through a straightforward counterpart of formula (2.28).

4.1.1 Martingale Invariance Property

We examine first an abstract setup: we are given a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ endowed with a filtration $\mathbb{G}$. Let $\mathbb{F}$ be an arbitrary sub-filtration of $\mathbb{G}$.

Definition 4.3 We say that a filtration $\mathbb{F}$ has the *martingale invariance property* with respect to a filtration $\mathbb{G}$ (or that (H)-hypothesis holds) if every $\mathbb{F}$ square-integrable martingale is a $\mathbb{G}$ square-integrable martingale.

This hypothesis implies that the $\mathbb{F}$-Brownian motion remains a Brownian motion in the enlarged filtration. It was studied, among others, by Brémaud and Yor [4] and Mazziotto and Szpirglas [22], and for financial purposes by Kusuoka [19]. This hypothesis is quite natural, despite its technical form. It is equivalent to:

(H*) For any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$.

This can be written in any of the equivalent forms in which G_t and H are bounded random variable (see, e.g., Dellacherie and Meyer [9]):

(H1) $\forall\, H \in \mathcal{F}_\infty,\ \forall\, G_t \in \mathcal{G}_t, \quad \mathbf{E}\,(HG_t \,|\, \mathcal{F}_t) = \mathbf{E}\,(H \,|\, \mathcal{F}_t)\, \mathbf{E}\,(G_t \,|\, \mathcal{F}_t),$

(H2) $\forall\, t \geq 0,\ \forall\, G_t \in \mathcal{G}_t, \quad \mathbf{E}\,(G_t \,|\, \mathcal{F}_\infty) = \mathbf{E}\,(G_t \,|\, \mathcal{F}_t),$

(H3) $\forall\, t \geq 0,\ \forall\, H \in \mathcal{F}_\infty, \quad \mathbf{E}\,(H \,|\, \mathcal{G}_t) = \mathbf{E}\,(H \,|\, \mathcal{F}_t).$

Lemma 4.9 *In our setting, condition* (H) *is equivalent to the following condition* (H′)

$$\mathbf{P}(\tau \leq t \,|\, \mathcal{F}_\infty) = \mathbf{P}(\tau \leq t \,|\, \mathcal{F}_t), \quad \forall t \in \mathbb{R}_+. \tag{4.45}$$

Proof. If (H2) holds, then (4.45) holds too. If (4.45) holds, then, for $s \leq t$,

$$\mathbf{P}(\tau \leq s \,|\, \mathcal{F}_t) = \mathbf{P}(\tau \leq s \,|\, \mathcal{F}_\infty \,|\, \mathcal{F}_t) = \mathbf{P}(\tau \leq s \,|\, \mathcal{F}_s \,|\, \mathcal{F}_t) = \mathbf{P}(\tau \leq s \,|\, \mathcal{F}_s) = \mathbf{P}(\tau \leq s \,|\, \mathcal{F}_\infty).$$

Then, the fact that $\mathcal{D}_t$ is generated by the sets $\{\tau \leq s\}, s \leq t$ proves that $\mathcal{F}_\infty$ and $\mathcal{D}_t$ are conditionally independent given $\mathcal{F}_t$. This result can be also found in [10]. □

4.1.2 Evaluation of Λ: Special Case

We find it convenient to introduce the following condition:

(G) F follows an increasing process.

Proposition 4.7 *Assume that* (G) *holds, i.e., F is increasing. If the process Λ given by the formula*

$$\Lambda_t = \int_{]0,t]} \frac{dF_u}{1 - F_{u-}} = \int_{]0,t]} \frac{d\mathbf{P}(\tau \le u \,|\, \mathcal{F}_u)}{1 - \mathbf{P}(\tau < u \,|\, \mathcal{F}_u)} \tag{4.46}$$

is $\mathbb{F}$-predictable, then Λ is the $\mathbb{F}$-martingale hazard process of a random time τ.

Proof. It suffices to check that $D_t - \Lambda_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale, where $\mathbb{G} = \mathbb{D} \vee \mathbb{F}$. Using (3.1), we obtain for $t < s$

$$\mathbf{E}\left(D_s - D_t \,|\, \mathcal{G}_t\right) = \mathbf{P}(t < \tau \le s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \frac{\mathbf{P}(t < \tau \le s \,|\, \mathcal{F}_t)}{\mathbf{P}(\tau > t \,|\, \mathcal{F}_t)} = 1\!\!1_{\{\tau > t\}} \frac{\mathbf{E}\left(F_s \,|\, \mathcal{F}_t\right) - F_t}{1 - F_t}.$$

On the other hand,

$$\mathbf{E}\left(\Lambda_{s\wedge\tau} - \Lambda_{t\wedge\tau} \,|\, \mathcal{G}_t\right) = \mathbf{E}\left(1\!\!1_{\{\tau > s\}}(\Lambda_s - \Lambda_t) \,|\, \mathcal{G}_t\right) + \mathbf{E}\left(1\!\!1_{\{t < \tau \le s\}}\, \tilde{\Lambda}_\tau \,|\, \mathcal{G}_t\right),$$

where, for a fixed t, we write $\tilde{\Lambda}_u = (\Lambda_u - \Lambda_t)1\!\!1_{]t,\infty[}(u)$ (so that $\tilde{\Lambda}$ follows a $\mathbb{F}$-predictable process). Therefore, an application of formula (3.3) gives

$$\mathbf{E}\left(1\!\!1_{\{t < \tau \le s\}}\, \tilde{\Lambda}_\tau \,|\, \mathcal{G}_t\right) = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}\, \mathbf{E}\left(\int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t\right).$$

Furthermore, (3.4) yields

$$\mathbf{E}\left(1\!\!1_{\{\tau > s\}}(\Lambda_s - \Lambda_t) \,|\, \mathcal{G}_t\right) = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}\, \mathbf{E}\left(1\!\!1_{\{\tau > s\}}\, (\Lambda_s - \Lambda_t) \,|\, \mathcal{F}_t\right).$$

Combining the formulae above, we get

$$\begin{aligned}
\mathbf{E}\left(\Lambda_{s\wedge\tau} - \Lambda_{t\wedge\tau} \,|\, \mathcal{G}_t\right) &= 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}\, \mathbf{E}\left(1\!\!1_{\{\tau > s\}}\, (\Lambda_s - \Lambda_t) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t\right) \\
&= 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}\, \mathbf{E}\left((1 - F_s)(\Lambda_s - \Lambda_t) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t\right),
\end{aligned}$$

where the last equality follows from the conditioning with respect to the σ-field $\mathcal{F}_s$.

4.1.3 Evaluation of Λ: General Case

Assume now that either (G) is not satisfied, and the process F is not increasing, or (G) holds, but the increasing process F is not $\mathbb{F}$-predictable.[6] As the next result shows, the $\mathbb{F}$-martingale hazard process Λ can nevertheless be found through a suitable modification of formula (4.46).

In the next result we do not need to assume that (G) holds. We write $\tilde{F}$ to denote the $\mathbb{F}$-compensator of the bounded $\mathbb{F}$-submartingale F. This means that $\tilde{F}$ is the unique $\mathbb{F}$-predictable, increasing process, with $\tilde{F}_0 = 0$, and such that the process $U = F - \tilde{F}$ follows a $\mathbb{F}$-martingale (the existence and uniqueness of $\tilde{F}$ is a consequence of the Doob-Meyer decomposition theorem). In standard examples, τ is a totally inaccessible $\mathbb{F}$-stopping time, and $\tilde{F}$ follows a $\mathbb{F}$-adapted process with continuous increasing sample paths.

[6] For instance, τ can be a $\mathbb{F}$-stopping time, which is not $\mathbb{F}$-predictable. If τ is a $\mathbb{F}$-stopping time, we have simply $F = H$, and the process D is not $\mathbb{F}$-predictable, unless the stopping time τ is a $\mathbb{F}$-predictable.

Proposition 4.8 (i) *The $\mathbb{F}$-martingale hazard process of a random time τ is given by the formula*

$$\Lambda_t = \int_{]0,t]} \frac{d\tilde{F}_u}{1 - F_{u-}}. \tag{4.47}$$

(ii) *If $\tilde{F}_t = \tilde{F}_{t\wedge\tau}$ for every $t \in \mathbb{R}_+$ (that is, the process $\tilde{F}$ is stopped at τ) then $\Lambda = \tilde{F}$.*

Proof. It is clear that the process Λ given by (4.47) is predictable. Therefore, we need only to verify that the process $\tilde{M}_t = D_t - \Lambda_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale. We proceed along the same lines as in the proof of Proposition 4.7.

We shall now prove part (ii). We assume that $\tilde{F}_{t\wedge\tau} = \tilde{F}_t$ for every $t \in \mathbb{R}_+$. This means, in particular, that the process $F_t - \tilde{F}_{t\wedge\tau}$ is a $\mathbb{F}$-martingale. We wish to show that if the process $D_t - \tilde{F}_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale, that is, for any $t \le s$

$$\mathbf{E}\,(D_s - \tilde{F}_{s\wedge\tau}\,|\,\mathcal{G}_t) = D_t - \tilde{F}_{t\wedge\tau},$$

or equivalently,

$$\mathbf{E}\,(D_s - D_t\,|\,\mathcal{G}_t) = \mathbf{E}\,(\tilde{F}_{s\wedge\tau} - \tilde{F}_{t\wedge\tau}\,|\,\mathcal{G}_t).$$

By virtue of (3.1), we have

$$\mathbf{E}\,(D_s - D_t\,|\,\mathcal{G}_t) = (1 - D_t)\,\frac{\mathbf{E}\,(D_s - D_t\,|\,\mathcal{F}_t)}{\mathbf{E}\,(1 - D_t\,|\,\mathcal{F}_t)}. \tag{4.48}$$

On the other hand,

$$\begin{aligned}
\mathbf{E}\,(\tilde{F}_{s\wedge\tau} - \tilde{F}_{t\wedge\tau}\,|\,\mathcal{G}_t) &= \mathbf{E}\,\big(\mathbb{1}_{\{\tau>t\}}(\tilde{F}_{s\wedge\tau} - \tilde{F}_{t\wedge\tau})\,\big|\,\mathcal{G}_t\big) = (1 - D_t)\,\frac{\mathbf{E}\,(\tilde{F}_{s\wedge\tau} - \tilde{F}_{t\wedge\tau}\,|\,\mathcal{F}_t)}{\mathbf{E}\,(1 - D_t\,|\,\mathcal{F}_t)} \\
&= (1 - D_t)\,\frac{\mathbf{E}\,(F_s - F_t\,|\,\mathcal{F}_t)}{\mathbf{E}\,(1 - D_t\,|\,\mathcal{F}_t)} = (1 - D_t)\,\frac{\mathbf{E}\,(D_s - D_t\,|\,\mathcal{F}_t)}{\mathbf{E}\,(1 - D_t\,|\,\mathcal{F}_t)},
\end{aligned}$$

where the second equality follows from (3.1), and the third one is a consequence of our assumption that the process $F_t - \tilde{F}_{t\wedge\tau}$ is a $\mathbb{F}$-martingale. □

Remarks. Under assumption (H), the process $\tilde{F}$ is never stopped at τ, unless τ is a $\mathbb{F}$-stopping time. To show this assume, on the contrary, that $\tilde{F}_t = \tilde{F}_{t\wedge\tau}$. Let us stress that under (H), the process $F_t - \tilde{F}_{t\wedge\tau}$ is not only a $\mathbb{F}$-martingale, but also a $\mathbb{G}$-martingale (this *martingale invariance property* was introduced in Section 4.1.1).

Since by virtue of part (ii) in Proposition 4.8 the process $D_t - \tilde{F}_{t\wedge\tau}$ is a $\mathbb{G}$-martingale, we conclude that $D - F$ follows a $\mathbb{G}$-martingale. In view of the definition of F, the last property reads

$$\mathbf{E}\,(D_s\,|\,\mathcal{G}_t) - \mathbf{E}\,(\mathbf{E}\,(D_s\,|\,\mathcal{F}_s)\,|\,\mathcal{G}_t) = D_t - \mathbf{E}\,(D_t\,|\,\mathcal{F}_t),$$

for $t \le s$, or equivalently

$$\mathbf{E}\,(D_s - D_t\,|\,\mathcal{G}_t) = \mathbf{E}\,\big(\mathbf{E}\,(D_s\,|\,\mathcal{F}_s)\,|\,\mathcal{G}_t) - \mathbf{E}\,(D_t\,|\,\mathcal{F}_t) = I_1 - I_2. \tag{4.49}$$

Under (H), we have

$$I_1 = \mathbf{E}\,(\mathbf{P}(\tau \le s\,|\,\mathcal{F}_s)\,|\,\mathcal{F}_t \vee \mathcal{D}_t) = \mathbf{E}\,(\mathbf{P}(\tau \le s\,|\,\mathcal{F}_\infty)\,|\,\mathcal{F}_t \vee \mathcal{D}_t) = \mathbf{E}\,(\mathbf{P}(\tau \le s\,|\,\mathcal{F}_\infty)\,|\,\mathcal{F}_t)$$

since the random variable $\mathbf{P}(\tau \le s\,|\,\mathcal{F}_\infty)$ is obviously $\mathcal{F}_\infty$-measurable, and the σ-fields $\mathcal{F}_\infty$ and $\mathcal{D}_t$ are conditionally independent given $\mathcal{F}_t$. Consequently, $I_1 = \mathbf{E}\,(\mathbf{E}\,(D_s\,|\,\mathcal{F}_\infty)\,|\,\mathcal{F}_t) = \mathbf{E}\,(D_s\,|\,\mathcal{F}_t)$. We conclude that (4.49) can be rewritten as follows

$$\mathbf{E}\,(D_s - D_t\,|\,\mathcal{G}_t) = \mathbf{E}\,(D_s\,|\,\mathcal{F}_t) - \mathbf{E}\,(D_t\,|\,\mathcal{F}_t).$$

Furthermore, applying (4.48) to the right-hand side of the last equality, we obtain

$$(1 - D_t)\,\frac{\mathbf{E}\,(D_s - D_t \,|\, \mathcal{F}_t)}{\mathbf{E}\,(1 - D_t \,|\, \mathcal{F}_t)} = \mathbf{E}\,(D_s - D_t \,|\, \mathcal{F}_t).$$

By letting s tend to ∞, we obtain $D_t = \mathbf{E}\,(D_t \,|\, \mathcal{F}_t)$ or more explicitly, $\mathbf{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau \le t\}}$ for every $t \in \mathbb{R}_+$. This shows that τ is a $\mathbb{F}$-stopping time.

4.1.4 Uniqueness of a Martingale Hazard Process Λ

We shall first examine the relationship between the concept of a $\mathbb{F}$-martingale hazard process Λ of τ and the classic notion of $\mathbb{G}$-*compensator* of τ.

Definition 4.4 A process A is a $\mathbb{G}$-*compensator* of τ if and only if the following conditions are satisfied: (i) A is a $\mathbb{G}$-predictable right-continuous increasing process, with $A_0 = 0$, (ii) the process $D - A$ is a $\mathbb{G}$-martingale.

It is well known that for any random time τ and any filtration $\mathbb{G}$ such that τ is a $\mathbb{G}$-stopping time there exists a unique $\mathbb{G}$-compensator A of τ. Moreover, $A_t = A_{t\wedge\tau}$, that is, A is stopped at τ. In the next auxiliary result, we deal with an arbitrary filtration $\mathbb{F}$ which, when combined with the natural filtration $\mathbb{D}$ of a $\mathbb{G}$-stopping time τ, generates the enlarged filtration $\mathbb{G}$.

Lemma 4.10 *Let $\mathbb{F}$ be an arbitrary filtration such that $\mathbb{G} = \mathbb{D} \vee \mathbb{F}$.*
(i) *Let A be a $\mathbb{G}$-predictable right-continuous increasing process satisfying $A_t = A_{t\wedge\tau}$. Then there exists a $\mathbb{F}$-predictable right-continuous increasing process Λ such that $A_t = \Lambda_{t\wedge\tau}$.*
(ii) *Let Λ be a $\mathbb{F}$-predictable right-continuous increasing process. Then $A_t = \Lambda_{t\wedge\tau}$ is a $\mathbb{G}$-predictable right-continuous increasing process.*

The next proposition summarizes the relationships between the $\mathbb{G}$-compensator of τ and the $\mathbb{F}$-martingale hazard process Λ of τ. Once again, $\mathbb{F}$ is an arbitrary filtration such that $\mathbb{G} = \mathbb{D} \vee \mathbb{F}$.

Proposition 4.9 (i) *Let Λ be a $\mathbb{F}$-martingale hazard process of τ. Then the process $A_t = \Lambda_{t\wedge\tau}$ is the $\mathbb{G}$-compensator of τ.*
(ii) *Let A be the $\mathbb{G}$-compensator of τ. Then there exists a $\mathbb{F}$-martingale hazard process Λ such that $A_t = \Lambda_{t\wedge\tau}$.*

Let us summarize the results above. First, for any random time τ on some probability space $(\Omega, \mathcal{G}, \mathbf{P})$, and an arbitrary filtration $\mathbb{F}$ there exists a $\mathbb{F}$-martingale hazard process Λ of τ. Furthermore, it is unique up to time τ, in the following sense: if Λ^1 and Λ^2 are two $\mathbb{F}$-martingale hazard processes of τ, then $\Lambda^1_{t\wedge\tau} = \Lambda^2_{t\wedge\tau}$. To ensure the uniqueness after τ of a $\mathbb{F}$-martingale hazard processes we need to impose additional restrictions on Λ.

Assume now that we are given a $\mathbb{G}$-stopping time τ for some filtration $\mathbb{G}$. Then there exist several distinct filtrations $\mathbb{F}$ such that $\mathbb{G} = \mathbb{D} \vee \mathbb{F}$. Assume that $\mathbb{G} = \mathbb{D} \vee \mathbb{F}^1 = \mathbb{D} \vee \mathbb{F}^2$, and denote by Λ^i a $\mathbb{F}^i$-martingale hazard process of τ. Then $\Lambda^1_{t\wedge\tau} = A_{t\wedge\tau} = \Lambda^2_{t\wedge\tau}$. It seems reasonable to search for the $\hat{\mathbb{F}}$-martingale hazard process where $\hat{\mathbb{F}}$ is a 'minimal' filtration such that $\mathbb{G} = \mathbb{D} \vee \hat{\mathbb{F}}$.

4.2 Relationships Between Hazard Processes Γ and Λ

Let us assume that the $\mathbb{F}$-hazard process Γ is well defined (in particular, τ is not a $\mathbb{F}$-stopping time). Recall that for any $\mathcal{F}_s$-measurable random variable Y we have (cf. (3.6))

$$\mathbf{E}\,(\mathbb{1}_{\{\tau > s\}}\, Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\, \mathbf{E}\,\big(Y e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t\big). \tag{4.50}$$

The natural question which arises is this context is: can we substitute Γ with the $\mathbb{F}$-martingale hazard function Λ in the formula above? Of course, the answer is trivial when it is known that the equality $\Lambda = \Gamma$ is valid, for instance, when condition (G) are satisfied and F is a continuous process. More precisely, we have the following result.

Proposition 4.10 *Under assumption* (G) *the following assertions are valid.*
(i) *If the $\mathbb{F}$-hazard process Γ is continuous, then the $\mathbb{F}$-martingale hazard process Λ is also continuous, and both processes coincide, namely,*

$$\Gamma_t = \Lambda_t = -\ln(1 - F_t), \quad \forall\, t \in \mathbb{R}_+. \tag{4.51}$$

(ii) *If the $\mathbb{F}$-hazard process Γ is a discontinuous process then the equality $\Lambda = \Gamma$ is not satisfied, but we have*

$$e^{-\Gamma_t} = e^{-\Lambda_t^c} \prod_{0<u\leq t} (1 - \Delta\Lambda_u), \tag{4.52}$$

where Λ^c is the continuous component of Λ, that is, $\Lambda_t^c = \Lambda_t - \sum_{0\leq u\leq t} \Delta\Lambda_u$.

If, in addition, the process $\Lambda = \Gamma$ is absolutely continuous then for any $\mathcal{F}_s$-measurable random variable Y we get

$$\mathbf{E}\,(\mathbb{1}_{\{\tau>s\}}\, Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbf{E}\,\big(Y e^{-\int_t^s \lambda_u\, du} \,|\, \mathcal{F}_t\big). \tag{4.53}$$

The following result is an obvious consequence of Proposition 3.1.

