INDIFFERENCE PRICING AND HEDGING OF DEFAULTABLE CLAIMS

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Contents

1 Introduction

In this note, we present a few alternative ways of pricing defaultable claims in the situation when perfect hedging is not possible. In Bielecki at al. (2004b), we have presented the mean-variance hedging framework. Now, we study the indifference price approach that was initiated by Hodges and Neuberger (1989). We shall refer to this approach as the "Hodges price" approach. This will lead us to solving portfolio optimization problems in incomplete market, and we shall use the dynamic programming approach. We also present the Hamilton-Jacobi-Bellman (HJB) equations, when appropriate, even though this method typically requires strong assumptions to give closed-form solutions. In particular, when dealing with the general dynamic programming approach, we need not make any Markovian assumption about the underlying processes; such assumptions are fundamental for the HJB methodology to work.

We emphasize that a very important aspect of our analysis is the distinction between the case when admissible portfolios are adapted to the filtration $\mathbb{F}$, and the case when admissible portfolios are adapted to the filtration $\mathbb{G}$.

In the first section, we define the Hodges indifference price associated to strategies adapted with the reference filtration $\mathbb{F}$, and we solve the problem for exponential preferences and for some particular defaultable claims. We shall use results obtained here to provide basis for a comparison between the historical spread and the risk-neutral one.

In the second section, using backward stochastic differential equations (BSDEs), we work with $\mathbb{G}$-adapted strategies, and we solve portfolio optimization problems for exponential utility functions. Our method relies on the ideas of Rouge and El Karoui (2000) and Musiela and Zariphopoulou (2004). The reader can refer to El Karoui and Mazliak (1997), El Karoui and Quenez (1997), El Karoui et al. (1997), or to the survey by Buckdahn (2000) for an introduction to the theory of backward stochastic differential equations and its applications in finance.

In the third section, we study a particular indifference price based on the quadratic criterion; we call such a price the *quadratic hedging price*. In particular, we compare the indifference prices obtained using strategies adapted to the reference filtration $\mathbb{F}$ to the indifference prices obtained using strategies based on the enlarged filtration $\mathbb{G}$. It is worthwhile to stress, though, that the quadratic utility alone is not quite adequate for the pricing purposes, although it represents a good criterion for hedging purposes. This is one of the reasons we presented the mean-variance approach to pricing and hedging in Bielecki et al. (2004b).

In the last section, we present in a very particular case the duality approach for exponential utilities.

2 Hedging in Incomplete Markets

The default-free asset is Z^1 with the dynamics

$$dZ_t^1 = Z_t^1(\nu dt + \sigma dW_t), \quad Z_0^1 > 0,$$

and the price process of the money market account has the dynamics

$$dZ_t^2 = Z_t^2 r dt, \quad Z_0^2 = 1,$$

where r is the constant interest rate. The default-free market is complete and arbitrage free: one can hedge perfectly any square-integrable contingent claim $X \in \mathcal{F}_T$. The default time is some random time τ, and the default process is denoted as $H_t = \mathbb{1}_{\{\tau \le t\}}$. The reference filtration is the Brownian filtration $\mathcal{F}_t = \sigma(W_u,\, u \le t)$ and the enlarged filtration is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathcal{H}_t = \sigma(H_u,\, u \le t)$.

We assume that the hazard process $F_t = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\}$ is absolutely continuous with respect to Lebesgue measure, so that $F_t = \int_0^t f_u \, du$ (hence, it is an increasing process). Therefore, the process

$$M_t = H_t - \int_0^{t\wedge\tau} \gamma_u \, du = H_t - \int_0^{t\wedge\tau} \frac{f_u}{1-F_u} \, du$$

is a $\mathbb{G}$-martingale, where γ is the *default intensity*. Note that the stochastic intensity γ is the intensity of the default time τ with respect to the reference filtration $\mathbb{F}$ generated by the Brownian motion W.

For a fixed $T > 0$, we introduce a *risk-neutral probability* $\mathbb{Q}$ for the market model (Z^1, Z^2) by setting $d\mathbb{Q}|_{\mathcal{G}_t} = \eta_t \, d\mathbb{P}|_{\mathcal{G}_t}$ for $t \in [0, T]$, where the Radon-Nikodym density η is the $\mathbb{F}$-martingale defined as

$$d\eta_t = -\theta \eta_t \, dW_t, \quad \eta_0 = 1,$$

where $\theta = \nu/\sigma$. Under $\mathbb{Q}$, the discounted process $\widetilde{Z}^1_t = e^{-rt} Z^1_t$ is a martingale. It should be emphasized that $\mathbb{Q}$ is not necessarily a martingale measure for defaultable assets. Let us recall, however, that if $\widetilde{\mathbb{Q}}$ is any equivalent martingale measure on $\mathbb{G}$ for the default-free and defaultable market, then the restriction of $\widetilde{\mathbb{Q}}$ to $\mathbb{F}$ is equal to the restriction of $\mathbb{Q}$ to $\mathbb{F}$. A *defaultable claim* is simply any random variable X, which is $\mathcal{G}_T$-measurable. Hence, default-free claims are formally considered as special cases of defaultable claims.