Corollary 4.10 *Suppose that assumption* (G) *holds and F follows a continuous process so that $\Gamma_t = \Lambda_t = -\ln(1 - F_t)$ for every $t \in \mathbb{R}_+$.*
i) Let $Y = h(\tau)$ for a Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$. Then

$$\mathbf{E}\,(Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau\leq t\}}\, h(\tau) + \mathbb{1}_{\{\tau>t\}}\, \mathbf{E}\,\Big(\int_t^\infty h(u) e^{\Lambda_t - \Lambda_u}\, d\Lambda_u \,\Big|\, \mathcal{F}_t\Big). \tag{4.54}$$

ii) Let Z be a $\mathbb{F}$-predictable process. Then for any $t \leq s$

$$\mathbf{E}\,(Z_\tau \mathbb{1}_{\{t<\tau\leq s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbf{E}\,\Big(\int_t^s Z_u e^{\Lambda_t - \Lambda_u}\, d\Lambda_u \,\Big|\, \mathcal{F}_t\Big). \tag{4.55}$$

We shall now examine the following question: does the continuity of the $\mathbb{F}$-martingale hazard process Λ imply the equality $\Lambda = \Gamma$? The first result in this direction gives only a partial answer to this question.

Proposition 4.11 *Under assumption* (G), *assume that any $\mathbb{F}$-martingale is continuous. If the $\mathbb{F}$-martingale hazard process Λ follows a continuous process then for arbitrary $t \leq s$ and any bounded $\mathcal{F}_s$-measurable random variable Y we have*

$$\mathbb{1}_{\{\tau>t\}}\mathbf{E}\,(Y e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau>t\}}\mathbf{E}\,(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t). \tag{4.56}$$

Proof. We shall show that for any $t \leq s$

$$\mathbf{P}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\mathbf{E}\,(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t). \tag{4.57}$$

To this end, let us introduce the $\mathbb{F}$-martingale $m_t = \mathbf{E}\,(Y e^{-\Lambda_s} \,|\, \mathcal{F}_t)$, where Y is a bounded $\mathcal{F}_s$-measurable random variable. Also let $\tilde{L}_t = \mathbb{1}_{\{\tau>t\}}\, e^{\Lambda_t}$. The $\mathbb{G}$-martingale property of $\tilde{L}$ follows easily from Itô's lemma and the assumed continuity of Λ. Indeed,

$$d\tilde{L}_t = (1 - D_{t-}) e^{\Lambda_t}\, d\Lambda_t - e^{\Lambda_t}\, dD_t = -e^{\Lambda_t}\, d\tilde{M}_t = -\tilde{L}_{t-}\, d\tilde{M}_t. \tag{4.58}$$

By virtue of part (i) in Lemma 3.6 the stopped process $\tilde{m}_t = m_{t\wedge\tau}$ is a continuous $\mathbb{G}$-martingale, so that it is orthogonal to the $\mathbb{G}$-martingale $Z_t = \tilde{L}_{t\wedge s}$ (which is obviously of finite variation). Therefore the product $\tilde{m}Z$ is a $\mathbb{G}$-martingale, and thus

$$\mathbf{E}\,(\tilde{m}_s Z_s \,|\, \mathcal{G}_t) = \tilde{m}_t Z_t = \mathbb{1}_{\{\tau>t\}}\, m_{t\wedge\tau} e^{\Lambda_t} = \mathbb{1}_{\{\tau>t\}}\, m_t e^{\Lambda_t} = \mathbb{1}_{\{\tau>t\}} \mathbf{E}\,(Y e^{\Lambda_t-\Lambda_s} \,|\, \mathcal{F}_t).$$

Furthermore,

$$\tilde{m}_s Z_s = \mathbb{1}_{\{\tau>s\}}\, m_{s\wedge\tau} e^{\Lambda_s} = \mathbb{1}_{\{\tau>s\}}\, Y m_s e^{\Lambda_s} = Y \mathbb{1}_{\{\tau>s\}}.$$

This shows that (4.57) is indeed satisfied. Combining (4.57) with (4.50), we get (4.56). □

It appears that under the assumptions of Proposition 4.11 we can establish the equality $\Gamma = \Lambda$ as the following result shows.

Proposition 4.12 *Under assumption* (G), *if* $\mathbb{F}$ *supports only continuous martingales, then:*
(i) *if* Λ *is a continuous process, then* Γ *is also continuous and* $\Lambda = \Gamma$,
(ii) *if* Λ *is a discontinuous process, then* Γ *is also a discontinuous process, and* $\Lambda \neq \Gamma$.

Proof. We know that the $\mathbb{F}$-martingale hazard process Λ is given by (4.47). Therefore, if Λ is continuous then also $\tilde{F}$ is continuous, and thus also $F = \tilde{M} + \tilde{F}$ follows an increasing continuous process. Consequently, Λ is given by (4.46) and thus $\Lambda_t = -\ln(1-F_t) = \Gamma_t$. The second statement follows by similar arguments. □

To the best of our knowledge, it is not known whether the (absolute) continuity of Λ implies the (absolute) continuity of Γ, in general (under (G), say). The following conjecture seems to be natural.

Conjecture (A). Under assumption (G), if the $\mathbb{F}$-martingale hazard process Λ is continuous, then $\Gamma = \Lambda$.

In view of Proposition 4.7, it would be enough to show that Γ is a continuous process, and the equality $\Gamma = \Lambda$ would then follow. The following example shows that Conjecture A is false, in general, when hypothesis (G) fails to hold.

Example 4.1 Let $(W_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion on $(\Omega, \mathbb{F}, \mathbf{P})$, where $\mathbb{F} = \mathbb{F}^W$ is the natural filtration of W. A random time τ is defined by $\tau = \sup\{\, t \le 1 : W_t = 0 \,\}$. We set $\mathbb{G} = \mathbb{D} \vee \mathbb{F}$. Then the $\mathbb{F}$-hazard process of τ equals $\Gamma_t = -\ln(1-F_t)$, where

$$F_t = \mathbf{P}(\tau \le t \,|\, \mathcal{F}_t) = \Phi\Big(\frac{|W_t|}{\sqrt{1-t}}\Big), \quad \text{and} \quad \Phi(x) := \sqrt{\frac{2}{\pi}} \int_0^x e^{-u^2/2}\, du.$$

On the other hand, the $\mathbb{F}$-martingale hazard process of τ is

$$\Lambda_t = \sqrt{\frac{2}{\pi}} \int_0^t \frac{dL_s^0}{\sqrt{1-s}},$$

where L^0 stands for the local time of W at 0. Indeed, we have the following well known result (see, for instance, Yor [26]).

Lemma 4.11 *We have*

$$F_t = \mathbf{P}(\tau \le t \,|\, \mathcal{F}_t) = \Phi\Big(\frac{|\,W_t\,|}{\sqrt{1-t}}\Big). \tag{4.59}$$

The $\mathbb{F}$*-compensator of* F *equals*

$$\widetilde{F}_t = \sqrt{\frac{2}{\pi}} \int_0^t \frac{dL_s^0}{\sqrt{1-s}},$$

where L^0 *is the local time at level* 0 *of the Brownian motion* W.

Proof. For $t < 1$, the set $\{\tau \le t\}$ is equal to $\{d_t > 1\}$, where

$$d_t = \inf\{\, u \ge t : W_u = 0\}.$$

Let us recall the following equality (cf. Yor [26])

$$d_t = t + \inf\{u \ge 0 : W_{u+t} - W_t = -W_t\} = t + \hat{\tau}_{-W_t} \stackrel{d}{=} t + \frac{W_t^2}{G^2}. \tag{4.60}$$

We denote here $\hat{\tau}_b := \inf\{u \ge 0 : \hat{W}_u = b\}$, where $\hat{W}$ is a Brownian motion independent of $\mathcal{F}_t^W$. Also, G is a Gaussian variable, with mean 0 and variance 1, independent of W_t. It is clear that

$$\mathbf{P}\Big(\frac{a^2}{G^2} > 1 - t\Big) = \Phi\Big(\frac{|\,a\,|}{\sqrt{1-t}}\Big). \tag{4.61}$$

The Itô-Tanaka formula combined with the identity $x\Phi'(x) + \Phi''(x) = 0$ lead to

$$\begin{aligned}
\mathbf{P}(\tau \le t \,|\, \mathcal{F}_t) = \Phi\left(\frac{|\,W_t\,|}{\sqrt{1-t}}\right) &= \int_0^t \Phi'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) d\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) + \frac{1}{2}\int_0^t \Phi''\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right)\frac{ds}{1-s} \\
&= \int_0^t \Phi'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right)\frac{\mathrm{sgn}(W_s)}{\sqrt{1-s}}\, dW_s + \int_0^t \Phi'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right)\frac{dL_s^0}{\sqrt{1-s}} \\
&= \int_0^t \Phi'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right)\frac{\mathrm{sgn}(W_s)}{\sqrt{1-s}}\, dW_s + \sqrt{\frac{2}{\pi}}\int_0^t \frac{dL_s^0}{\sqrt{1-s}}\,.
\end{aligned}$$

But in view of (4.60)-(4.61)

$$\mathbf{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbf{P}(d_t > 1 \,|\, \mathcal{F}_t) = \Phi\left(\frac{|\,W_t\,|}{\sqrt{1-t}}\right),$$

and thus the $\mathbb{F}$-compensator of F is

$$\tilde{F}_t = \sqrt{\frac{2}{\pi}}\int_0^t \frac{dL_s^0}{\sqrt{1-s}}.$$

This ends the proof. □

Using the well-known property of the local time

$$L_t^0 = \int_0^t \mathbb{1}_{\{W_s=0\}}\, dL_s^0$$

we find that $L_t^0 = L_{t\wedge\tau}^0$, so that $\tilde{F}_t = \tilde{F}_{t\wedge\tau}$. Consequently, in view of part (ii) in Proposition 4.8 we conclude that the $\mathbb{F}$-martingale hazard process Λ equals $\tilde{F}$. Notice that condition (G) is not satisfied in the present setup (thus (H) does not hold). Furthermore, both Γ and Λ are continuous processes, but Λ follows an increasing process and Γ has non-zero continuous martingale part, so that clearly Γ and Λ do not coincide. To conclude, if (G) fails to hold, the continuity of Γ and Λ is not sufficient for the equality $\Gamma = \Lambda$ to hold. Notice also that the $\mathbb{G}$-compensator A of D, which satisfies $A_t = \Lambda_{t\wedge\tau}$ is also equal $\tilde{F}$. Let us finally observe that $\mathbb{F}$-martingale hazard process Λ is also the $\hat{\mathbb{F}}$-martingale hazard process of τ, where $\hat{\mathbb{F}}$ stands for the natural filtration of $|W_t|$ (of course, $\hat{\mathbb{F}}$ is a strict subfiltration of $\mathbb{F}$). Similarly, we have $F_t = \mathbf{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbf{P}(\tau \le t \,|\, \hat{\mathcal{F}}_t) = \hat{F}_t$ for every t, and thus $\Gamma_t = \hat{\Gamma}_t$.

4.3 Martingale Representation Theorem

We shall first consider the following setup: we are given an underlying filtration $\mathbb{F}$ and the enlarged fitration $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$, where $\mathbb{D}$ is generated by a random time τ. In addition, we assume that the assumptions of Proposition 4.8 are valid, so that F follows an increasing $\mathbb{F}$-predictable process and the $\mathbb{F}$-martingale hazard process Λ of a random time τ is given by the formula

$$\Lambda_t = \int_{]0,t]} \frac{dF_u}{1 - F_{u-}}. \tag{4.62}$$

By the definition of the $\mathbb{F}$-martingale hazard process the compensated process $\tilde{M}_t = D_t - \Lambda_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale. Recall also that by virtue of Lemma 3.7 the process L given by the formula

$$L_t := \mathbb{1}_{\{\tau>t\}}(1 - F_t)^{-1} = \frac{1 - D_t}{1 - F_t}$$

is always a $\mathbb{G}$-martingale. We shall check that

$$dL_t = -(1 - F_t)^{-1}\, d\tilde{M}_t. \tag{4.63}$$

To this end, we note that

$$A_t := \Lambda_{t\wedge\tau} = \int_{]0,t]} \frac{1 - D_{u-}}{1 - F_{u-}}\, dF_u = \int_{]0,t]} L_{u-}\, dF_u. \tag{4.64}$$

Moreover, since

$$1 - D_t = L_t(1 - F_t)$$

we get by Itô's lemma (notice that L is also a process of finite variation)

$$dD_t = L_{t-}\, dF_t - (1 - F_t)\, dL_t = A_t - (1 - F_t)\, dL_t$$

The following result is a counterpart of Proposition 3.4.

Proposition 4.13 *Let Z be a $\mathbb{F}$-predictable process such that the random variable Z_τ is integrable. Then the $\mathbb{G}$-martingale $M_t^Z := E(Z_\tau \,|\, \mathcal{G}_t)$ admits the following decomposition*

$$M_t^Z = m_0 + \int_{]0,t]} L_{t-} dm_u + \int_{]0,t]} (Z_u - D_u)\, d\tilde{M}_u, \tag{4.65}$$

where m is a $\mathbb{F}$-martingale, namely,

$$m_t = \mathbf{E}\Big(\int_0^\infty Z_u\, dF_u \,\Big|\, \mathcal{F}_t \Big)$$

so that in particular $m_0 = M_0^Z$. Moreover

$$D_t = (1 - F_t)^{-1} \mathbf{E}\Big(\int_t^\infty Z_u\, dF_u \,\Big|\, \mathcal{F}_t \Big).$$

Proof. By virtue of Proposition 3.1, we have (cf. (3.3))

$$M_t^Z = \mathbf{E}\,(Z_\tau \,|\, \mathcal{G}_t) = D_t Z_\tau + (1 - D_t) D_t = D_t Z_\tau + \hat{D}_t$$

where

$$\hat{D}_t := (1 - D_t) D_t = L_t \big(m_t - \int_{]0,t]} Z_u\, dF_u \big).$$

Notice that L follows a process of finite variation so that

$$\begin{aligned} d\hat{D}_t &= L_{t-}(dm_t - Z_t\, dF_t) + D_t(1 - F_t)\, dL_t \\ &= L_{t-}\, dm_t - Z_t\, dD_t + D_t(1 - F_t)\, dL_t \\ &= L_{t-}\, dm_t - Z_t\, dD_t - D_t\, d\tilde{M}_t, \end{aligned}$$

where we have used (4.63) and (4.64). Consequently,

$$dM_t^Z = Z_t\, dD_t + d\hat{D}_t = L_{t-}\, dm_t + (Z_t - D_t)\, d\tilde{M}_t$$

This gives the desired expresssion (4.65). □

Formula (4.65) can also be rewritten as follows

$$M_t^Z = m_0 + \int_{]0,t\wedge\tau]} (1 - F_{t-})^{-1}\, dm_u + \int_{]0,t]} (Z_u - D_u)\, d\tilde{M}_u. \tag{4.66}$$

4.4 Case of the Martingale Invariance Property

In the next result, we assume that the martingale invariance property (H) holds. Recall that condition (G) is then also satisfied. It should be stressed, however, that we shall work directly with the $\mathbb{F}$-martingale hazard process Λ. Therefore, Proposition 4.14 covers also the case when the $\mathbb{F}$-hazard process Γ does not exist (for instance, when τ is a $\mathbb{F}$-stopping time). It appears that equality (4.57) is valid for a $\mathcal{G}_s$-measurable random variable Y, provided that a suitable continuity condition is satisfied. The following result is due to Duffie et al. [13].

Proposition 4.14 *Assume that hypothesis* (H) *is satisfied and the $\mathbb{F}$-martingale hazard process Λ of τ is continuous. For a fixed $s > 0$, let Y be a $\mathcal{G}_s$-measurable integrable random variable. If the* (*right-continuous*) *process V, given by the formula*

$$V_t = \mathbf{E}\left(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\right), \quad \forall\, t \in [0, s], \tag{4.67}$$

is continuous at τ, that is, $\Delta V_{s\wedge\tau} = V_{s\wedge\tau} - V_{(s\wedge\tau)-} = 0$, then for any $t < s$ we have

$$\mathbf{E}\left(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}}\, \mathbf{E}\left(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\right).$$

Proof. We shall first check that

$$U_t := \mathbb{1}_{\{\tau>t\}} V_t = \mathbf{E}\left(\Delta V_\tau \mathbb{1}_{\{t<\tau\leq s\}} + \mathbb{1}_{\{\tau>s\}} Y \,\middle|\, \mathcal{G}_t\right), \tag{4.68}$$

or equivalently

$$U_t = \mathbf{E}\left(\int_{]t,s]} \Delta V_u\, dD_u + \mathbb{1}_{\{\tau>s\}} Y \,\middle|\, \mathcal{G}_t\right). \tag{4.69}$$

In view of (4.67), $V_t = e^{\Lambda_t} m_t$, where m is a $\mathbb{F}$-martingale: $m_t := \mathbf{E}\left(Y e^{-\Lambda_s} \,|\, \mathcal{F}_t\right)$ for $t \in [0, s]$ (notice that m is also a $\mathbb{G}$-martingale). Using Itô's product rule, we obtain

$$dV_t = m_{t-}\, de^{\Lambda_t} + e^{\Lambda_t}\, dm_t = V_{t-} e^{-\Lambda_t}\, de^{\Lambda_t} + e^{\Lambda_t}\, dm_t. \tag{4.70}$$

On the other hand, an application of Itô's formula yields

$$dU_t = (1 - D_{t-})\, dV_t - V_{t-}\, dD_t - \Delta V_t \Delta D_t.$$

Combining the last formula with (4.70), we obtain

$$dU_t = (1 - D_{t-})\big(V_{t-} e^{-\Lambda_t}\, de^{\Lambda_t} + e^{\Lambda_t}\, dm_t\big) - V_{t-}\, dD_t - \Delta V_t\, dD_t.$$

After rearranging, we get

$$dU_t = -\Delta V_t\, dD_t + dC_t, \tag{4.71}$$

where C stands for a $\mathbb{G}$-martingale. More precisely,

$$dC_t = (1 - D_{t-}) e^{\Lambda_t}\, dm_t + dD_t,$$

where in turn

$$dD_t = -V_{t-}\big(dD_t - (1 - D_{t-})e^{-\Lambda_t}\, de^{\Lambda_t}\big) = -V_{t-}\, d(D_t - \Lambda_{t\wedge\tau}) = -V_{t-}\, d\tilde{M}_t,$$

so that D is a $\mathbb{G}$-martingale. Since obviously $U_s = 1\!\!1_{\{\tau>s\}}Y$, (4.71) implies (4.69). If V is continuous at τ then (4.68) yields

$$\mathbf{E}\,(1\!\!1_{\{\tau>s\}}\,Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}V_t = 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\,\big(Ye^{\Lambda_t-\Lambda_s} \,|\, \mathcal{F}_t\big).$$

This completes the proof. □

The following corollary provides a sufficient condition for the martingale hazard process Λ to determine the conditional survival probability of τ given the σ-field $\mathcal{G}_t$.

Corollary 4.11 *Under the assumptions of Proposition* 4.14, *if the process V given by the formula*

$$V_t = \mathbf{E}\,\big(e^{\Lambda_t-\Lambda_s} \,|\, \mathcal{F}_t\big), \quad \forall\, t \in [0, s], \tag{4.72}$$

is continuous at τ, that is, $\Delta V_{s\wedge\tau} = V_{s\wedge\tau} - V_{(s\wedge\tau)-} = 0$, then for any $t \le s$ we have

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\mathbf{E}\,\big(e^{\Lambda_t-\Lambda_s} \,|\, \mathcal{F}_t\big).$$

Let us stress that the case when $\mathbb{F} = \mathbb{G}$ (that is, the case when τ is a $\mathbb{G}$-stopping time) is also covered by Corollary 4.11.

4.4.1 Application to the Valuation of Defaultable Claims

Assume that B is a $\mathbb{F}$-adapted continuous process of finite variation given by the formula

$$B_t = \exp\Big(\int_0^t r_u\, du\Big)$$

for some $\mathbb{F}$-adapted integrable process r. The process B is referred to as the savings account. Let us set

$$S_t := B_t\,\mathbf{E}\Big(\int_{]t,T]} B_u^{-1} Z_u\, dD_u + B_T^{-1} X\, 1\!\!1_{\{T<\tau\}} \,\Big|\, \mathcal{G}_t\Big), \tag{4.73}$$

where Z is a $\mathbb{G}$-predictable process and X is a $\mathcal{G}_T$-measurable random variable. Then we have the following result due to Duffie et al. [13].

Proposition 4.15 *Assume that a random time τ admits an absolutely continuous $\mathbb{F}$-martingale hazard process Λ. For a given $\mathbb{G}$-predictable process Z and $\mathcal{G}_T$-measurable random variable X, define the process V by setting*

$$V_t = \tilde{B}_t\,\mathbf{E}\Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u\, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{G}_t\Big), \tag{4.74}$$

where $\tilde{B}$ is the 'savings account' corresponding to the default-adjusted short-term rate $R_t = r_t + \lambda_t$, that is,

$$\tilde{B}_t = \exp\Big(\int_0^t (r_u + \lambda_u)\, du\Big).$$

Then,

$$1\!\!1_{\{\tau>t\}}V_t = B_t\,\mathbf{E}\,\Big(B_\tau^{-1}(Z_\tau + \Delta V_\tau) 1\!\!1_{\{t<\tau\le T\}} + B_T^{-1} X\, 1\!\!1_{\{T<\tau\}} \,\Big|\, \mathcal{G}_t\Big).$$

Corollary 4.12 *Let the processes S and V be defined by* (4.73) *and* (4.74), *respectively. Then* (i)

$$S_t = 1\!\!1_{\{\tau>t\}}\Big(V_t - B_t\,\mathbf{E}\left(B_\tau^{-1}1\!\!1_{\{\tau\le T\}}\Delta V_\tau \,|\, \mathcal{G}_t\right)\Big),$$

(ii) *if $\Delta V_\tau = 0$, then $S_t = 1\!\!1_{\{\tau>t\}}V_t$ for every $t\in[0,T]$.*

Conjecture (B). Under assumption (H), if Z is a $\mathbb{F}$-predictable process and X is a $\mathcal{F}_T$-measurable random variable, then the continuity condition $\Delta V_\tau = 0$ is satisfied.

Instead of proving conjecture (B), to establish the equality $S_t = 1\!\!1_{\{\tau>t\}}V_t$, which is the convenient form of the valuation formula (4.73), it is enough to show first that $\Lambda=\Gamma$ and use the following result.

Proposition 4.16 *Assume that the condition* (H) *is valid and τ admits an absolutely continuous $\mathbb{F}$-martingale hazard function Λ. Let Z be a $\mathbb{F}$-predictable process and X be a $\mathcal{F}_T$-measurable random variable. If $\Gamma=\Lambda$ then $S_t = 1\!\!1_{\{\tau>t\}}V_t$ for $t\le T$, where processes S and V are given by* (4.73) *and* (4.74), *respectively.*

Proof. First, in view of (H), we have, for any $u>t$ (for the first equality, see Lemma 3.3)

$$\mathbf{P}(\tau\ge u\,|\,\mathcal{F}_\infty\vee\mathcal{D}_t) = 1\!\!1_{\{\tau>t\}}\frac{\mathbf{P}(\tau\ge u\,|\,\mathcal{F}_\infty)}{\mathbf{P}(\tau\ge t\,|\,\mathcal{F}_\infty)} = 1\!\!1_{\{\tau>t\}}\frac{\mathbf{P}(\tau\ge u\,|\,\mathcal{F}_u)}{\mathbf{P}(\tau\ge t\,|\,\mathcal{F}_t)} = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t-\Gamma_u}.$$

Therefore, if Z is a $\mathbb{F}$-predictable process and X is a $\mathcal{F}_T$-measurable random variable, we obtain

$$\begin{aligned}
S_t &= B_t\,\mathbf{E}\Big(\int_t^T B_u^{-1}Z_u\lambda_u 1\!\!1_{\{u\le\tau\}}\,du + B_T^{-1}X1\!\!1_{\{T<\tau\}}\,\Big|\,\mathcal{G}_t\Big)\\
&= B_t\,\mathbf{E}\Big(\int_t^T B_u^{-1}Z_u\lambda_u\,\mathbf{P}\{\tau\ge u\,|\,\mathcal{F}_\infty\vee\mathcal{D}_t\}\,du\,\Big|\,\mathcal{G}_t\Big) + \mathbf{E}\Big(B_T^{-1}X\,\mathbf{P}\{\tau>T\,|\,\mathcal{F}_\infty\vee\mathcal{D}_t\}\,\Big|\,\mathcal{G}_t\Big)\\
&= 1\!\!1_{\{\tau>t\}}B_t\,\mathbf{E}\Big(\int_t^T B_u^{-1}Z_u\lambda_u e^{\Gamma_t-\Gamma_u}\,du\,\Big|\,\mathcal{F}_t\vee\mathcal{D}_t\Big) + 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\Big(B_T^{-1}Xe^{\Gamma_t-\Gamma_T}\,\Big|\,\mathcal{F}_t\vee\mathcal{D}_t\Big)\\
&= 1\!\!1_{\{\tau>t\}}B_t\,\mathbf{E}\Big(\int_t^T B_u^{-1}Z_u\lambda_u e^{\Gamma_t-\Gamma_u}\,du\,\Big|\,\mathcal{F}_t\Big) + 1\!\!1_{\{\tau>t\}}\,\mathbf{E}\Big(B_T^{-1}Xe^{\Gamma_t-\Gamma_T}\,\Big|\,\mathcal{F}_t\Big),
\end{aligned}$$

where the last equality is an immediate consequence of (H) (see, e.g., (H.3)). If $\Gamma=\Lambda$, we may replace Γ by Λ in the last formula. □

4.5 Martingale Hazard Process of a Stopping Time

Assume now that the random time τ is a $\mathbb{F}$-stopping time. This holds if and only if $\mathcal{D}_t\subset\mathcal{F}_t$ for every $t\in\mathbb{R}_+$, or equivalently, if $\mathcal{F}_t=\mathcal{G}_t$ for every $t\in\mathbb{R}_+$. For obvious reasons, condition (H) is then satisfied. Moreover, $F=D$ so that the $\mathbb{F}$-hazard process Γ of τ is not well defined. In view of our further purposes, we find it convenient to refer to τ as a $\mathbb{G}$-stopping time, rather than a $\mathbb{F}$-stopping time. If τ is a $\mathbb{G}$-predictable stopping time then we get simply $\Lambda=D$. On the other hand, if τ is a totally inaccessible $\mathbb{G}$-stopping time, then the $\mathbb{G}$-compensator of the associated jump process D follows a continuous process.[7] Once again, we have $\Lambda\neq\Gamma$ since obviously the $\mathbb{G}$-hazard process Γ is not well defined.[8] Nevertheless, as already noticed in the previous section, the process Λ can be used in the evaluation of certain conditional expectations, provided that some additional assumptions are fulfilled. The following result is a straightforward consequence of Proposition 4.14.

[7] See, for instance, Theorem V.T40 in Dellacherie [8].

[8] However, it is plausible that Λ coincides with the $\hat{\mathbb{F}}$-hazard process $\hat{\Gamma}$ for a certain sub-filtration $\hat{\mathbb{F}}$ of $\mathbb{G}$. Of course, τ cannot be a stopping time with respect to $\hat{\mathbb{F}}$ (see the example below).

Corollary 4.13 *Assume that the $\mathbb{G}$-martingale hazard process Λ of a $\mathbb{G}$-stopping time is an absolutely continuous process. For a fixed $s > 0$, let Y be a $\mathcal{G}_s$-measurable random variable. If the process V, given by the formula*

$$V_t = \mathbf{E}\left(Ye^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\right) = \mathbf{E}\left(Ye^{-\int_t^s \lambda_u \, du} \,|\, \mathcal{G}_t\right), \quad \forall \, t \in [0, s],$$

is continuous at τ, then for any $t < s$ we have

$$\mathbf{E}\left(\mathbb{1}_{\{\tau > s\}} Y \,|\, \mathcal{G}_t\right) = \mathbb{1}_{\{\tau > t\}} \mathbf{E}\left(Ye^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\right) = \mathbf{E}\left(Ye^{-\int_t^s \lambda_u \, du} \,|\, \mathcal{G}_t\right).$$

Example 4.2 Let τ be a random time, given on some probability space $(\Omega, \mathcal{G}, \mathbf{P})$, such that the cumulative distribution function F of τ is continuous. If we take $\mathbb{G} = \mathbb{D}$, then τ is a totally inaccessible $\mathbb{G}$-stopping time, and its $\mathbb{G}$-martingale hazard process Λ satisfies

$$\Lambda_{t \wedge \tau} = \int_0^{t \wedge \tau} \frac{dF(u)}{1 - F(u)}$$

(for instance, $\Lambda_{t \wedge \tau} = t \wedge \tau$ when τ has a unit exponential law under $\mathbf{P}$). It is thus clear that we have $\Lambda_t = \Gamma^0(t \wedge \tau) = \Lambda^0(t \wedge \tau)$, where Γ^0 (Λ^0) is the (martingale) hazard function of τ. We set

$$\Lambda_t = \int_0^t \frac{dF(u)}{1 - F(u)} = \Gamma^0(t) = \Lambda^0(t), \quad \forall \, t \in \mathbb{R}_+.$$

For the process Λ given above, the process V given by (4.72) does not have a jump at τ (for any fixed $s > 0$) and thus for arbitrary $t < s$ we have

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \mathbf{E}\left(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\right) = \mathbb{1}_{\{\tau > t\}} \frac{1 - F(s)}{1 - F(t)},$$

as expected. To conclude, the $\mathbb{G}$-martingale hazard process Λ of τ coincides in fact with the $\mathbb{F}^0$-martingale hazard process of τ (which in turn coincides with the hazard function of τ). It would be thus more natural to consider τ as a random time with respect to the trivial filtration $\mathbb{F}^0$, and to consider the following representation of the 'enlarged' filtration: $\mathbb{G} = \mathbb{D} \vee \mathbb{F}^0$.

Conjecture (C). If Λ is continuous and equality

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \mathbf{E}\left(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\right)$$

holds for every $t \leq s$, then we can find a suitable decomposition $\mathbb{G} = \mathbb{D} \vee \mathbb{F}$, such that Λ is a $\mathbb{F}$-martingale hazard process, and

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \mathbf{E}\left(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\right)$$

for every $t \leq s$.

4.6 Random Time with a Given Hazard Process

In this section, we shall examine the commonly used construction of a random time for a given 'hazard process' Ψ. The $\mathbb{F}$-adapted continuous process Ψ can be equally well considered as the $\mathbb{F}$-hazard process Γ, or the $\mathbb{F}$-martingale hazard process Λ. Indeed, in the standard construction of τ the following properties hold:
(i) Ψ coincides with the $\mathbb{F}$-hazard process Γ of τ,
(ii) Ψ is the $\mathbb{F}$-martingale hazard process of a random time τ, and
(iii) Ψ is $\mathbb{G}$-martingale hazard process of τ considered as a $\mathbb{G}$-stopping time.
It should be stressed that τ constructed below is a random time (but not a stopping time) with

respect to the filtration $\mathbb{F}$, and it is a totally inaccessible stopping time with respect to the enlarged filtration $\mathbb{G}$.

Let Ψ be a $\mathbb{F}$-adapted, continuous, increasing process given on a filtered probability space $(\tilde{\Omega}, \mathbb{F}, \tilde{\mathbf{P}})$ such that $\Psi_0 = 0$ and $\Psi_\infty = +\infty$. For instance, Ψ can be given by the formula

$$\Psi_t = \int_0^t \psi_u \, du, \quad \forall\, t \in \mathbb{R}_+, \tag{4.75}$$

where ψ is a non-negative $\mathbb{F}$-progressively measurable process. Our goal is to construct a random time τ, on an enlarged probability space $(\Omega, \mathcal{G}, \mathbf{P})$, in such a way that Ψ is a $\mathbb{F}$-(martingale) hazard process of τ. To this end, we assume that ξ is a random variable on some probability space[9] $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{P}})$, with the uniform probability law on $[0,1]$. We may take the product space $\Omega = \tilde{\Omega} \times \hat{\Omega}$, $\mathcal{G} = \mathcal{F}_\infty \otimes \hat{\mathcal{F}}$ and $\mathbf{P} = \tilde{\mathbf{P}} \otimes \hat{\mathbf{P}}$. We introduce the random time τ by setting

$$\tau = \inf\{\, t \in \mathbb{R}_+ \,:\, e^{-\Psi_t} \le \xi \,\} = \inf\{\, t \in \mathbb{R}_+ \,:\, \Psi_t \ge -\ln \xi \,\}. \tag{4.76}$$

As usual, we set $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t$ for every t. We shall now check that properties (i)-(iii) also hold.

Proof of (i). We shall first check that (i) holds. To this end, we shall find the process $F_t = \mathbf{P}(\tau \le t \,|\, \mathcal{F}_t)$. Since clearly $\{\tau > t\} = \{e^{-\Psi_t} > \xi\}$, we get $\mathbf{P}(\tau > t \,|\, \mathcal{F}_\infty) = e^{-\Psi_t}$. Consequently,

$$1 - F_t = \mathbf{P}(\tau > t \,|\, \mathcal{F}_t) = \mathbf{E}\left(\mathbf{P}(\tau > t \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t\right) = e^{-\Psi_t},$$

and thus F is a $\mathbb{F}$-adapted continuous increasing process. Notice that

$$F_t = 1 - e^{-\Psi_t} = \mathbf{P}(\tau \le t \,|\, \mathcal{F}_\infty) = \mathbf{P}(\tau \le t \,|\, \mathcal{F}_t). \tag{4.77}$$

We conclude that Ψ coincides with the $\mathbb{F}$-hazard process Γ.

Proof of (ii). The next step is to check that Ψ is the $\mathbb{F}$-martingale hazard process Λ. This can be done either directly, or through equality $\Lambda = \Gamma$. Since Ψ is a continuous process, to show that $\Lambda = \Gamma$, it is enough to check that condition (H) holds, and to apply Corollary 4.10.

¿From (4.77), (H) is valid. We fix t and we consider an arbitary $u \le t$. Since for any $u \in \mathbb{R}_+$

$$\mathbf{P}(\tau \le u \,|\, \mathcal{F}_\infty) = 1 - e^{-\Psi_u}, \tag{4.78}$$

we indeed obtain

$$\mathbf{P}(\tau \le u \,|\, \mathcal{F}_t) = \mathbf{E}\left(\mathbf{P}(\tau \le u \,|\, \mathcal{F}_\infty) \,\middle|\, \mathcal{F}_t\right) = 1 - e^{-\Psi_u} = \mathbf{P}(\tau \le u \,|\, \mathcal{F}_\infty).$$

Alternatively, we may also check directly that (H) holds. Since $\{\tau \le s\} = \{\Psi_s \ge -\ln \xi\} \in \hat{\mathcal{F}} \vee \mathcal{F}_s$, it is clear that $\mathcal{F}_t \subset \mathcal{D}_t \vee \mathcal{F}_t \subset \hat{\mathcal{F}} \vee \mathcal{F}_t$. Therefore, for any $\mathcal{F}_\infty$-measurable bounded random variable ξ, we have

$$\mathbf{E}\,(\xi \,|\, \mathcal{D}_t \vee \mathcal{F}_t) = \mathbf{E}\,(\xi \,|\, \hat{\mathcal{F}} \vee \mathcal{F}_t) = \mathbf{E}\,(\xi \,|\, \mathcal{F}_t), \tag{4.79}$$

where the second equality is a consequence of the independence of $\hat{\mathcal{F}}$ and $\mathcal{F}_\infty$. This shows that (H) holds.

We conclude that the $\mathbb{F}$-martingale hazard process Λ of τ coincides with Γ. To be more specific, we have $\Psi_t = \Lambda_t = \Gamma_t = -\ln(1 - F_t)$. Furthermore, the martingale invariance property holds: any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale.

Proof of (iii). Let us now check directly that Ψ is a $\mathbb{F}$-martingale hazard process of a random time τ. Since Ψ is a $\mathbb{F}$-predictable process (and thus a $\mathbb{G}$-predictable process), we shall simultaneously

[9] Of course, it is enough to assume that we may define on $(\Omega, \mathcal{G}, \mathbf{P})$ a random variable ξ which is uniformly distributed on $[0,1]$, and is independent of the process Ψ (we then set $\hat{\mathcal{F}} = \sigma(\xi)$).

show that Ψ is also the $\mathbb{G}$-martingale hazard process of a $\mathbb{G}$-stopping time τ. We need to verify that the process $D_t - \Psi_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale. First, by virtue of Lemma 3.2 we have for $t \le s$

$$\mathbf{E}\,(D_s - D_t \,|\, \mathcal{G}_t) = \mathbf{E}\,(\mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbf{E}\,(\mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \frac{\mathbf{P}(t<\tau\le s \,|\, \mathcal{F}_t)}{\mathbf{P}(\tau>t \,|\, \mathcal{F}_t)}.$$

Using (4.77), we get $\mathbf{P}(t<\tau\le s \,|\, \mathcal{F}_t) = \mathbf{E}\,(F_s \,|\, \mathcal{F}_t) - F_t$. Therefore

$$\mathbf{E}\,(D_s - D_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \frac{\mathbf{E}\,(F_s \,|\, \mathcal{F}_t) - F_t}{1-F_t}. \tag{4.80}$$

On the other hand, if we set $Y = \Psi_{s\wedge\tau} - \Psi_{t\wedge\tau}$, then in view of (i) we get

$$Y = \mathbb{1}_{\{\tau>t\}} Y = \ln\Big(\frac{1-F_{s\wedge\tau}}{1-F_{t\wedge\tau}}\Big) = \int_{]t,s\wedge\tau]} \frac{dF_u}{1-F_u}.$$

Using again (3.1), we obtain

$$\mathbf{E}\big(Y \,|\, \mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}}\, \frac{\mathbf{E}\,(Y \,|\, \mathcal{F}_t)}{\mathbf{P}(\tau>t \,|\, \mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}}\, \frac{\mathbf{E}\,\big(\int_{]t,s\wedge\tau]}(1-F_u)^{-1} dF_u \,|\, \mathcal{F}_t\big)}{1-F_t} = \mathbb{1}_{\{\tau>t\}}\, \frac{\mathbf{E}\,(F_s \,|\, \mathcal{F}_t) - F_t}{1-F_t}.$$

We conclude that the process $D_t - \Psi_{t\wedge\tau}$ is indeed a $\mathbb{G}$-martingale.

Notice that the role played by the 'hazard process' Ψ in (i) and (iii) is slightly different. If we consider Ψ as a $\mathbb{F}$-hazard process of τ, then using Corollary 3.1 we deduce that for any $\mathcal{F}_s$-measurable random variable Y

$$\mathbf{E}\,(\mathbb{1}_{\{\tau>s\}}\, Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbf{E}\,\big(Y e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t\big). \tag{4.81}$$

On the other hand, if Ψ is considered as the $\mathbb{G}$-martingale hazard process then, in view of Corollary 4.13, for any $\mathcal{G}_s$-measurable random variable Y such that the associated process V is continuous at τ we obtain

$$\mathbf{E}\,(\mathbb{1}_{\{\tau>s\}}\, Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbf{E}\,\big(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\big). \tag{4.82}$$

If Y is actually $\mathcal{F}_s$-measurable then we have (see (4.79))

$$\mathbf{E}\,\big(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\big) = \mathbf{E}\,\big(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t \vee \mathcal{D}_t\big) = \mathbf{E}\,\big(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\big)$$

so that the associated process V is necessarily continuous at τ, and formulae (4.81) and (4.82) coincide.

Remarks. Assume that Ψ satisfies (4.75). Then (4.80) can be rewritten as follows

$$\mathbf{P}(t<\tau\le s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbf{E}\,\big(1 - e^{-\int_t^s \psi_u\, du} \,|\, \mathcal{F}_t\big). \tag{4.83}$$

Using (4.77), we find that the cumulative distribution function of a random time τ under $\mathbf{P}$ equals

$$\mathbf{P}(\tau \le t) = 1 - \mathbf{E}\,\Big(e^{-\int_0^t \psi_u\, du}\Big) = 1 - e^{-\int_0^t \gamma^0(u)\, du},$$

where we write γ^0 to denote the unique $\mathbb{F}^0$-intensity of τ.

5 Analysis of Several Random Times

In this section, we shall deal with several random times. Suppose we are given random times $\tau_1, \ldots, \tau_n$, defined on a common probability space $(\Omega, \mathcal{G}, \mathbf{P})$ endowed with a filtration $\mathbb{F}$. We are interested in the study of the hazard functions and processes associated with these random times. In particular, one of our goals will be to examine the relationship between the $(\mathbb{F}, \mathbb{G})$-martingale hazard processes of stopping times τ_i and the $(\mathbb{F},\mathbb{G})$-martingale hazard process of their minimum $\tau = \min(\tau_1,\dots,\tau_n)$. For $i=1,\dots,n$ we set

$$D^i_t = \mathbb{1}_{\{\tau_i \le t\}}, \quad \forall\, t \in \mathbb{R}_+,$$

and we introduce the associated filtration $\mathbb{D}^i$ generated by the process D^i. We introduce the enlarged filtration $\mathbb{G} := \mathbb{D}^1 \vee \ldots \vee \mathbb{D}^n \vee \mathbb{F}$. It is thus evident that $\tau_1,\dots,\tau_n$ are stopping times with respect to the filtration $\mathbb{G}$.

5.1 Ordered Random Times

Consider two $\mathbb{F}$-adapted increasing continuous processes, Ψ^1 and Ψ^2, which satisfy $\Psi^2_0 = \Psi^1_0 = 0$ and $\Psi^2_t > \Psi^1_t$ for every $t \in \mathbb{R}_+$. Let ξ be a random variable which is uniformly distributed on $[0,1]$, and is independent of the process Ψ. For $i = 1, 2$ we set (see Section 4.6)

$$\tau_i = \inf\{\, t \in \mathbb{R}_+ \,:\, e^{-\Psi^i_t} \le \xi \,\} = \inf\{\, t \in \mathbb{R}_+ \,:\, \Psi^i_t \ge -\ln \xi \,\}. \tag{5.1}$$

so that obviously $\tau_1 < \tau_2$ with probability 1.

We shall write $\mathbb{G}^i = \mathbb{D}^i \vee \mathbb{F}$, for $i = 1, 2$, and $\mathbb{G} = \mathbb{D}^1 \vee \mathbb{D}^2 \vee \mathbb{F}$. An analysis of each random time τ_i with respect to its 'natural' enlarged filtration $\mathbb{G}^i$ can be done along the same lines as in the previous section. Recall that for two filtration $\mathbb{F} \subset \mathbb{G}$, a $\mathbb{F}$-predictable right-continuous increasing process Λ^i is a $(\mathbb{F},\mathbb{G})$-martingale hazard process of a random time τ_i if the process

$$M^i_t = D^i_t - \Lambda^i_{t\wedge\tau_i}$$

follows a $\mathbb{G}$-martingale. It is clear that for each i the process Ψ^i represents: (i) the $(\mathbb{F},\mathbb{G}^i)$-hazard process Γ^i of a random time τ_i, (ii) the $(\mathbb{F},\mathbb{G}^i)$-martingale hazard process Λ^i of a random time τ_i, and (iii) the $\mathbb{G}^i$-martingale hazard process of τ_i when τ_i is considered as a $\mathbb{G}^i$-stopping time.

We shall focus on the study of hazard processes with respect to the enlarged filtrations. We find it convenient to introduce the following auxiliary notation:[10] $\mathbb{F}^i = \mathbb{D}^i \vee \mathbb{F}$, so that $\mathbb{G} = \mathbb{D}^1 \vee \mathbb{F}^2$ and $\mathbb{G} = \mathbb{D}^2 \vee \mathbb{F}^1$. Let us start by an analysis of τ_1. We are looking for the $(\mathbb{F}^2,\mathbb{G})$-hazard process $\tilde\Gamma^1$ of τ_1, as well as for the $(\mathbb{F}^2,\mathbb{G})$-martingale hazard process $\tilde\Lambda^1$ of τ_1. We shall first check that $\tilde\Gamma^1 \ne \Gamma^1$. Indeed, by virtue of the definition of a hazard process we have, for $t \in \mathbb{R}_+$,

$$e^{-\Gamma^1_t} = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t) = e^{-\Psi^1_t}.$$

and

$$e^{-\tilde\Gamma^1_t} = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}^2_t) = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^2_t).$$

Equality $\tilde\Gamma^1 = \Gamma^1$ would thus imply the following relationship, for every $t \in \mathbb{R}_+$,

$$\mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^2_t) = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t), \tag{5.2}$$

The relationship above is manifestly not valid, however. In effect, the inequality $\tau_2 \le t$ implies $\tau_1 \le t$, therefore on the set $\{\tau_2 \le t\}$, which clearly belongs to the σ-field $\mathcal{D}^2_t$, we have $\mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^2_t) = 0$, and this contradicts (5.2). This shows also that the $(\mathbb{F}^2,\mathbb{G})$-hazard process $\tilde\Gamma^1$ is well defined only strictly before τ_2.

On the other hand, since $\mathbb{G} = \mathbb{G}^1 \vee \mathbb{D}^2$, it is clear that the process $D^1_t - \Psi^1_{t\wedge\tau_1}$, which is of course stopped at τ_1, is not only a $\mathbb{G}^1$-martingale, but also a $\mathbb{G}$-martingale. We conclude that Ψ^1 coincides with the $(\mathbb{F}^2,\mathbb{G})$-martingale hazard process $\tilde\Lambda^1$ of τ_1. A similar reasoning shows that Ψ^1 represents also the $\mathbb{G}$-martingale hazard process $\hat\Lambda^1$ of τ_1.

[10] Though in the present setup $\mathbb{F}^i = \mathbb{G}^i$, this double notation will appear useful in what follows.

As one might easily guess, the properties of τ_2 with respect to the filtration $\mathbb{F}^1$ are slightly different. First, we have

$$e^{-\tilde{\Gamma}^2_t} = \mathbf{P}(\tau_2 > t \,|\, \mathcal{F}^1_t) = \mathbf{P}(\tau_2 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^1_t).$$

We claim that $\tilde{\Gamma}^2 \neq \Gamma^2$, that is, the equality

$$\mathbf{P}(\tau_2 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^1_t) = \mathbf{P}(\tau_2 > t \,|\, \mathcal{F}_t) \tag{5.3}$$

is not valid, in general. Indeed, the inequality $\tau_1 > t$ implies $\tau_2 > t$, and thus on set $\{\tau_1 > t\}$, which belongs to $\mathcal{D}^1_t$, we have $\mathbf{P}(\tau_2 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^1_t) = 1$, in contradiction with (5.3). Notice that the process $\tilde{\Gamma}^2$ is not well defined after time τ_1.

Furthermore, the process $D^2_t - \Psi^2_{t\wedge\tau_2}$ is a $\mathbb{G}^2$-martingale; it does not follow a $\mathbb{G}$-martingale, however (otherwise, the equality $\tilde{\Gamma}^2 = \Gamma^2 = \Psi^2$ would hold on $[0, \tau_2]$, but this is clearly not true). The exact evaluation of the $(\mathbb{F}^1, \mathbb{G})$-martingale hazard process $\tilde{\Lambda}^2$ of τ_2 seems to be rather difficult. Let us only mention that it is reasonable to expect that $\tilde{\Lambda}^2$ it is discontinuous at τ_1.

Let us finally notice that τ_1 is a totally inaccessible stopping time not only with respect to $\mathbb{G}^1$, but also with respect to the filtration $\mathbb{G}$. On the other hand, τ_2 is a totally inaccessible stopping time with respect to $\mathbb{G}^1$, but it is a predictable stopping time with respect to $\mathbb{G}$. Indeed, we may easily find an announcing sequence τ_2^n of $\mathbb{G}$-stopping times, for instance,

$$\tau_2^n = \inf\{\, t \geq \tau_1 \,:\, \Psi^2_t \geq -\ln\xi - \tfrac{1}{n}\,\}.$$

Therefore the $\mathbb{G}$-martingale hazard process $\hat{\Lambda}^2$ of τ_2 coincides with the $\mathbb{G}$-predictable process $D^2_t = \mathbb{1}_{\{\tau_2 \leq t\}}$. Let us set $\tau = \tau_1 \wedge \tau_2$. In the present setup, it is evident that $\tau = \tau_1$, and thus the $\mathbb{G}$-martingale hazard process $\hat{\Lambda}$ of τ is equal to Ψ^1. It is also equal to the sum of $\mathbb{G}$-martingale hazard processes $\hat{\Lambda}^i$ of τ_i, $i = 1, 2$. Indeed, we have

$$\hat{\Lambda}_{t\wedge\tau} = \Psi^1_{t\wedge\tau} = \Psi^1_{t\wedge\tau} + D^2_{t\wedge\tau} = \hat{\Lambda}^1_{t\wedge\tau} + \hat{\Lambda}^2_{t\wedge\tau}.$$

We shall see in the next section that this property is universal (though not always very useful).

5.2 Properties of the Minimum of Several Random Times

We shall examine the following problem: given a finite family of random times τ_i, $i = 1, \ldots, n$, and the associated hazard processes, find the hazard process of the random time $\tau = \tau_1 \wedge \ldots \wedge \tau_n$. The problem above cannot be solved in such a generality, that is, without the knowledge of the joint law of $(\tau_1, \ldots, \tau_n)$. Indeed, as we shall see in what follows the solution depends heavily on specific assumptions on random times and the choice of filtrations (we follow Duffie [11] and Kusuoka [19]).

5.2.1 Hazard Function of the Minimum of Several Random Times

Let us first consider a simple result, in which we focus on the calculation of the hazard function of the minimum of several independent random times.

Lemma 5.1 *Let τ_i, $i = 1, \ldots, n$, be n random times defined on a common probability space $(\Omega, \mathcal{G}, \mathbf{P})$. Assume that τ_i admits the hazard function Γ^i. If τ_i, $i = 1, \ldots, n$, are mutually independent random variables, then the hazard function Γ of τ is equal to the sum of hazard functions Γ^i, $i = 1, \ldots, n$.*

Proof. For any $t \in \mathbb{R}_+$ we have

$$\begin{aligned} e^{-\Gamma(t)} &= 1 - F(t) = \mathbf{P}(\tau > t) = \mathbf{P}(\min(\tau_1, \ldots, \tau_n) > t) = \prod_{i=1}^n \mathbf{P}(\tau_i > t) \\ &= \prod_{i=1}^n (1 - F_i(t)) = \prod_{i=1}^n e^{-\Gamma^i(t)} = e^{-\sum_{i=1}^n \Gamma^i(t)}. \qquad \square \end{aligned}$$

Let us now focus on the case of continuous distribution functions $F_i,\ i = 1, \dots, n$. In this case, we get also $\Lambda(t) = \sum_{i=1}^n \Lambda^i(t)$. In particular, if τ_i admits the intensity $\gamma^i(t) = \lambda^i(t) = f_i(t)(1-F_i(t))^{-1}$, for each i, then the process

$$D_t - \sum_{i=1}^n \int_0^{t\wedge\tau} \gamma^i(u)\,du = 1\!\!1_{\{\tau\le t\}} - \sum_{i=1}^n \int_0^{t\wedge\tau} \lambda^i(u)\,du$$

follows a $\mathbb{D}$-martingale, where $\mathbb{D} = \mathbb{D}^1 \vee \dots \vee \mathbb{D}^n$.

Conversely, if the hazard function of τ satisfies $\Lambda(t) = \Gamma(t) = \sum_{i=1}^n \Gamma^i(t) = \sum_{i=1}^n \Lambda^i(t)$ for every t then we obtain

$$\mathbf{P}(\tau_1 > t, \dots, \tau_n > t) = \prod_{i=1}^n \mathbf{P}(\tau_i > t), \quad \forall\, t \in \mathbb{R}_+.$$

5.2.2 Martingale Hazard Process of the Minimum of Several Random Times

We borrow from Duffie [11] the following simple result (see Lemma 1 therein).

Lemma 5.2 *Let $\tau_i,\ i = 1, \dots, n,$ be random times such that $\mathbf{P}(\tau_i = \tau_j) = 0$ for $i \ne j$. Then the $(\mathbb{F}, \mathbb{G})$-martingale hazard process Λ of $\tau = \tau_1 \wedge \dots \wedge \tau_n$ is equal to the sum of $(\mathbb{F}, \mathbb{G})$-martingale hazard processes Λ^i – that is,*

$$\Lambda_t = \sum_{i=1}^n \Lambda^i_t, \quad \forall\, t \in \mathbb{R}_+. \tag{5.4}$$

If Λ is a continuous process then the process $\tilde{L}$ given by the formula $\tilde{L}_t = (1 - D_t)e^{\Lambda_t}$ is a $\mathbb{G}$-martingale.

Proof. By assumption, for any $i = 1, \dots, n,$ the process $\tilde{M}^i_t := D^i_t - \Lambda^i_{t\wedge\tau_i}$ is a $\mathbb{G}$-martingale. Therefore, by the well-known properties of martingales the stopped process[11]

$$(\tilde{M}^i_t)^\tau = D^i_{t\wedge\tau} - \Lambda^i_{t\wedge\tau_i\wedge\tau} = D^i_{t\wedge\tau} - \Lambda^i_{t\wedge\tau}$$

also follows a $\mathbb{G}$-martingale for any fixed i. On the other hand, since $\mathbf{P}(\tau_i = \tau_j) = 0$ for $i \ne j$, we have

$$\sum_{i=1}^n D^i_{t\wedge\tau} = D_t = 1\!\!1_{\{\tau\le t\}}.$$

Therefore, the process

$$\tilde{M}_t := D_t - \sum_{i=1}^n \Lambda^i_{t\wedge\tau} = \sum_{i=1}^n (\tilde{M}^i_t)^\tau$$

obviously follows a $\mathbb{G}$-martingale, as a sum of $\mathbb{G}$-martingales. We conclude that the $(\mathbb{F}, \mathbb{G})$-martingale hazard process Λ of τ satisfies (5.4). The second statement is an easy consequence of Itô's formula, which gives

$$\tilde{L}_t = 1 - \int_{]0,t]} \tilde{L}_{u-}\, d\tilde{M}_u. \tag{5.5}$$

This ends the proof. □

The striking feature of Lemma 5.2 is that the $(\mathbb{F}, \mathbb{G})$-martingale hazard process of τ can be easily found without the knowledge the joint probability law of random times $\tau_1, \dots, \tau_n$. It should thus be observed that in order to make use of the notion of a $(\mathbb{F}, \mathbb{G})$-martingale hazard process Λ we need

[11] Of course, if $\tau_i,\ i = 1 \dots, n$ are stopping times with respect to the filtration $\mathbb{G}$, then τ is also a $\mathbb{G}$-stopping time.

to show in addition that Λ actually possesses desired properties. For instance, it would be useful to know whether the equality

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \mathbf{E}\left(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\right) \tag{5.6}$$

holds for every $t \leq s$, or more generally whether we have

$$\mathbf{E}\left(\mathbb{1}_{\{\tau > s\}} Y \,|\, \mathcal{G}_t\right) = \mathbb{1}_{\{\tau > t\}} \mathbf{E}\left(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\right) \tag{5.7}$$

for any bounded $\mathcal{G}_s$-measurable (or $\mathcal{F}_s$-measurable) random variable Y.

From now on, we shall assume that hypothesis (H) is satisfied. Combining Lemma 5.2 with Corollary 4.11, we get immediately the following result, which gives only a partial answer to the last question, however (see also Proposition 5.2 for related results).

Proposition 5.1 *Let τ_i, $i = 1, \ldots, n$, be random times such that $\mathbf{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. Assume that hypothesis (H) is satisfied and that each random time τ_i admits a continuous $(\mathbb{F}, \mathbb{G})$-martingale hazard process Λ^i. Let us set $\Lambda = \sum_{i=1}^n \Lambda^i$, and let Y be a bounded $\mathcal{G}_s$-measurable random variable. Assume that the process V given by the formula*

$$V_t = \mathbf{E}\left(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\right), \quad \forall\, t \in [0, s], \tag{5.8}$$

is continuous at τ. Then for any $t < s$ we have

$$\mathbf{E}\left(\mathbb{1}_{\{\tau > s\}} Y \,|\, \mathcal{G}_t\right) = \mathbb{1}_{\{\tau > t\}} \mathbf{E}\left(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\right). \tag{5.9}$$

In the case of absolutely continuous processes Λ^i we have

$$V_t = \mathbf{E}\left(Y e^{-\sum_{i=1}^n \int_t^s \lambda_u^i \, du} \,|\, \mathcal{F}_t\right), \tag{5.10}$$

and

$$\mathbf{E}\left(\mathbb{1}_{\{\tau > s\}} Y \,|\, \mathcal{G}_t\right) = \mathbb{1}_{\{\tau > t\}} \mathbf{E}\left(Y e^{-\sum_{i=1}^n \int_t^s \lambda_u^i \, du} \,|\, \mathcal{F}_t\right). \tag{5.11}$$

At the first glance Proposition 5.1 seems to be a very useful and powerful result, since apparently it covers the case of independent and dependent random times. Notice, however, that the assumptions in Proposition 5.1 are rather restrictive: (i) any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale, (ii) each $(\mathbb{F}, \mathbb{G})$-martingale hazard process Λ^i is continuous. In addition, we deal here with a rather delicate issue of checking the continuity of V at τ. Therefore, the number of circumstances when Proposition 5.1 can be easily applied is in fact rather limited. One of them is examined in the foregoing example, in which random times τ_i are assumed conditionally independent given the filtration $\mathbb{F}$.

Example 5.3 Let ψ^1 and ψ^2 be two $\mathbb{F}$-progressively measurable non-negative stochastic processes defined on a common probability space $(\Omega, \mathcal{G}, \mathbf{P})$, endowed with the filtration $\mathbb{F}$. We assume that

$$\int_0^\infty \psi_u^1 \, du = \int_0^\infty \psi_u^2 \, du = \infty$$

and we set

$$\tau_i = \inf \left\{ t \in \mathbb{R}_+ \, : \, \int_0^t \psi_u^i \, du \geq -\ln \xi^i \right\}, \tag{5.12}$$

where ξ^1, ξ^2 are mutually independent random variables, defined on $(\Omega, \mathcal{G}, \mathbf{P})$, which are also independent of processes ψ^i, $i = 1, 2$, and are uniformly distributed on the unit interval $[0, 1]$. For each i, the enlarged filtration $\mathbb{G}^i := \mathbb{D}^i \vee \mathbb{F}$ thus satisfies $\mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{D}_t^i \subset \mathcal{F}_t \vee \sigma(\xi^i)$ for every t.

From Section 4.6 we know that the process $\Psi^i = \int_0^t \psi^i_u \, du$ represents the $\mathbb{F}$-hazard process of τ_i. In particular, for any $\mathcal{F}_s$-measurable random variable Y we have for every $t \le s$ (cf. (4.83))

$$\mathbf{E}\,(1\!\!1_{\{\tau_i > s\}} Y \,|\, \mathcal{G}^i_t) = 1\!\!1_{\{\tau_i > t\}}\, \mathbf{E}\,\big(Y e^{-\int_t^s \psi^i_u \, du} \,|\, \mathcal{F}_t\big). \tag{5.13}$$

In addition, the process Ψ^i is also the $(\mathbb{F}, \mathbb{G}^i)$-martingale hazard process of a random time τ_i. Finally, τ_i is a totally inaccessible stopping time with respect to $\mathbb{G}^i$, and the continuity condition of Corollary 4.11 is satisfied. Indeed, for any fixed $s > 0$, the process

$$V^i_t = \mathbf{E}\,\big(e^{\Psi^i_t - \Psi^i_s} \,|\, \mathcal{F}_t\big) = \mathbf{E}\,\big(e^{\Psi^i_t - \Psi^i_s} \,|\, \mathcal{G}^i_t\big), \quad \forall\, t \in [0, s],$$

is obviously continuous at τ_i. We conclude that for any $t \le s$

$$\mathbf{P}(\tau_i > s \,|\, \mathcal{G}^i_t) = 1\!\!1_{\{\tau_i > t\}}\, \mathbf{E}\,\big(e^{-\int_t^s \psi^i_u \, du} \,|\, \mathcal{F}_t\big) = 1\!\!1_{\{\tau_i > t\}}\, \mathbf{E}\,\big(e^{-\int_t^s \psi^i_u \, du} \,|\, \mathcal{G}^i_t\big). \tag{5.14}$$

We introduce the filtration $\mathbb{G}$ by setting $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1 \vee \mathbb{D}^2$. Then τ_1, τ_2, as well as $\tau = \min(\tau_1, \tau_2)$ are $\mathbb{G}$-stopping times. It is not obvious, however, that the process Ψ^i is the $(\mathbb{F}, \mathbb{G})$-martingale hazard process of τ_i. We know that Ψ^i is a $\mathbb{G}$-adapted continuous process such that $\tilde{M}^i_t = D^i_t - \Psi^i_{t \wedge \tau_i}$ is a $\mathbb{G}^i$-martingale. To conclude, we need to show that $\tilde{M}^i$ is also a $\mathbb{G}$-martingale.

Let us consider, for instance, $i = 1$. The random variable $\tilde{M}^1_t$ is manifestly $\mathcal{G}_t$-measurable. It is thus enough to check that for any $t \le s$

$$\mathbf{E}\,(D^1_s - \Psi^1_{s \wedge \tau_1} \,|\, \mathcal{G}_t) = \mathbf{E}\,(D^1_s - \Psi^1_{s \wedge \tau_1} \,|\, \mathcal{G}^1_t).$$

Notice that the σ-fields $\mathcal{G}^1_s$ and $\mathcal{D}^2_t$ are conditionally independent given $\mathcal{G}^1_t$. Consequently,

$$\mathbf{E}\,(D^1_s - \Psi^1_{s \wedge \tau_1} \,|\, \mathcal{G}_t) = \mathbf{E}\,(D^1_s - \Psi^1_{s \wedge \tau_1} \,|\, \mathcal{G}^1_t \vee \mathcal{D}^2_t) = \mathbf{E}\,(D^1_s - \Psi^1_{s \wedge \tau_1} \,|\, \mathcal{G}^1_t).$$

Since we have shown that Ψ^1 is the $(\mathbb{F}, \mathbb{G})$-martingale hazard process of τ_1, we have (under mild assumption on $\mathcal{G}_s$-measurable random variable Y)

$$\mathbf{E}\,(1\!\!1_{\{\tau_1 > s\}}\, Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau_1 > t\}}\; \mathbf{E}\,\big(Y e^{\Psi^1_t - \Psi^1_s} \,|\, \mathcal{G}_t\big).$$

In particular, we have for any $t \le s$ (cf. (5.14))

$$\mathbf{P}(\tau_1 > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau_1 > t\}}\; \mathbf{E}\,\big(e^{\Psi^1_t - \Psi^1_s} \,|\, \mathcal{G}_t\big) = 1\!\!1_{\{\tau_i > t\}}\, \mathbf{E}\,\big(e^{-\int_t^s \psi^i_u \, du} \,|\, \mathcal{F}_t\big) \tag{5.15}$$

since the process

$$\tilde{V}^1_t := \mathbf{E}\,\big(e^{\Psi^1_t - \Psi^1_s} \,|\, \mathcal{G}_t\big) = \mathbf{E}\,\big(e^{\Psi^1_t - \Psi^1_s} \,|\, \mathcal{F}_t\big), \quad \forall\, t \in [0, s],$$

is continuous at τ_1.

In view of Lemma 5.2, the $(\mathbb{F}, \mathbb{G})$-martingale hazard process Ψ of τ, when stopped at τ, is the sum of $(\mathbb{F}, \mathbb{G})$-martingale hazard processes Ψ^i, $i = 1, 2,$ associated with random times τ_i, $i = 1, 2,$ also stopped at τ. We have for $t \le s$

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}}\; \mathbf{E}\,\big(e^{\Psi_t - \Psi_s} \,|\, \mathcal{G}_t\big) = 1\!\!1_{\{\tau > t\}}\, \mathbf{E}\,\big(e^{-\int_t^s (\psi^1_u + \psi^2_u)\, du} \,|\, \mathcal{F}_t\big). \tag{5.16}$$

In should be stressed that the last formula is a consequence of the assumption that the underlying random variables ξ^1 and ξ^2 are independent. The case of dependent random variables ξ^1 and ξ^2 is much more involved; let us only observe that we cannot expect formula (5.16) to hold in this case. Indeed, it seems plausible that the $\mathbb{G}$-martingale hazard process of τ_1 will have a jump at τ_2 on the set $\{\tau_2 < \tau_1\}$, and conversely, the $\mathbb{G}$-martingale hazard process of τ_2 will be discontinuous at τ_1 on the set $\{\tau_1 < \tau_2\}$. Consequently, one may conjecture that the sum of these processes will have a discontinuity at τ, and thus it will not be possible to use the $\mathbb{G}$-martingale hazard process of τ to directly represent the survival probability $\mathbf{P}(\tau > s \,|\, \mathcal{G}_t)$ through a counterpart of formula (5.16).

At the intuitive level, if the underlying random variables ξ^1 and ξ^2 are not independent, the observed occurrence of τ_2 (τ_1, resp.) has a sudden impact on our assessments of the likelihood of the occurrence of τ_1 (τ_2, resp.) in a given time interval in the future. A very special case of such a situation, when $\xi^1 = \xi^2$, was examined in Section 5.1. The general case remains, to our knowledge, an open problem.

Remarks. Alternatively, we may check that Ψ^1 is also the $(\mathbb{G}^2, \mathbb{G})$-martingale hazard process of τ_1. Since Ψ^1 is a continuous $\mathbb{G}^2$-adapted process and $\mathbb{G} = \mathbb{G}^2 \vee \mathbb{D}^1$, it is enough to verify that Ψ^1 coincides with the $\mathbb{G}^2$-hazard process of τ_1, or equivalently, that

$$\mathbf{P}(\tau_1 > t \,|\, \mathcal{G}^2_t) = e^{-\Psi^1_t}, \quad \forall\, t \in \mathbb{R}_+.$$

The last equality is clear, however, since the σ-fields $\mathcal{G}^1_t$ and $\mathcal{D}^2_t$ are conditionally independent given $\mathcal{F}_t$, and thus (the event $\{\tau_1 > t\}$ belongs, of course, to $\mathcal{G}^1_t$)

$$\mathbf{P}(\tau_1 > t \,|\, \mathcal{G}^2_t) = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^2_t) = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t) = e^{-\Psi^1_t}.$$

5.2.3 Case of a Brownian Filtration

In this section, we consider once again the case of the Brownian filtration, that is, we assume that $\mathbb{F} = \mathbb{F}^W$ for some Brownian motion W. We postulate that W remains a martingale (and thus a Brownian motion) with respect to the enlarged filtration $\mathbb{G} = \mathbb{D}^1 \vee \ldots \vee \mathbb{D}^n \vee \mathbb{F}$. In view of the martingale representation property of the Brownian filtration this means, of course, that any $\mathbb{F}$-local martingale follows also a local martingale with respect to $\mathbb{G}$ (or indeed with respect to any filtration $\mathbb{F} \subseteq \tilde{\mathbb{F}} \subseteq \mathbb{G}$), so that (H) holds. It is worthwhile to stress that the case when $\mathbb{F}$ is a trivial filtration is also covered by the results of this section, however.

The first result is a generalization of the martingale representation property established in Corollary 3.7 (see also Proposition 3.5). Recall that in Corollary 3.7 we have assumed that the $\mathbb{F}$-hazard process Γ of a random time τ is an increasing continuous process. Also, by virtue of results of Section 4.2 (see Proposition 4.10) under the assumptions of Corollary 3.7 we have $\Gamma = \Lambda$, that is, the $\mathbb{F}$-hazard process Γ and the $(\mathbb{F}, \mathbb{G})$-martingale hazard process Λ coincide.

In the present setup, we find it convenient to make assumptions directly about the $(\mathbb{F}, \mathbb{G})$-martingale hazard processes Λ^i of random times τ_i, $i = 1, \ldots, n$. As before, we assume that $\mathbf{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. Recall that by virtue of the definition of the $(\mathbb{F}, \mathbb{G})$-martingale hazard process Λ^i of a random time τ_i the process

$$\tilde{M}^i_t = D^i_t - \Lambda^i_{t \wedge \tau_i} \tag{5.17}$$

follows a $\mathbb{G}$-martingale. It is thus easily seen that the process

$$\tilde{L}^i_t = (1 - D^i_t) e^{\Lambda^i_t}$$

also follows a $\mathbb{G}$-martingale, since clearly (cf. (4.58) or (5.5))

$$\tilde{L}^i_t = 1 - \int_{]0,t]} \tilde{L}^i_{u-} \, d\tilde{M}^i_u. \tag{5.18}$$

It is easily seen that $\tilde{L}^i$ and $\tilde{L}^j$ are mutually orthogonal $\mathbb{G}$-martingales for any $i \neq j$ (a similar remark applies to $\tilde{M}^i$ and $\tilde{M}^j$).

For a fixed k with $0 \leq k \leq n$, we introduce the filtration $\tilde{\mathbb{G}} = \mathbb{D}^1 \vee \ldots \vee \mathbb{D}^k \vee \mathbb{F}$. Then obviously $\tilde{\mathbb{G}} = \mathbb{G}$ if $k = n$, and by convention $\tilde{\mathbb{G}} = \mathbb{F}$ for $k = 0$. It is clear that for any fixed k and arbitrary $i \leq k$ processes $\tilde{L}^i$ and $\tilde{M}^i$ are $\tilde{\mathbb{G}}$-adapted. More specifically, $\tilde{L}^i$ and $\tilde{L}^j$ are mutually orthogonal $\tilde{\mathbb{G}}$-martingales for $i, j \leq k$ provided that $i \neq j$.

A trivial modification of Lemma 5.2 shows that the $(\mathbb{F}, \tilde{\mathbb{G}})$-martingale hazard process of the random time $\tilde{\tau} := \tau_1 \wedge \ldots \wedge \tau_k$ equals $\tilde{\Lambda} = \sum_{i=1}^{k} \Lambda^i$. In other words, the process $\tilde{D}_t - \sum_{i=1}^{k} \Lambda^i_{t\wedge\tilde{\tau}}$ is a $\tilde{\mathbb{G}}$-martingale, where we set $\tilde{D}_t = 1\!\!1_{\{\tilde{\tau}\le t\}}$.

Proposition 5.2 *Assume that the $\mathbb{F}$-Brownian motion W remains a Brownian motion with respect to the enlarged filtration $\mathbb{G}$. Let Y be a bounded $\mathcal{F}_T$-measurable random variable, and let $\tilde{\tau} = \tau_1 \wedge \ldots \wedge \tau_k$. Then for any $t \le s \le T$ we have*

$$\mathbf{E}\,(1\!\!1_{\{\tilde{\tau}>s\}} Y \,|\, \mathcal{G}_t) = \mathbf{E}\,(1\!\!1_{\{\tilde{\tau}>s\}} Y \,|\, \tilde{\mathcal{G}}_t) = 1\!\!1_{\{\tilde{\tau}>t\}}\, \mathbf{E}\,\big(Y e^{\tilde{\Lambda}_t - \tilde{\Lambda}_s} \,|\, \mathcal{F}_t\big), \tag{5.19}$$

so that

$$\mathbf{P}(\tilde{\tau} > s \,|\, \mathcal{G}_t) = \mathbf{P}(\tilde{\tau} > s \,|\, \tilde{\mathcal{G}}_t) = 1\!\!1_{\{\tilde{\tau}>t\}} \mathbf{E}\,(e^{\tilde{\Lambda}_t - \tilde{\Lambda}_s} \,|\, \mathcal{F}_t). \tag{5.20}$$

In particular, for $\tau = \tau_1 \wedge \ldots \wedge \tau_n$ we have

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \mathbf{E}\,(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t), \tag{5.21}$$

where $\Lambda = \sum_{i=1}^{n} \Lambda^i$.

Proof. For a fixed $s \le T$, we set

$$\tilde{Y}_t = \mathbf{E}\,(Y e^{-\tilde{\Lambda}_s} \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T].$$

Let the process U be given by the formula

$$U_t = (1 - \tilde{D}_{t\wedge s}) e^{\tilde{\Lambda}_{t\wedge s}} = \prod_{i=1}^{k} \tilde{L}^i_{t\wedge s}, \quad \forall\, t \in [0, T]. \tag{5.22}$$

Under the present assumptions the process $\tilde{Y}$ is a continuous $\mathbb{G}$-martingale and thus also a $\tilde{\mathbb{G}}$-martingale. The process U, which is manifestly of finite variation, is also a $\tilde{\mathbb{G}}$-martingale as a product of mutually orthogonal $\tilde{\mathbb{G}}$-martingales $\tilde{L}^1, \ldots, \tilde{L}^k$ (stopped at s). Therefore, their product $U\tilde{Y}$ is a $\tilde{\mathbb{G}}$-martingale. Consequently, for any $t \le s$

$$\mathbf{E}\,(1\!\!1_{\{\tilde{\tau}>s\}} Y \,|\, \tilde{\mathcal{G}}_t) = \mathbf{E}\,(U_T \tilde{Y}_T \,|\, \tilde{\mathcal{G}}_t) = U_t \tilde{Y}_t = (1 - \tilde{D}_t) e^{\tilde{\Lambda}_t} \mathbf{E}\,(Y e^{-\tilde{\Lambda}_s} \,|\, \mathcal{F}_t)$$

as expected. It is clear that we may replace $\tilde{\mathbb{G}}$ by $\mathbb{G}$ in the reasoning above. □

Let us set $\tilde{\mathbb{F}} := \mathbb{D}^{k+1} \vee \ldots \vee \mathbb{D}^n \vee \mathbb{F}$ for a fixed, but arbitrary, $k = 0, \ldots, n$. The next result generalizes Proposition 3.5.

Proposition 5.3 *Assume that:*
(i) *the Brownian motion W remains a Brownian motion with respect to $\mathbb{G}$,*
(ii) *for each $i = 1, \ldots, n$ the $\mathbb{F}$-martingale hazard process Λ^i is continuous. Consider a $\tilde{\mathbb{F}}$-martingale $M_t = E(X \,|\, \tilde{\mathcal{F}}_t)$, $t \in [0, T]$, where X is a $\tilde{\mathcal{F}}_T$-measurable random variable, integrable with respect to* **P**. *Then M admits the following integral representation*

$$M_t = M_0 + \int_0^t \xi_u \, dW_u + \sum_{i=k+1}^{n} \int_{]0,t]} \zeta^i_u \, d\tilde{M}^i_u, \tag{5.23}$$

where ξ and ζ^i, $i = k+1, \ldots, n$ are $\tilde{\mathbb{F}}$-predictable processes.

Proof. The proof is similar to the proof of Proposition 3.5. We start by noticing that it is enough to consider a random variable X of the form $X = Y \prod_{j=1}^{r} (1 - H^{i_j}_{s_j})$ for some $r \le n - k$, where

$0 < s_1 < \cdots < s_r \leq T$, and $k+1 \leq i_1 < \cdots < i_r \leq n$. Finally, Y is assumed to be a $\mathcal{F}_T$-measurable integrable random variable. We introduce the $\mathbb{F}$-martingale

$$\tilde{Y}_t = \mathbf{E}\Big(Y \exp\Big(\sum_{i=1}^{r} \Lambda^i_{s_i}\Big)\,\Big|\,\mathcal{F}_t\Big).$$

Since $\mathbb{F}$ is generated by a Brownian motion W, invoking the martingale representation property of the Brownian motion, we conclude that $\tilde{Y}$ follows a continuous process that admits the integral representation

$$\tilde{Y}_t = \tilde{Y}_0 + \int_0^t \tilde{\xi}_u\, dW_u, \quad \forall\, t \in [0,T],$$

for some $\mathbb{F}$-predictable process $\tilde{\xi}$. Furthermore, W remains a martingale with respect to $\mathbb{G}$ and thus $\tilde{Y}$ is also a $\mathbb{G}$-martingale.[12] Therefore, $\tilde{Y}$ is orthogonal to each $\mathbb{G}$-martingale of finite variation $\tilde{M}^i$. Using Itô's formula and (5.18), we obtain

$$Y\prod_{j=1}^{r}(1-H^{i_j}_{s_j}) = \tilde{Y}_T \prod_{j=1}^{r} \tilde{L}^{i_j}_{s_j} = \tilde{Y}_0 + \int_0^T \prod_{j=1}^{r} \tilde{L}^{i_j}_{(u\wedge s_j)-}\, d\tilde{Y}_u + \sum_{l=1}^{r} \int_{]0,s_l]} \tilde{Y}_{u-} \prod_{j=1}^{r} \tilde{L}^{i_j}_{(u\wedge s_j)-}\, d\tilde{M}^{i_l}_u.$$

The last formula leads to (5.23). □

Remarks. If the random variable X is merely $\mathcal{G}_T$-measurable, we may still apply Proposition 5.3 to the $\tilde{\mathbb{F}}$-martingale $M_t = E(X \,|\, \tilde{\mathcal{F}}_t)$ since clearly $M_t = E(\tilde{X} \,|\, \tilde{\mathcal{F}}_t)$, where $\tilde{X} := E(X \,|\, \tilde{\mathcal{F}}_T)$ is a $\tilde{\mathcal{F}}_T$-measurable random variable. This shows that representation (5.23) holds for any $\tilde{\mathbb{F}}$-martingale.

It is also interesting to observe that we may in fact substitute in Proposition 5.2 the Brownian filtration $\mathbb{F}$ with the filtration $\tilde{\mathbb{F}} := \mathbb{D}^{k+1} \vee \ldots \vee \mathbb{D}^n \vee \mathbb{F}$. First, it is clear that $\tilde{\Lambda} = \sum_{i=1}^k \Lambda^i$ is also the $(\tilde{\mathbb{F}}, \mathbb{G})$-martingale hazard process of $\tilde{\tau}$. Second, Proposition 5.3 shows that the process

$$\hat{Y}_t := \mathbf{E}\,(Ye^{-\tilde{\Lambda}_s} \,|\, \tilde{\mathcal{F}}_t), \quad \forall\, t \in [0,T],$$

where Y is a $\tilde{\mathcal{F}}_T$-measurable random variable, admits the following integral representation

$$\hat{Y}_t = \hat{Y}_0 + \int_0^t \xi_u\, dW_u + \sum_{i=k+1}^{n} \int_{]0,t]} \zeta^i_u\, d\tilde{M}^i_u, \tag{5.24}$$

where ξ and ζ^i, $i = k+1, \ldots, n$ are $\tilde{\mathbb{F}}$-predictable processes. Therefore, $\hat{Y}$ is a $\mathbb{G}$-martingale orthogonal to the $\mathbb{G}$-martingale U given by (5.22). Arguing in a much the same way as in the proof of Proposition 5.2 we thus obtain the following result.

Corollary 5.1 *Let Y be a bounded $\tilde{\mathcal{F}}_T$-measurable random variable. Let $\tilde{\tau} = \tau_1 \wedge \ldots \wedge \tau_k$. Then for any $t \leq s \leq T$ we have*

$$\mathbf{E}\,(\mathbb{1}_{\{\tilde{\tau}>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tilde{\tau}>t\}}\, \mathbf{E}\,\big(Ye^{\tilde{\Lambda}_t - \tilde{\Lambda}_s} \,|\, \tilde{\mathcal{F}}_t\big). \tag{5.25}$$

In particular, we have

$$\mathbf{P}(\tilde{\tau} > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tilde{\tau}>t\}} \mathbf{E}\,(e^{\tilde{\Lambda}_t - \tilde{\Lambda}_s} \,|\, \tilde{\mathcal{F}}_t). \tag{5.26}$$

5.3 Change of a Probability Measure

In this section, in which we follow Kusuoka [19], we shall extend the results of Section 3.3 to the case of several random times. We preserve the assumptions of Section 5.2.3, in particular, the filtration $\mathbb{F}$ is generated by a Brownian motion W which is also a $\mathbb{G}$-martingale (the case of a trivial filtration $\mathbb{F}$ is also covered by the results of this section though).

[12] Since $\tilde{Y}$ is manifestly $\tilde{\mathbb{F}}$-adapted, it follows also a martingale with respect to $\tilde{\mathbb{F}}$.

For a fixed $T > 0$, we shall examine the properties of $\tilde{\tau}$ under a probability measure $\mathbf{P}^*$, which is equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{G}_T)$. To this end, we introduce the associated $\mathbb{G}$-martingale η by setting, for $t \in [0,T]$,

$$\eta_t := \frac{d\mathbf{P}^*}{d\mathbf{P}}_{\,|\mathcal{G}_t} = \mathbf{E}_{\mathbf{P}}(X \,|\, \mathcal{G}_t), \quad \mathbf{P}\text{-a.s.}, \tag{5.27}$$

where X is a $\mathcal{G}_T$-measurable random variable, integrable with respect to $\mathbf{P}$, and such that $\mathbf{P}(X > 0) = 1$. By virtue of Proposition 5.3 (with $k = 0$), the Radon-Nikodým density process η admits the following representation

$$\eta_t = 1 + \int_0^t \xi_u \, dW_u + \sum_{i=1}^n \int_{]0,t]} \zeta_u^i \, d\tilde{M}_u^i, \tag{5.28}$$

where ξ and ζ^i, $i = 1, \dots, n$ are $\mathbb{G}$-predictable stochastic processes. It can be shown that η is a strictly positive process, so that we may rewrite (5.28) as follows

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} \big(\beta_u \, dW_u + \sum_{i=1}^n \kappa_u^i \, d\tilde{M}_u^i \big), \tag{5.29}$$

where β and κ^i, $i = 1, \dots, n$ are $\mathbb{G}$-predictable processes, with $\kappa^i > -1$. The following result extends Proposition 3.6 (its proof goes along exactly the same lines as the proof of Proposition 3.6 and thus it is left to the reader).

Proposition 5.4 *Let $\mathbf{P}^*$ be a probability measure on $(\Omega, \mathcal{G}_T)$ equivalent to $\mathbf{P}$. If the Radon-Nikodým density of $\mathbf{P}^*$ with respect to $\mathbf{P}$ is given by (5.29) then the process*

$$W_t^* = W_t - \int_0^t \beta_u \, du, \quad \forall \, t \in [0,T], \tag{5.30}$$

follows a $\mathbb{G}$-Brownian motion under $\mathbf{P}^*$, and for each $i = 1, \dots, n$ the process

$$M_t^{i*} := \tilde{M}_t^i - \int_{]0,t\wedge\tau_i]} \kappa_u^i \, d\Lambda_u^i = D_t^i - \int_{]0,t\wedge\tau_i]} (1 + \kappa_u^i) \, d\Lambda_u^i, \quad \forall \, t \in [0,T], \tag{5.31}$$

is a $\mathbb{G}$-martingale orthogonal to W^* under $\mathbf{P}^*$. Moreover, processes M^{i*} and M^{j*} follow mutually orthogonal $\mathbb{G}$-martingales under $\mathbf{P}^*$ for any $i \neq j$.

Though the process M^{i*} follows a $\mathbb{G}$-martingale under $\mathbf{P}^*$, it should be stressed that the process $\int_{]0,t]} (1 + \kappa_u^i) \, d\Lambda_u^i$ is not necessarily the $(\mathbb{F}, \mathbb{G})$-martingale hazard process of τ_i under $\mathbf{P}^*$, since it is not $\mathbb{F}$-adapted but merely $\mathbb{G}$-adapted, in general. This lack of measurability can be partially improved, however. For instance, for any fixed i we can choose a suitable version of the process κ^i, namely, a process κ^{i*} that coincides with κ^i on a random interval $[0, \tau_i]$, and such that κ^{i*} is a predictable process with respect to the enlarged filtration $\mathbb{F}^{i*} := \mathbb{D}^1 \vee \ldots \vee \mathbb{D}^{i-1} \vee \mathbb{D}^{i+1} \vee \ldots \mathbb{D}^n \vee \mathbb{F}$. Since

$$D_t^i - \int_{]0,t\wedge\tau_i]} (1 + \kappa_u^{i*}) \, d\Lambda_u^i = D_t^i - \int_{]0,t\wedge\tau_i]} (1 + \kappa_u^i) \, d\Lambda_u^i,$$

we conclude that for each fixed i the process

$$\Lambda_t^{i*} = \int_{]0,t]} (1 + \kappa_u^{i*}) \, d\Lambda_u^i$$

represents the $(\mathbb{F}^{i*}, \mathbb{G})$-martingale hazard process of τ_i under $\mathbf{P}^*$. This does not mean, however, that the equality

$$\mathbf{P}^*(\tau_i > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau_i > t\}} \mathbf{E}_{\mathbf{P}^*}(e^{\Lambda_t^{i*} - \Lambda_s^{i*}} \,|\, \mathcal{F}_t^{i*})$$

is valid for every $s \le t \le T$. We prefer to examine the last question in a slightly more general setting. For a fixed $k \le n$ let us consider the random time[13] $\tilde\tau = \tau_1 \wedge \ldots \wedge \tau_k$. As in Section 5.2.3, we shall write $\tilde{\mathbb{F}} = \mathbb{D}^{k+1} \vee \ldots \vee \mathbb{D}^n \vee \mathbb{F}$. For any $i = 1, \ldots, n$ we denote by $\tilde\kappa^i$ ($\tilde\beta$, resp.) the $\tilde{\mathbb{F}}$-predictable process such that $\tilde\kappa^i = \kappa^i$ ($\tilde\beta = \beta$, resp.) on the random set $[0, \tilde\tau]$.

Lemma 5.3 *The* $(\tilde{\mathbb{F}}, \mathbb{G})$*-martingale hazard process of the random time* $\tilde\tau$ *under* $\mathbf{P}^*$ *is given by the formula*

$$\Lambda^*_t = \sum_{i=1}^k \int_{]0,t]} (1 + \tilde\kappa^i_u)\, d\Lambda^i_u. \tag{5.32}$$

Proof. Let us set

$$\tilde W^*_t = W_t - \int_0^t \tilde\beta_u\, du,$$

and

$$\tilde M^{i*}_t = D^i_t - \int_{]0,t\wedge\tau_i]} (1 + \tilde\kappa^i_u)\, d\Lambda^i_u$$

for $i = 1, \ldots, n$. The processes $\tilde W^*$ and $\tilde M^{i*}$ follow $\mathbb{G}$-martingales under $\mathbf{P}^*$, provided that they are stopped at $\tilde\tau$ (since $\tilde W^*_{t\wedge\tilde\tau} = W^*_{t\wedge\tilde\tau}$ and $\tilde M^{i*}_{t\wedge\tilde\tau} = M^{i*}_{t\wedge\tilde\tau}$). Consequently, the process

$$\tilde D_t - \sum_{i=1}^k \int_{]0,t\wedge\tilde\tau]} (1 + \tilde\kappa^i_u)\, d\Lambda^i_u = \sum_{i=1}^k (\tilde M^{i*}_t)^{\tilde\tau}$$

is a $\mathbb{G}$-martingale. □

In view of Corollary 5.1 and Lemma 5.3, it would be natural to conjecture that for any $t \le s \le T$ we have

$$\mathbf{E}_{\mathbf{P}^*}(\mathbb{1}_{\{\tilde\tau > s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tilde\tau > t\}}\, \mathbf{E}_{\mathbf{P}^*}\big(Y e^{\Lambda^*_t - \Lambda^*_s} \,|\, \tilde{\mathcal{F}}_t\big), \tag{5.33}$$

where Y is a bounded $\tilde{\mathcal{F}}_T$-measurable random variable. It appears, however, that that the last formula is not valid, in general, unless we substitute the probability measure $\mathbf{P}^*$ in the right-hand side of (5.33) with some related probability measure. To this end, we introduce the following auxiliary density processes $\hat\eta^\ell$ for $\ell = 1, 2, 3$

$$\hat\eta^1_t = 1 + \int_{]0,t]} \hat\eta^1_{u-}\big(\tilde\beta_u\, dW_u + \sum_{i=k+1}^n \tilde\kappa^i_u\, d\tilde M^i_u\big), \tag{5.34}$$

$$\hat\eta^2_t = 1 + \int_{]0,t]} \hat\eta^2_{u-}\big(\tilde\beta_u\, dW_u + \sum_{i=1}^n \tilde\kappa^i_u\, d\tilde M^i_u\big), \tag{5.35}$$

and

$$\hat\eta^3_t = 1 + \int_{]0,t]} \hat\eta^3_{u-}\big(\tilde\beta_u\, dW_u + \sum_{i=1}^k \kappa^i_u\, d\tilde M^i_u + \sum_{i=k+1}^n \tilde\kappa^i_u\, d\tilde M^i_u\big). \tag{5.36}$$

It is useful to observe that the process $\hat\eta^1$ is $\tilde{\mathbb{F}}$-adapted (since, in particular, each process $\tilde M^i$ is adapted to the filtration $\mathbb{D}^i \vee \mathbb{F}$). On the other hand, processes $\hat\eta^2$ and $\hat\eta^3$ are merely $\mathbb{G}$-adapted, but not necessarily $\tilde{\mathbb{F}}$-adapted, in general. We find in convenient to introduce the $\tilde{\mathbb{F}}$-adapted modifications $\tilde\eta^2$, $\tilde\eta^3$ of $\hat\eta^2$, $\hat\eta^3$ by setting $\tilde\eta^\ell_t = \mathbf{E}\,(\hat\eta^\ell_T \,|\, \tilde{\mathcal{F}}_t)$ for $t \le T$, $\ell = 2, 3$. From the uniqueness of the martingale representation property established in Proposition 5.3 we deduce that for $\ell = 2, 3$ we have (for $\ell = 1$, (5.37) simply coincides with (5.34))

$$\tilde\eta^\ell_t = 1 + \int_{]0,t]} \tilde\eta^\ell_{u-}\big(\tilde\beta_u\, dW_u + \sum_{i=k+1}^n \tilde\kappa^i_u\, d\tilde M^i_u\big). \tag{5.37}$$

[13] Since the order of random times is not essential here, the analysis below covers also the case of a single random time τ_i for any $i = 1, \ldots, n$.

We define a probability measure $\tilde{\mathbf{P}}_\ell$ on $(\Omega, \mathcal{G}_T)$ by setting, for $\ell = 1, 2, 3$

$$\hat{\eta}^\ell_t := \frac{d\tilde{\mathbf{P}}_\ell}{d\mathbf{P}}_{\,|\mathcal{G}_t}, \quad \mathbf{P}\text{-a.s.}, \tag{5.38}$$

for $t \in [0,T]$. It is thus clear that

$$\tilde{\eta}^\ell_t = \mathbf{E}\,(\hat{\eta}^\ell_T \,|\, \tilde{\mathcal{F}}_t) = \frac{d\tilde{\mathbf{P}}_\ell}{d\mathbf{P}}_{\,|\tilde{\mathcal{F}}_t}, \quad \mathbf{P}\text{-a.s.} \tag{5.39}$$

The following result, due to Kusuoka [19], is a counterpart of Corollary 5.1.

Proposition 5.5 *Let $\ell \in \{1,2,3\}$ and let Y be a bounded $\tilde{\mathcal{F}}_T$-measurable random variable. Then for any $t \le s \le T$ we have*

$$\mathbf{E}_{\mathbf{P}^*}(\mathbb{1}_{\{\tilde{\tau}>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tilde{\tau}>t\}}\, \mathbf{E}_{\tilde{\mathbf{P}}_\ell}\big(Y e^{\Lambda^*_t - \Lambda^*_s} \,|\, \tilde{\mathcal{F}}_t\big), \tag{5.40}$$

where the process Λ^* is given by formula (5.32). *In particular, we have*

$$\mathbf{P}^*(\tilde{\tau} > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tilde{\tau}>t\}} \mathbf{E}_{\tilde{\mathbf{P}}_\ell}(e^{\Lambda^*_t - \Lambda^*_s} \,|\, \tilde{\mathcal{F}}_t). \tag{5.41}$$

The proof of Proposition 5.5 parallels the demonstration of Proposition 5.2 (see also remarks preceding Corollary 5.1). We need some preliminary results, however. First, we shall establish a counterpart of the integral representation (5.23).

Lemma 5.4 *Let Y be a $\tilde{\mathbb{F}}$-martingale under $\tilde{\mathbf{P}}_\ell$ for some $\ell \in \{1,2,3\}$. Then there exist $\tilde{\mathbb{F}}$-predictable processes $\tilde{\xi}$ and $\tilde{\zeta}^i$, $i = k+1, \ldots, n$, such that*

$$Y_t = Y_0 + \int_0^t \tilde{\xi}_u \, d\tilde{W}^*_u + \sum_{i=k+1}^n \int_{]0,t]} \tilde{\zeta}^i_u \, d\tilde{M}^{i*}_u. \tag{5.42}$$

Proof. The proof combines the calculations already employed in the proof of Corollary 3.9 with the martingale representation property under $\mathbf{P}$ established in Proposition 5.3. We fix ℓ, and we write $\tilde{\eta}_t = \mathbf{E}\,(\eta^\ell_T \,|\, \tilde{\mathcal{F}}_t)$ (of course, $\tilde{\eta}_t = \hat{\eta}^1_t$ if we take $\ell = 1$). We introduce an auxiliary process $\tilde{Y}$, which follows a $\tilde{\mathbb{F}}$-martingale under $\mathbf{P}$, by setting

$$\tilde{Y}_t = \int_{]0,t]} \tilde{\eta}^{-1}_{u-} \, d(\tilde{\eta}_u Y_u) - \int_{]0,t]} \tilde{\eta}^{-1}_{u-} Y_{u-} \, d\tilde{\eta}_u$$

for $t \in [0,T]$. Since Itô's formula yields

$$\tilde{\eta}^{-1}_{u-} \, d(\tilde{\eta}_u Y_u) = dY_u + \tilde{\eta}^{-1}_{u-} Y_{u-} \, d\tilde{\eta}_u + \tilde{\eta}^{-1}_{u-} \, d[Y, \tilde{\eta}]_u,$$

the process Y admits the following useful representation

$$Y_t = Y_0 + \tilde{Y}_t - \int_{]0,t]} \tilde{\eta}^{-1}_{u-} \, d[Y, \tilde{\eta}]_u. \tag{5.43}$$

On the other hand, invoking Proposition 5.3, we obtain the following integral representation for the process $\tilde{Y}$

$$\tilde{Y}_t = \int_0^t \xi_u \, dW_u + \sum_{i=k+1}^n \int_{]0,t]} \zeta^i_u \, d\tilde{M}^i_u,$$

where ξ and ζ^i, $i = 1, \ldots, k$ are $\tilde{\mathbb{F}}$-predictable processes. Consequently, we have

$$\begin{aligned} dY_t &= \xi_t\, dW_t + \sum_{i=k+1}^n \zeta_t^i\, d\tilde{M}_t^i - \tilde{\eta}_{t-}^{-1}\, d[Y, \tilde{\eta}]_t \\ &= \xi_t\, d\tilde{W}_t^* + \sum_{i=k+1}^n \zeta_t^i (1 + \tilde{\kappa}_t^i)^{-1}\, d\tilde{M}_t^{i*}. \end{aligned}$$

To establish the last equality, notice that (5.37) combined with (5.31) yield

$$\tilde{\eta}_{t-}^{-1}\, d[Y, \tilde{\eta}]_t = \xi_t \tilde{\beta}_t\, dt + \sum_{i=k+1}^n \zeta_t^i \tilde{\kappa}_t^i (1 + \tilde{\kappa}_t^i)^{-1}\, dD_t^i,$$

where the last equality follows in turn from the following relationship

$$\Delta[Y, \tilde{\eta}]_t = \tilde{\eta}_{t-} \sum_{i=k+1}^n \big(\zeta_t^i \tilde{\kappa}_t^i\, dD_t^i - \tilde{\kappa}_t^i \Delta[Y, \tilde{\eta}]_t \big).$$

We conclude that Y satisfies (3.41) with $\tilde{\xi} = \xi$ and $\tilde{\zeta}^i = \zeta^i (1 + \tilde{\kappa}^i)^{-1}$ for $i = k+1, \ldots, n$. □

Corollary 5.2 *Let Y be a bounded $\tilde{\mathcal{F}}_T$-measurable random variable. For a fixed $s \le T$ we define the process $\hat{Y}$ by setting*

$$\hat{Y}_t = \mathbf{E}_{\tilde{\mathbf{P}}_\ell} (Y e^{-\Lambda_s^*} \,|\, \tilde{\mathcal{F}}_t), \quad \forall\, t \in [0, T]. \tag{5.44}$$

The process $\hat{Y}$ admits the following integral representation under $\tilde{\mathbf{P}}_\ell$

$$\hat{Y}_t = \hat{Y}_0 + \int_0^t \hat{\xi}_u\, d\tilde{W}_u^* + \sum_{i=k+1}^n \int_{]0,t]} \hat{\zeta}_u^i\, d\tilde{M}_u^{i*}, \tag{5.45}$$

where $\hat{\xi}$ and $\hat{\zeta}^i$, $i = k+1, \ldots, n$ are $\tilde{\mathbb{F}}$-predictable processes. The stopped process $\hat{Y}_{t \wedge \tilde{\tau}}$ follows a $\mathbb{G}$-martingale orthogonal under $\mathbf{P}^*$ to $\mathbb{G}$-martingales M^{i*}, $i = 1, \ldots, k$.

Proof. It is enough to apply Lemma 5.4 and to notice that the stopped process $\hat{Y}_{t \wedge \tilde{\tau}}$ satisfies

$$\hat{Y}_{t \wedge \tilde{\tau}} = \hat{Y}_0 + \int_0^t \hat{\xi}_u\, dW_{u \wedge \tilde{\tau}}^* + \sum_{i=k+1}^n \int_{]0,t]} \hat{\zeta}_u^i\, dM_{u \wedge \tilde{\tau}}^{i*},$$

and to recall that $\tilde{W}_{t \wedge \tilde{\tau}}^* = W_{t \wedge \tilde{\tau}}^*$ and $\tilde{M}_{t \wedge \tilde{\tau}}^{i*} = M_{t \wedge \tilde{\tau}}^{i*}$. □

We are in the position to establish Proposition 5.5.

Proof of Proposition 5.5. For a fixed $s \le T$, let $\hat{Y}$ be the process defined through formula (5.44). Furthermore, let U be the process given by the expression (notice that the process U is stopped at $\tilde{\tau} \wedge s$)

$$\begin{aligned} U_t &= (1 - \tilde{H}_{t \wedge s}) e^{\Lambda_{t \wedge s}^*} = (1 - \tilde{H}_{t \wedge s}) \prod_{i=1}^k e^{\int_0^{t \wedge s} (1 + \tilde{\kappa}_u^i) d\Lambda_u^i} \\ &= \prod_{i=1}^k (1 - D_{t \wedge s}^i) e^{\int_0^t (1 + \kappa_u^i) d\Lambda_u^i} = \prod_{i=1}^k L_{t \wedge s}^{i*}, \end{aligned}$$

where

$$L_t^{i*} = (1 - D_t^i) e^{\int_0^t (1 + \kappa_u^i) d\Lambda_u^i}$$

so that (cf. (5.18))

$$L_t^{i*} = 1 - \int_{]0,t]} L_{u-}^{i*} \, dM_u^{i*}. \tag{5.46}$$

In view of (5.45), the above representation of the process U and (5.46), the processes U and $\hat{Y}_{t\wedge\tilde{\tau}}$ are mutually orthogonal $\mathbb{G}$-martingales under $\mathbf{P}^*$. Therefore,

$$\mathbf{E}_{\mathbf{P}^*}(1\!\!1_{\{\tilde{\tau}>s\}} Y \,|\, \mathcal{G}_t) = \mathbf{E}_{\mathbf{P}^*}(U_T \hat{Y}_T \,|\, \mathcal{G}_t) = U_t \hat{Y}_t = (1-\tilde{H}_t) e^{\Lambda_t^*} \mathbf{E}_{\tilde{\mathbf{P}}_\ell}(Y e^{-\Lambda_s^*} \,|\, \tilde{\mathcal{F}}_t).$$

The last expression yields the asserted formulae (5.40)-(5.41). □

Remarks. It is also possible to consider the filtration generated by the random time $\tilde{\tau}$, rather then the filtration $\mathbb{D}^1 \vee \ldots \vee \mathbb{D}^k$. Consequently, instead of the filtration $\mathbb{G}$ we would have $\hat{\mathbb{G}} = \tilde{\mathbb{D}} \vee \tilde{\mathbb{F}}$. Since the stopped process $\tilde{H}_t - \Lambda^*_{t\wedge\tilde{\tau}}$ (as usual, $\tilde{H}_t = 1\!\!1_{\{t\le\tilde{\tau}\}}$) is a $\mathbb{G}$-martingale, and it is manifestly a $\tilde{\mathbb{G}}$-adapted process, it follows also a $\tilde{\mathbb{G}}$-martingale. Let us consider the following property: any $\tilde{\mathbb{F}}$-martingale remains a $\mathbb{G}$-martingale (or a $\hat{\mathbb{G}}$-martingale). It seems plausible to conjecture that this property is not valid under $\mathbf{P}^*$, in general, but it holds under $\tilde{\mathbf{P}}_\ell$ for $\ell = 1,2,3$.

5.4 Kusuoka's Example

The following example, borrowed from Kusuoka [19], shows that formula (5.33) does not hold, in general. We assume that under the original probability measure $\mathbf{P}$ the random times τ_i, $i=1,2$ are mutually independent random variables, with exponential laws with the parameters λ_1 and λ_2, respectively. The joint probability law of the pair (τ_1,τ_2) under $\mathbf{P}$ admits the density function $f(x,y) = \lambda_1\lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}$ for $(x,y) \in \mathbb{R}^2_+$. Our goal is to examine these random times under a specific equivalent change of probability measure $\mathbf{P}^*$. In words, under $\mathbf{P}^*$ the original intensity λ_1 of the random time τ_1 jumps to some fixed value α_1 after the occurrence of τ_2 (the behaviour of the intensity of τ_2 is analogous). Such a specification of the stochastic intensity of dependent random times appears in a natural way in certain practical applications related to the valuation of defaultable claims. Notice that the filtration $\mathbb{F}$ is here assumed to be a trivial filtration.

We shall now formally define the probability measure $\mathbf{P}^*$. Let α_1 and α_2 be positive real numbers. For a fixed $T>0$, we introduce an equivalent probability measure $\mathbf{P}^*$ on $(\Omega,\mathcal{G})$ by setting

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \eta_T, \quad \mathbf{P}\text{-a.s.}, \tag{5.47}$$

where η_t, $t\in[0,T]$, satisfies

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{]0,t]} \eta_{u-} \kappa_u^i \, d\tilde{M}_u^i, \tag{5.48}$$

where

$$\kappa_t^1 = 1\!\!1_{\{\tau_2<t\}}\Big(\frac{\alpha_1}{\lambda_1} - 1\Big), \quad \kappa_t^2 = 1\!\!1_{\{\tau_1<t\}}\Big(\frac{\alpha_2}{\lambda_2} - 1\Big).$$

It is useful to notice that $\eta_T = \eta_T^1 \eta_T^2$, where for every $t\in[0,T]$

$$\eta_t^i = 1 + \int_{]0,t]} \eta_{u-}^i \kappa_u^i \, d\tilde{M}_u^i, \tag{5.49}$$

for $i=1,2$, or more explicitly, for every $t\in[0,T]$

$$\eta_t^1 = 1\!\!1_{\{\tau_1\le\tau_2\}} + 1\!\!1_{\{\tau_2\le t<\tau_1\}} e^{-(\alpha_1-\lambda_1)(t-\tau_2)} + 1\!\!1_{\{\tau_2<\tau_1\le t\}} \frac{\alpha_1}{\lambda_1} e^{-(\alpha_1-\lambda_1)(\tau_1-\tau_2)}, \tag{5.50}$$

$$\eta_t^2 = 1\!\!1_{\{\tau_2\le\tau_1\}} + 1\!\!1_{\{\tau_1\le t<\tau_2\}} e^{-(\alpha_2-\lambda_2)(t-\tau_1)} + 1\!\!1_{\{\tau_1<\tau_2\le t\}} \frac{\alpha_2}{\lambda_2} e^{-(\alpha_2-\lambda_2)(\tau_2-\tau_1)}. \tag{5.51}$$

It is obvious that the process κ^1 (κ^2, resp.) is $\mathbb{D}^2$-predictable ($\mathbb{D}^1$-predictable, resp.) Then $\Lambda_t^{i*} = \int_0^t \lambda_u^{i*}\, du$, where the processes λ^{i*}, $i = 1, 2$ are given by the formulae

$$\lambda_t^{*1} = \lambda_1(1 - D_t^2) + \alpha_1 D_t^2 = \lambda_1 \mathbb{1}_{\{\tau_2 > t\}} + \alpha_1 \mathbb{1}_{\{\tau_2 \le t\}},$$

and

$$\lambda_t^{*2} = \lambda_2(1 - D_t^1) + \alpha_2 D_t^1 = \lambda_2 \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \le t\}}.$$

This means that the processes

$$D_t^1 - \int_0^{t\wedge\tau_1} \big(\lambda_1 \mathbb{1}_{\{\tau_2 > u\}} + \alpha_1 \mathbb{1}_{\{\tau_2 \le u\}}\big)\, du = D_t^1 - \lambda_1(t \wedge \tau_1 \wedge \tau_2) - \alpha_1((t\wedge\tau_1) \vee \tau_2 - \tau_2)$$

and

$$D_t^2 - \int_0^{t\wedge\tau_2} \big(\lambda_2 \mathbb{1}_{\{\tau_1 > u\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \le u\}}\big)\, du = D_t^2 - \lambda_2(t \wedge \tau_1 \wedge \tau_2) - \alpha_2((t\wedge\tau_2) \vee \tau_1 - \tau_1)$$

are $\mathbf{P}^*$-martingales with respect to the joint filtration $\mathbb{G} = \mathbb{D}^1 \vee \mathbb{D}^2$.

In view of the assumed symmetry, it is enough to consider the random time $\tilde\tau = \tau_1$ (i.e., we have $n = 2$ and $k = 1$). Notice that in the present setup we have $\tilde\kappa_t^2 = 0$ since obviously $\kappa_t^2 = 0$ on the random interval $[0, \tau_1]$. Therefore, the probability measure $\tilde{\mathbf{P}}_1$ given by formulae (5.34)-(5.38) coincides with the original probability measure $\mathbf{P}$. It is useful to notice that κ^1 is $\mathbb{D}^1$-predictable so that $\tilde\kappa_t^1 = \kappa_t^1$ for every t. Consequently, the probability measures $\tilde{\mathbf{P}}_2$ and $\tilde{\mathbf{P}}_3$ coincide with the probability measure $\mathbf{P}_1^*$ given by formulae (5.49) and (5.57) below.

Our aim is to evaluate directly the conditional expectation $\mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}_t^1 \vee \mathcal{D}_t^2)$ for $t \le s \le T$, and subsequently to verify that

$$\mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}_t^1 \vee \mathcal{D}_t^2) = \mathbb{1}_{\{\tau_1 > t\}} \mathbf{E}_{\tilde{\mathbf{P}}_1}(e^{\Lambda_t^{i*} - \Lambda_s^{i*}} \,|\, \mathcal{D}_t^2) = \mathbb{1}_{\{\tau_1 > t\}} \mathbf{E}_{\mathbf{P}}(e^{\Lambda_t^{i*} - \Lambda_s^{i*}} \,|\, \mathcal{D}_t^2),$$

where the second equality is an obvious consequence of the equality $\tilde{\mathbf{P}}_1 = \mathbf{P}$. Finally, we shall check that, in general,

$$\mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}_t^1 \vee \mathcal{D}_t^2) \ne \mathbb{1}_{\{\tau_1 > t\}} \mathbf{E}_{\mathbf{P}^*}(e^{\Lambda_t^{i*} - \Lambda_s^{i*}} \,|\, \mathcal{D}_t^2).$$

5.4.1 Unconditional Law of τ_1 under $\mathbf{P}^*$

We find it convenient to derive first the unconditional law of τ_1 under $\mathbf{P}^*$. In view of (5.50)-(5.51), the marginal density $f_{\tau_1}^*$ of the random time τ_1 under $\mathbf{P}^*$ equals[14]

$$\begin{aligned}
f_{\tau_1}^*(t) &= \int_0^t \lambda_1\lambda_2 \frac{\alpha_1}{\lambda_1} e^{-(\alpha_1-\lambda_1)(t-y)} e^{-(\lambda_1 t + \lambda_2 y)}\, dy \\
&\quad + \int_t^T \lambda_1\lambda_2 \frac{\alpha_2}{\lambda_2} e^{-(\alpha_2-\lambda_2)(y-t)} e^{-(\lambda_1 t + \lambda_2 y)}\, dy \\
&\quad + \int_t^\infty \lambda_1\lambda_2 e^{-(\alpha_2-\lambda_2)(T-t)} e^{-(\lambda_1 t + \lambda_2 y)}\, dy \\
&= \frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\alpha_1\lambda_2 e^{-\alpha_1 t} + (\lambda_1-\alpha_1)(\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)t}\Big)
\end{aligned}$$

for every $t \le T$. For $t > T$, we have

$$\begin{aligned}
f_{\tau_1}^*(t) &= \int_0^T \lambda_1\lambda_2 e^{-(\alpha_1-\lambda_1)(T-y)} e^{-(\lambda_1 t + \lambda_2 y)}\, dy + \int_T^\infty \lambda_1\lambda_2 e^{-(\lambda_1 t + \lambda_2 y)}\, dy \\
&= \frac{\lambda_1 e^{-\lambda_1 t}}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-(\alpha_1-\lambda_1)T} + (\lambda_1-\alpha_1) e^{-\lambda_2 T}\Big).
\end{aligned}$$

[14]For simplicity of exposition, we shall assume that $\lambda_1 + \lambda_2 - \alpha_1 \ne 0$ and $\lambda_1 + \lambda_2 - \alpha_2 \ne 0$.

Consequently, for any $s \in [0,T]$, we get

$$\mathbf{P}^*(\tau_1 > s) = \frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-\alpha_1 s} + (\lambda_1-\alpha_1)e^{-(\lambda_1+\lambda_2)s}\Big). \tag{5.52}$$

For $s > T$, we have

$$\mathbf{P}^*(\tau_1 > s) = \frac{e^{-\lambda_1 s}}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-(\alpha_1-\lambda_1)T} + (\lambda_1-\alpha_1)e^{-\lambda_2 T}\Big). \tag{5.53}$$

5.4.2 Conditional Law of τ_1 under $\mathbf{P}^*$

Our next goal is to derive an explicit formula for the conditional probability $I := \mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}^1_t \vee \mathcal{D}^2_t)$.

Lemma 5.5 *For every $t \le s \le T$ we have*

$$I = 1\!\!1_{\{\tau_1>t,\tau_2>t\}}\,\frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-\alpha_1(s-t)} + (\lambda_1-\alpha_1)e^{-(\lambda_1+\lambda_2)(s-t)}\Big) + 1\!\!1_{\{\tau_2\le t<\tau_1\}}\, e^{-\alpha_1(s-t)}.$$

Proof. In view of results of Section 3.1.1, for arbitrary $t \le s$ we have

$$I = \mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}^1_t \vee \mathcal{D}^2_t) = (1-D^1_t)\,\frac{\mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}^2_t)}{\mathbf{P}^*(\tau_1 > t \,|\, \mathcal{D}^2_t)}, \tag{5.54}$$

where in turn

$$\mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}^2_t) = (1-D^2_t)\,\frac{\mathbf{P}^*(\tau_1 > s, \tau_2 > t)}{\mathbf{P}^*(\tau_2 > t)} + D^2_t\,\mathbf{P}^*(\tau_1 > s \,|\, \tau_2). \tag{5.55}$$

Combining (5.54) with (5.55), we obtain

$$I = (1-D^1_t)(1-D^2_t)\,\frac{\mathbf{P}^*(\tau_1 > s, \tau_2 > t)}{\mathbf{P}^*(\tau_1 > t, \tau_2 > t)} + (1-D^1_t)D^2_t\,\frac{\mathbf{P}^*(\tau_1 > s \,|\, \tau_2)}{\mathbf{P}^*(\tau_1 > t \,|\, \tau_2)},$$

or more explicitly,

$$I = 1\!\!1_{\{\tau_1>t,\tau_2>t\}}\,\frac{\mathbf{P}^*(\tau_1 > s, \tau_2 > t)}{\mathbf{P}^*(\tau_1 > t, \tau_2 > t)} + 1\!\!1_{\{\tau_2\le t<\tau_1\}}\,\frac{\mathbf{P}^*(\tau_1 > s \,|\, \tau_2)}{\mathbf{P}^*(\tau_1 > t \,|\, \tau_2)} = 1\!\!1_{\{\tau_1>t,\tau_2>t\}} I_1 + 1\!\!1_{\{\tau_2\le t<\tau_1\}} I_2.$$

In order to evaluate I_1, observe first that

$$\begin{aligned}
\mathbf{P}^*(\tau_1 > s, \tau_2 \le t) &= \int_s^T\!\!\int_0^t \lambda_1\lambda_2\frac{\alpha_1}{\lambda_1} e^{-(\alpha_1-\lambda_1)(x-y)} e^{-(\lambda_1 x+\lambda_2 y)}\,dxdy \\
&\quad + \int_T^\infty\!\!\int_0^t \lambda_1\lambda_2 e^{-(\alpha_1-\lambda_1)(T-y)} e^{-(\lambda_1 x+\lambda_2 y)}\,dxdy \\
&= \frac{\lambda_2 e^{-\alpha_1 s}}{\lambda_1+\lambda_2-\alpha_1}\Big(1 - e^{-(\lambda_1+\lambda_2-\alpha_1)t}\Big).
\end{aligned}$$

Combining the last formula with (5.52), we obtain

$$\begin{aligned}
\mathbf{P}^*(\tau_1 > s, \tau_2 > t) &= \mathbf{P}^*(\tau_1 > s) - \mathbf{P}^*(\tau_1 > s, \tau_2 \le t) \\
&= \frac{\lambda_1-\alpha_1}{\lambda_1+\lambda_2-\alpha_1} e^{-(\lambda_1+\lambda_2)s} + \frac{\lambda_2}{\lambda_1+\lambda_2-\alpha_1} e^{-\alpha_1 s-(\lambda_1+\lambda_2-\alpha_1)t}.
\end{aligned}$$

We conclude that

$$I_1 = \frac{\mathbf{P}^*(\tau_1 > s, \tau_2 > t)}{\mathbf{P}^*(\tau_1 > t, \tau_2 > t)} = \frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-\alpha_1(s-t)} + (\lambda_1-\alpha_1)e^{-(\lambda_1+\lambda_2)(s-t)}\Big).$$

It remains to evaluate I_2. To this end, it is enough to check that for $t \le s \le T$ we have

$$I_3 = \mathbb{1}_{\{\tau_2 \le t\}} \mathbf{P}^*(\tau_1 > s \,|\, \tau_2) = \mathbb{1}_{\{\tau_2 \le t\}} \frac{(\lambda_1 + \lambda_2 - \alpha_2)\lambda_2 e^{-\alpha_1(s-\tau_2)}}{\lambda_1 \alpha_2 e^{(\lambda_1+\lambda_2-\alpha_2)\tau_2} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}. \tag{5.56}$$

Indeed, the last formula would thus yield immediately

$$\mathbb{1}_{\{\tau_2 \le t < \tau_1\}} \frac{\mathbf{P}^*(\tau_1 > s \,|\, \tau_2)}{\mathbf{P}^*(\tau_1 > t \,|\, \tau_2)} = \mathbb{1}_{\{\tau_2 \le t < \tau_1\}} e^{-\alpha_1(s-t)},$$

the desired result. To evaluate I_3, we may, for instance, notice that for any $u \le s$

$$\begin{aligned} \mathbf{P}^*(\tau_1 > s \,|\, \tau_2 = u) &= \frac{1}{f^*_{\tau_2}(u)} \Big(\int_s^T \frac{\alpha_1}{\lambda_1} e^{-(\alpha_1-\lambda_1)(x-u)} f(x,u)\, dx + \int_T^\infty e^{-(\alpha_1-\lambda_1)(T-u)} f(x,u)\, dx \Big) \\ &= \frac{(\lambda_1 + \lambda_2 - \alpha_2)\lambda_2 e^{-(\lambda_1+\lambda_2-\alpha_1)u} e^{-\alpha_1 s}}{\lambda_1 \alpha_2 e^{-\alpha_2 u} + (\lambda_2 - \alpha_2)(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)u}}, \end{aligned}$$

which gives (5.56) upon simplification. An alternative, though somewhat lengthy, way do to the calculations for I_3 would be to use directly the Bayes formula

$$\mathbb{1}_{\{\tau_2 \le t\}} \mathbf{P}^*(\tau_1 > s \,|\, \tau_2) = \mathbb{1}_{\{\tau_2 \le t\}} \frac{\mathbf{E}_{\mathbf{P}}(\eta_s \mathbb{1}_{\{\tau_1 > s\}} \,|\, \tau_2)}{\mathbf{E}_{\mathbf{P}}(\eta_s \,|\, \tau_2)}$$

and to check that for arbitrary $t \le s \le T$ we have

$$\mathbb{1}_{\{\tau_2 \le t\}} \mathbf{E}_{\mathbf{P}}(\eta_s \mathbb{1}_{\{\tau_1 > s\}} \,|\, \tau_2) = \mathbb{1}_{\{\tau_2 \le t\}} e^{-\alpha_1 s} e^{-(\lambda_1 - \alpha_1)\tau_2}$$

and

$$\mathbb{1}_{\{\tau_2 \le t\}} \mathbf{E}_{\mathbf{P}}(\eta_s \,|\, \tau_2) = \mathbb{1}_{\{\tau_2 \le t\}} \frac{f^*_{\tau_2}(\tau_2)}{f_{\tau_2}(\tau_2)},$$

where $f_{\tau_2}(u) = \lambda_2 e^{-\lambda_2 u}$. Details are left to the reader. □

Remarks. Observe that to find $\mathbf{P}^*(\tau_1 > s, \tau_2 > t)$ for $t \le s \le T$, it suffices in fact to notice that

$$J := \mathbf{P}^*(\tau_1 > s, \tau_2 > t) = \mathbf{E}_{\mathbf{P}}(\eta_T \mathbb{1}_{\{\tau_1 > s, \tau_2 > t\}}) = \mathbf{E}_{\mathbf{P}}(\eta_s \mathbb{1}_{\{\tau_1 > s, \tau_2 > t\}})$$

and

$$\eta_s \mathbb{1}_{\{\tau_1 > s, \tau_2 > t\}} = \eta^1_s \mathbb{1}_{\{\tau_1 > s, \tau_2 > t\}} = \mathbb{1}_{\{\tau_1 > s, \tau_2 > s\}} + \mathbb{1}_{\{t < \tau_2 < s < \tau_1\}} e^{-(\alpha_1 - \lambda_1)(s - \tau_2)}.$$

Therefore,

$$\begin{aligned} J &= \int_s^\infty \int_s^\infty \lambda_1 \lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}\, dxdy + \int_s^\infty \int_t^s \lambda_1 \lambda_2 e^{-(\alpha_1 - \lambda_1)(s-y)} e^{-(\lambda_1 x + \lambda_2 y)}\, dxdy \\ &= \frac{\lambda_1 - \alpha_1}{\lambda_1 + \lambda_2 - \alpha_1} e^{-(\lambda_1+\lambda_2)s} + \frac{\lambda_2}{\lambda_1 + \lambda_2 - \alpha_1} e^{-\alpha_1 s - (\lambda_1 + \lambda_2 - \alpha_1)t}. \end{aligned}$$

Let us introduce a probability measure $\mathbf{P}^*_1$ by setting

$$\frac{d\mathbf{P}^*_1}{d\mathbf{P}} = \eta^1_T, \quad \mathbf{P}\text{-a.s.} \tag{5.57}$$

It is interesting to observe that the marginal density $\tilde{f}_{\tau_1}$ of τ_1 under $\mathbf{P}^*_1$ coincides with $f^*_{\tau_1}$, since

$$\begin{aligned} \tilde{f}_{\tau_1}(t) &= \int_0^t \lambda_1 \lambda_2 \frac{\alpha_1}{\lambda_1} e^{-(\alpha_1 - \lambda_1)(t-y)} e^{-(\lambda_1 t + \lambda_2 y)}\, dxdy + \int_t^\infty \lambda_1 \lambda_2 e^{-(\lambda_1 t + \lambda_2 y)}\, dxdy \\ &= \frac{1}{\lambda_1 + \lambda_2 - \alpha_1} \Big(\alpha_1 \lambda_2 e^{-\alpha_1 t} + (\lambda_1 - \alpha_1)(\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2)t} \Big) \end{aligned}$$

for every $t \le T$. It is also obvious that $\tilde{f}_{\tau_1} = f^*_{\tau_1}$ for $t > T$. In fact, one may also deduce easily from the calculations in the proof of Lemma 5.5 and remarks above that

$$\begin{aligned} I &= \mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}^1_t \vee \mathcal{D}^2_t) = \mathbf{P}^*_1(\tau_1 > s \,|\, \mathcal{D}^1_t \vee \mathcal{D}^2_t) \\ &= 1\!\!1_{\{\tau_1 > t, \tau_2 > t\}} \frac{\mathbf{P}^*_1(\tau_1 > s, \tau_2 > t)}{\mathbf{P}^*_1(\tau_1 > t, \tau_2 > t)} + 1\!\!1_{\{\tau_2 \le t < \tau_1\}} \frac{\mathbf{P}^*_1(\tau_1 > s \,|\, \tau_2)}{\mathbf{P}^*_1(\tau_1 > t \,|\, \tau_2)}. \end{aligned}$$

5.4.3 Intensity of τ_1 under $\mathbf{P}^*$

We shall now focus on the intensity process of τ_1 under $\mathbf{P}^*$. We have

$$\Lambda^{1*}_t = \int_0^t \big(\lambda_1 1\!\!1_{\{\tau_2 > u\}} + \alpha_1 1\!\!1_{\{\tau_2 \le u\}}\big)\, du = \lambda_1(t \wedge \tau_2) + \alpha_1(t \vee \tau_2 - \tau_2). \tag{5.58}$$

The first equality in next result is merely a special case of Proposition 5.5. In particular, equality (5.59) shows that we may apply Λ^{1*} to evaluate the conditional probability. Inequality (5.60) makes it clear that the process Λ^{1*} does not coincide with the $\mathbb{D}^2$-hazard process of τ_1 under $\mathbf{P}^*$, however.

Lemma 5.6 *Let $I := \mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}^1_t \vee \mathcal{D}^2_t)$. For every $t \le s \le T$ we have*

$$I = 1\!\!1_{\{\tau_1 > t\}} \mathbf{E}_{\mathbf{P}}(e^{\Lambda^{1*}_t - \Lambda^{1*}_s} \,|\, \mathcal{D}^2_t) = 1\!\!1_{\{\tau_1 > t\}} \mathbf{E}_{\mathbf{P}^*_1}(e^{\Lambda^{1*}_t - \Lambda^{1*}_s} \,|\, \mathcal{D}^2_t) = 1\!\!1_{\{\tau_1 > t\}} \mathbf{E}_{\tilde{\mathbf{P}}_1}(e^{\Lambda^{1*}_t - \Lambda^{1*}_s} \,|\, \mathcal{D}^2_t) \tag{5.59}$$

and

$$\mathbf{P}^*(\tau_1 > s \,|\, \mathcal{D}^1_t \vee \mathcal{D}^2_t) \ne 1\!\!1_{\{\tau_1 > t\}} \mathbf{E}_{\mathbf{P}^*}(e^{\Lambda^{1*}_t - \Lambda^{1*}_s} \,|\, \mathcal{D}^2_t). \tag{5.60}$$

Proof. Let us check that $\tilde{I} = I$, where I is given by Lemma 5.5, and

$$\tilde{I} = 1\!\!1_{\{\tau_1 > t\}} \mathbf{E}_{\mathbf{P}}(e^{\Lambda^{1*}_t - \Lambda^{1*}_s} \,|\, \mathcal{D}^2_t).$$

It is enough to verify that

$$\mathbf{E}_{\mathbf{P}}\big(e^{\Lambda^{1*}_t - \Lambda^{1*}_s} \,\big|\, \mathcal{D}^2_t\big) = 1\!\!1_{\{\tau_2 > t\}} \frac{1}{\lambda_1 + \lambda_2 - \alpha_1} \Big(\lambda_2 e^{-\alpha_1(s-t)} + (\lambda_1 - \alpha_1) e^{-(\lambda_1+\lambda_2)(s-t)}\Big) + 1\!\!1_{\{\tau_2 \le t\}}\, e^{-\alpha_1(s-t)}.$$

If we denote $Y = e^{\Lambda^{1*}_t - \Lambda^{1*}_s}$ then the general formula yields

$$\mathbf{E}_{\mathbf{P}}\big(e^{\Lambda^{1*}_t - \Lambda^{1*}_s} \,\big|\, \mathcal{D}^2_t\big) = 1\!\!1_{\{\tau_2 > t\}} \frac{\mathbf{E}_{\mathbf{P}}(Y 1\!\!1_{\{\tau_2 > t\}})}{\mathbf{P}(\tau_2 > t)} + 1\!\!1_{\{\tau_2 \le t\}}\, \mathbf{E}_{\mathbf{P}}(Y \,|\, \tau_2). \tag{5.61}$$

Standard calculations show that

$$\begin{aligned} \mathbf{E}_{\mathbf{P}}\big(Y 1\!\!1_{\{\tau_2 > t\}}\big) &= \mathbf{E}_{\mathbf{P}}\Big(1\!\!1_{\{\tau_2 > t\}}\, e^{\lambda_1(t - s\wedge\tau_2) + \alpha_1(\tau_2 - s\vee\tau_2)}\Big) \\ &= \int_t^s e^{\lambda_1(t-u) + \alpha_1(u-s)} \lambda_2 e^{-\lambda_2 u}\, du + \int_s^\infty e^{\lambda_1(t-s)} \lambda_2 e^{-\lambda_2 u}\, du \\ &= \frac{\lambda_2 e^{\lambda_1 t - \alpha_1 s}}{\lambda_1 + \lambda_2 - \alpha_1} \Big(e^{-(\lambda_1+\lambda_2-\alpha_1)t} - e^{-(\lambda_1+\lambda_2-\alpha_1)s}\Big) + e^{\lambda_1 t} e^{-(\lambda_1+\lambda_2)s} \end{aligned}$$

and, of course, $\mathbf{P}(\tau_2 > t) = e^{-\lambda_2 t}$. Consequently, we obtain

$$\frac{\mathbf{E}_{\mathbf{P}}(Y 1\!\!1_{\{\tau_2 > t\}})}{\mathbf{P}(\tau_2 > t)} = \frac{1}{\lambda_1 + \lambda_2 - \alpha_1} \Big(\lambda_2 e^{-\alpha_1(s-t)} + (\lambda_1 - \alpha_1) e^{-(\lambda_1+\lambda_2)(s-t)}\Big),$$

as expected. Furthermore, it follows easily from (5.58) that

$$1\!\!1_{\{\tau_2 \le t\}}\, \mathbf{E}_{\mathbf{P}}(e^{\Lambda^{1*}_t - \Lambda^{1*}_s} \,|\, \tau_2) = 1\!\!1_{\{\tau_2 \le t\}}\, e^{-\alpha_1(s-t)}.$$

This ends the proof of the first equality in (5.59). The second equality in (5.59) follows from the calculations above and the fact that the law of τ_2 under $\mathbf{P}_1^*$ is identical with its law under $\mathbf{P}^*$. The last equality in (5.59) is trivial since $\tilde{\mathbf{P}}_1 = \mathbf{P}$.

We shall now consider (5.60) for $t = 0$ (the general case is left to the reader as exercise). More precisely, we wish to show that for $s \le T$

$$\mathbf{P}^*(\tau_1 > s) \neq \mathbf{E}_{\mathbf{P}^*}(e^{-\Lambda_s^{1*}}), \tag{5.62}$$

where the left-hand side is given by (5.52). We have

$$\mathbf{E}_{\mathbf{P}^*}\big(e^{-\Lambda_s^{1*}}\big) = \mathbf{E}_{\mathbf{P}^*}\big(e^{-\lambda_1(s\wedge\tau_2)-\alpha_1(s\vee\tau_2-\tau_2)}\big) = \int_0^s e^{-\lambda_1 u-\alpha_1(s-u)} f_{\tau_2}^*(u)\,du + \int_s^\infty e^{-\lambda_1 s} f_{\tau_2}^*(u)\,du.$$

Consequently

$$\mathbf{E}_{\mathbf{P}^*}\big(e^{-\Lambda_s^{1*}}\big) = \int_0^s e^{-\lambda_1 u-\alpha_1(s-u)} f_{\tau_2}^*(u)\,du + e^{-\lambda_1 s}\mathbf{P}^*(\tau_2 > s),$$

where (cf. Section 5.4.1)

$$f_{\tau_2}^*(u) = \frac{1}{\lambda_1+\lambda_2-\alpha_2}\Big(\alpha_2\lambda_1 e^{-\alpha_2 u} + (\lambda_2-\alpha_2)(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)u}\Big)$$

for $u \le s \le T$, and

$$\mathbf{P}^*(\tau_2 > s) = \frac{1}{\lambda_1+\lambda_2-\alpha_2}\Big(\lambda_1 e^{-\alpha_2 s} + (\lambda_2-\alpha_2)e^{-(\lambda_1+\lambda_2)s}\Big).$$

Straightforward calculations yield

$$\begin{aligned}\mathbf{E}_{\mathbf{P}^*}\big(e^{-\Lambda_s^{1*}}\big) \quad = \quad & \frac{1}{\lambda_1+\lambda_2-\alpha_2}\Big[\frac{\lambda_1\alpha_2}{\lambda_1-\alpha_1+\alpha_2}\Big(e^{-\alpha_1 s} - e^{-(\lambda_1+\alpha_2)s}\Big)\\ & + \frac{(\lambda_2-\alpha_2)(\lambda_1+\lambda_2)}{2\lambda_1+\lambda_2-\alpha_1}\Big(e^{-\alpha_1 s} - e^{-(2\lambda_1+\lambda_2)s}\Big)\\ & + \Big(\lambda_1 e^{-(\lambda_1+\alpha_2)s} + (\lambda_2-\alpha_2)e^{-(2\lambda_1+\lambda_2)s}\Big)\Big]\end{aligned}$$

which shows, when combined with (5.52), that inequality (5.62) is valid. □

Remarks. If $\lambda_i = \alpha_i$ for $i = 1, 2$, the last formula gives, as it should

$$\mathbf{E}_{\mathbf{P}^*}\big(e^{-\Lambda_s^{1*}}\big) = \mathbf{P}(\tau_1 > s) = e^{-\lambda_1 s}.$$

In fact, the equalities above are also valid when $\lambda_2 \neq \alpha_2$, but $\lambda_1 = \alpha_1$. Let us now assume that $\lambda_2 = \alpha_2$, but $\lambda_1 \neq \alpha_1$ (this corresponds to the equality of probability measures $\mathbf{P}^* = \mathbf{P}_1^*$). Then we get

$$\mathbf{E}_{\mathbf{P}_1^*}\big(e^{-\Lambda_s^{1*}}\big) = \frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-\alpha_1 s} + (\lambda_1-\alpha_1)e^{-(\lambda_1+\lambda_2)s}\Big) = \mathbf{P}_1^*(\tau_1 > s) = \mathbf{P}^*(\tau_1 > s).$$

This coincides with the second equality in (5.59), in the special case of $t = 0$.

5.4.4 Validity of Hypothesis (G) under $\mathbf{P}^*$

We shall now check the validity of hypothesis (G). In the present context, we consider the random time $\tau = \tau_1$, we take $\mathbb{F} = \mathbb{D}^2$, and $t \le T$. Therefore, condition (G) reads as follows.

(G) The process $F_t = \mathbf{P}^*(\tau_1 \le t \,|\, \mathcal{D}_t^2)$, $t \in [0, T]$, admits a modification with increasing sample paths.

Let us denote $G_t = 1 - F_t = \mathbf{P}^*(\tau_1 > t \,|\, \mathcal{D}_t^2)$. It follows from the proof of Lemma 5.5 that

$$G_t = (1 - D_t^2)\,\frac{\mathbf{P}^*(\tau_1 > t, \tau_2 > t)}{\mathbf{P}^*(\tau_2 > t)} + D_t^2\,\mathbf{P}^*(\tau_1 > t \,|\, \tau_2), \tag{5.63}$$

where

$$\mathbf{P}^*(\tau_1 > t, \tau_2 > t) = \frac{\lambda_1 - \alpha_1}{\lambda_1 + \lambda_2 - \alpha_1} e^{-(\lambda_1+\lambda_2)t} + \frac{\lambda_2}{\lambda_1 + \lambda_2 - \alpha_1} e^{-\alpha_1 t - (\lambda_1+\lambda_2-\alpha_1)t} = e^{-(\lambda_1+\lambda_2)t}$$

and (cf. (5.52)-(5.53))

$$\mathbf{P}^*(\tau_2 > t) = \frac{1}{\lambda_1 + \lambda_2 - \alpha_2}\Big(\lambda_1 e^{-\alpha_2 t} + (\lambda_2 - \alpha_2) e^{-(\lambda_1+\lambda_2)t}\Big), \quad \forall\, t \le T,$$

$$\mathbf{P}^*(\tau_2 > t) = \frac{e^{-\lambda_2 t}}{\lambda_1 + \lambda_2 - \alpha_2}\Big(\lambda_1 e^{-(\alpha_2-\lambda_2)T} + (\lambda_2 - \alpha_2) e^{-\lambda_1 T}\Big), \quad \forall\, t > T.$$

On the other hand, for every $u \le t \le T$ we have (cf. (5.56))

$$\mathbf{P}^*(\tau_1 > t \,|\, \tau_2 = u) = \frac{(\lambda_1 + \lambda_2 - \alpha_2)\lambda_2 e^{-\alpha_1(t-u)}}{\lambda_1 \alpha_2 e^{(\lambda_1+\lambda_2-\alpha_2)u} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}\,.$$

Similar calculations yield

$$\mathbf{P}^*(\tau_1 > t \,|\, \tau_2 = u) = \frac{(\lambda_1 + \lambda_2 - \alpha_2)\lambda_2 e^{-(\alpha_1-\lambda_1)T} e^{\alpha_1 u - \lambda_1 t}}{\lambda_1 \alpha_2 e^{(\lambda_1+\lambda_2-\alpha_2)u} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}$$

for $u \le T < t,$ and finally

$$\mathbf{P}^*(\tau_1 > t \,|\, \tau_2 = u) = \frac{(\lambda_1 + \lambda_2 - \alpha_2) e^{\lambda_2 u - \lambda_1 t}}{\lambda_1 \lambda_2 e^{-(\alpha_2-\lambda_2)T} + (\lambda_2 - \alpha_2) e^{-\lambda_1 T}}$$

for $T < u \le t$. Combining the formulae above, we obtain

$$\begin{aligned} G_t \;=\; & \mathbb{1}_{\{t<\tau_2\le T\}}\,\frac{c}{\lambda_1 e^{ct} + \lambda_2 - \alpha_2} + \mathbb{1}_{\{T<t<\tau_2\}}\,\frac{c e^{\lambda_1(T-t)}}{\lambda_1 e^{cT} + \lambda_2 - \alpha_2} \\ & + \mathbb{1}_{\{\tau_2=u\le t\le T\}}\,\frac{c\lambda_2 e^{\alpha_1(u-t)}}{\lambda_1\alpha_2 e^{cu} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)} \\ & + \mathbb{1}_{\{\tau_2=u\le T<t\}}\,\frac{c\lambda_2 e^{(\lambda_1-\alpha_1)T} e^{\alpha_1 u - \lambda_1 t}}{\lambda_1\alpha_2 e^{cu} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)} \\ & + \mathbb{1}_{\{T<\tau_2=u\le t\}}\,\frac{c e^{\lambda_1 T} e^{\lambda_2 u - \lambda_1 t}}{\lambda_1\lambda_2 e^{cT} + (\lambda_2 - \alpha_2)}\,, \end{aligned}$$

where we denote $c = \lambda_1 + \lambda_2 - \alpha_2$. In particular, for every $t \in [0, T]$ we have

$$G_t = \mathbb{1}_{\{t<\tau_2\le T\}}\,\frac{c}{\lambda_1 e^{ct} + \lambda_2 - \alpha_2} + \mathbb{1}_{\{\tau_2\le t\le T\}}\,\frac{c\lambda_2 e^{-\alpha_1(t-\tau_2)}}{\lambda_1\alpha_2 e^{c\tau_2} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}.$$

Since both terms in the right-hand side of the last formula can be shown to follow decreasing functions, it is enough to examine the jump at $\tau_2,$ which equals

$$\Delta = \frac{c}{\lambda_1 e^{c\tau_2} + \lambda_2 - \alpha_2} - \frac{c\lambda_2}{\lambda_1\alpha_2 e^{c\tau_2} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}.$$

Straightforward calculations show that $\Delta \le 0$ if and only if $\lambda_2 \le \alpha_2$.

5.4.5 Validity of Hypothesis (H) under $\mathbf{P}^*$

Recall that we have $\mathbb{G} = \mathbb{D}^1 \vee \mathbb{D}^2$. If we choose $\tau = \tau_1$ then the filtration $\tilde{\mathbb{F}} = \mathbb{D}^2$ generated by τ_2 plays the role of the additional filtration $\mathbb{F}$. Therefore, condition (H) introduced in Section 4.1.1 can be restated as follows.

(H) Any $\mathbb{D}^1$-martingale under $\mathbf{P}^*$ follows also a $\mathbb{G}$-martingale under $\mathbf{P}^*$.

Condition (H) is equivalent to the following condition (cf. hypothesis (H3) in Section 4.1.1).

(H3) Equality $\mathbf{E}_{\mathbf{P}^*}(\xi \,|\, \mathcal{D}_t^1 \vee \mathcal{D}_t^2) = \mathbf{E}_{\mathbf{P}^*}(\xi \,|\, \mathcal{D}_t^2)$ holds for any bounded $\mathcal{D}_\infty^2$-measurable random variable ξ, and any $t \in \mathbb{R}_+$.

From the calculations done in preceding sections, it is clear that the last condition is not satisfied in Kusuoka's example. For instance, we may take $t < s$ and $\xi = 1\!\!1_{\{\tau_2 > s\}}$. Using a suitable modification of formula (5.52), we get

$$\mathbf{E}_{\mathbf{P}^*}(\xi \,|\, \mathcal{D}_t^2) = \mathbf{P}^*(\tau_2 > s \,|\, \mathcal{D}_t^2) = 1\!\!1_{\{\tau_2 > t\}} \frac{\mathbf{P}^*(\tau_2 > s)}{\mathbf{P}^*(\tau_2 > t)} = 1\!\!1_{\{\tau_2 > t\}} \frac{\lambda_1 e^{-\alpha_2 s} + (\lambda_2 - \alpha_2) e^{-(\lambda_1+\lambda_2)s}}{\lambda_1 e^{-\alpha_2 t} + (\lambda_2 - \alpha_2) e^{-(\lambda_1+\lambda_2)t}}.$$

On the other hand, Lemma 5.5 yields

$$\begin{aligned}
\mathbf{E}_{\mathbf{P}^*}(\xi \,|\, \mathcal{D}_t^1 \vee \mathcal{D}_t^2) &= \mathbf{P}^*(\tau_2 > s \,|\, \mathcal{D}_t^1 \vee \mathcal{D}_t^2) \\
&= 1\!\!1_{\{\tau_1 > t, \tau_2 > t\}} \frac{\mathbf{P}^*(\tau_1 > t, \tau_2 > s)}{\mathbf{P}^*(\tau_1 > t, \tau_2 > t)} + 1\!\!1_{\{\tau_1 \le t < \tau_2\}} \frac{\mathbf{P}^*(\tau_2 > s \,|\, \tau_1)}{\mathbf{P}^*(\tau_2 > t \,|\, \tau_1)} \\
&= 1\!\!1_{\{\tau_1 > t, \tau_2 > t\}} \frac{1}{\lambda_1 + \lambda_2 - \alpha_2} \Big(\lambda_1 e^{-\alpha_2(s-t)} + (\lambda_2 - \alpha_2) e^{-(\lambda_1+\lambda_2)(s-t)} \Big) \\
&\quad + 1\!\!1_{\{\tau_1 \le t < \tau_1\}} \, e^{-\alpha_2(s-t)}.
\end{aligned}$$

It is thus clear that $\mathbf{E}_{\mathbf{P}^*}(\xi \,|\, \mathcal{D}_t^1 \vee \mathcal{D}_t^2) \neq \mathbf{E}_{\mathbf{P}^*}(\xi \,|\, \mathcal{D}_t^2)$, and thus the martingale invariance property (H) does not hold under $\mathbf{P}^*$ (it is obvious that it holds under the original probability measure $\mathbf{P}$).

References

[1] Bielecki, T. and Rutkowski, M.: Modelling of credit risk: intensity based approach. Forthcoming in: *Advances in Mathematical Finance,* Cambridge University Press, 2000.

[2] Blanchet-Scalliet, C. and Jeanblanc, M.: Dynamics of defaultable zero-coupon. Working paper, 2000.

[3] Brémaud, P.: *Point Processes and Queues. Martingale Dynamics.* Springer-Verlag, Berlin, 1981.

[4] Brémaud, P. and Yor, M.: Changes of filtration and of probability measures. Z.f.W. 45, 269-295, 1978.

[5] Chou, C.S. and Meyer, P.A.: Sur la représentation des martingales comme intégrales stochastiques dans les processus ponctuels. Séminaire de Probabilités IX, Lect. Notes in Math., Springer-Verlag, Berlin, 1975, pp.226-236.

[6] Davis, M.H.A.: The representation of martingales of jump processes. *SIAM Journal on Control* 14, 623-638, 1976.

[7] Dellacherie, C.: Un exemple de la théorie générale des processus. Séminaire de Probabilités IV, Lecture Notes in Math. 124, Springer-Verlag, Berlin, 1970, pp.60-70.

[8] Dellacherie, C.: *Capacités et processus stochastiques*, Springer-Verlag, Berlin, 1972.

[9] Dellacherie, C. and Meyer, P.A.: *Probabilités et potentiel. Chapitres I-IV*, Hermann, Paris, 1975.

[10] Dellacherie, C. and Meyer, P.A.: A propos du travail de Yor sur les grossissements des tribus. Séminaire de Probabilités XII, Lect. Notes in Math. 649, Springer-Verlag, Berlin, 1978, pp.69-78.

[11] Duffie, D.: First-to-default valuation. Working paper, 1998.

[12] Duffie, D. and Lando, D.: Term structure of credit spreads with incomplete accounting information. Working paper, 1997.

[13] Duffie, D., Schroder, M. and Skiadas, C.: Recursive valuation of defaultable securities and the timing of resolution of uncertainty. *Annals of Applied Probability* 6, 1075-1090, 1997.

[14] Elliott, R.J.: *Stochastic Calculus and Applications*, Springer-Verlag, Berlin, 1982.

[15] Elliott, R.J.: Innovation projections of a jump process and local martingale. *Math. Proc. Cambridge Phil. Soc.* 81, 77-90, 1977.

[16] Elliott, R.J., Jeanblanc, M. and Yor, M.: On models of default risk. *Mathematical Finance* 10, 179-195, 2000.

[17] Jeanblanc, M. and Rutkowski, M.: Modelling of default risk: an overview. In: *Mathematical Finance: Theory and Practice,* Higher Education Press, Beijing, 171-269, 2000.

[18] Jeanblanc, M. and Rutkowski, M.: Default risk and hazard processes. Presented at the *Congress of the Bachelier Finance Society,* Paris, 2000.

[19] Kusuoka, S.: A remark on default risk models. *Advances in Mathematical Economics* 1, 69-82, 1999.

[20] Last, G. and Brandt, A.: *Marked Point Processes on the Real Line. The Dynamic Approach.* Springer-Verlag, Berlin, 1995.

[21] Liptser, R.S. and Shiryaev, A.N.: *Theory of Martingales.* Kluver, 1996.

[22] Mazziotto, G. and Szpirglas, J.: Modèle général de filtrage non linéaire et équations différentielles stochastiques associées. *Ann. Inst. Henri Poincaré* 15, 147-173, 1979.

[23] Protter, P.: *Stochastic Integration and Differential Equations.* Springer-Verlag, New York, 1990.

[24] Rogers, L.C.G. and Williams, D.: *Diffusions, Markov Processes and Martingales, Vol. 2: Itô Calculus*, Wiley, 1987.

[25] Rutkowski, M.: 'On models of default risk' by R.Elliott, M.Jeanblanc and M.Yor, working paper, 1999.

[26] Yor, M.: *Some Aspects of Brownian Motion. Part II: Some Recent Martingale Problems*, Lectures in Mathematics. ETH Zürich. Birkhäuser, 1997.