2.1 Hodges Indifference Price

We present a general framework of the Hodges and Neuberger (1989) approach with some strictly increasing, strictly concave and continuously differentiable mapping u, defined on $\mathbb{R}$. We solve explicitly the problem in the case of exponential utility for portfolios adapted to the reference filtration.

The Hodges approach to pricing of unhedgeable claims is a utility-based approach and can be summarized as follows: the issue at hand is to assess the value of some (defaultable) claim X as seen from the perspective of an economic agent who optimizes his behavior relative to some utility function, say u. In order to provide such an assessment one can argue that one should first consider the following possible modes of agent's behavior and the associated optimization problems:

Problem $(\mathcal{P})$: Optimization in the default-free market.

The agent invests his initial wealth $v > 0$ in the default-free financial market using a self-financing strategy. The associated optimization problem is,

$$(\mathcal{P}) \;:\; \mathcal{V}(v) := \sup_{\phi \in \Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V^v_T(\phi)\big)\big\},$$

where the wealth process $V_t = V^v_t(\phi)$, $t \in \mathbb{R}_+$, is solution of

$$dV_t = rV_t \, dt + \phi_t(dZ^1_t - rZ^1_t dt), \quad V_0 = v. \tag{1}$$

Recall that $\Phi(\mathbb{F})$ is the class of all admissible, $\mathbb{F}$-adapted, self-financing trading strategies (for the definition of this class, see Chapter 2).

Problem $(\mathcal{P}^X_{\mathbb{F}})$: Optimization in the default-free market using $\mathbb{F}$-adapted strategies and buying the defaultable claim.

The agent buys the contingent claim X at price p, and invests the remaining wealth $v - p$ in the financial market, using a trading strategy $\phi \in \Phi(\mathbb{F})$. The resulting *global terminal wealth* will be

$$V^{v-p,X}_T(\phi) = V^{v-p}_T(\phi) + X.$$

The associated optimization problem is

$$(\mathcal{P}^X_{\mathbb{F}})\ :\ \mathcal{V}_X(v-p) := \sup_{\phi\in\Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V^{v-p}_T(\phi)+X\big)\big\},$$

where the process $V^{v-p}(\phi)$ is solution of (1) with the initial condition $V^{v-p}_0(\phi)=v-p$. We emphasize that the class $\Phi(\mathbb{F})$ of admissible strategies is the same as in the problem $(\mathcal{P})$, that is, we restrict here our attention to trading strategies that are adapted to the reference filtration $\mathbb{F}$.

Problem $(\mathcal{P}^X_{\mathbb{G}})$: Optimization in the default-free market using $\mathbb{G}$-adapted strategies and buying the defaultable claim.

The agent buys the contingent claim X at price p, and invests the remaining wealth $v-p$ in the financial market, using a strategy adapted to the enlarged filtration $\mathbb{G}$. The associated optimization problem is

$$(\mathcal{P}^X_{\mathbb{G}})\ :\ \mathcal{V}^{\mathbb{G}}_X(v-p) := \sup_{\phi\in\Phi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V^{v-p}_T(\phi)+X\big)\big\},$$

where $\Phi(\mathbb{G})$ is the class of all $\mathbb{G}$-admissible trading strategies (for the definition of the class $\Phi(\mathbb{G})$, see Chapter 2).

Next, the utility based assessment of the value (price) of the claim X, as seen from the agent's perspective, is given in terms of the following definition.

Definition 2.1 For a given initial endowment v, the $\mathbb{F}$-*Hodges buying price* of a defaultable claim X is the real number $p^*_{\mathbb{F}}(v)$ such that $\mathcal{V}(v)=\mathcal{V}_X\big(v-p^*_{\mathbb{F}}(v)\big)$. Similarly, the $\mathbb{G}$-*Hodges buying price* of X is the real number $p^*_{\mathbb{G}}(v)$ such that $\mathcal{V}(v)=\mathcal{V}^{\mathbb{G}}_X\big(v-p^*_{\mathbb{G}}(v)\big)$.

Remark. We can define the $\mathbb{F}$-*Hodges selling price* $p^{\mathbb{F}}_*(v)$ of X by considering $-p$, where p is the buying price of $-X$, as specified in Definition 2.1.

If the contingent claim X is $\mathcal{F}_T$-measurable, then the $\mathbb{F}$- and the $\mathbb{G}$-Hodges selling and buying prices coincide with the hedging price of X, i.e., $p^*_{\mathbb{F}}(v)=p^*_{\mathbb{G}}(v)=\pi_0(X)=\mathbb{E}_{\mathbb{P}}(\zeta_T X)$, where we denote $\zeta_t=\eta_t R_t$ with $R_t=(Z^2_t)^{-1}=e^{-rt}$. Indeed, assume that there exists a self-financing portfolio $\widehat{\phi}$ such that $X=V^{\pi_0(X)}_T(\widehat{\phi})$, and let h be the $\mathbb{F}$-Hodges buying price. Suppose first that $h<\pi_0(X)$. Then for any ϕ we obtain

$$V^{v-h}_T(\phi)+X = V^{v-h}_T(\phi)+V^{\pi_0(X)}_T(\widehat{\phi}) = V^{v-h+\pi_0(X)}_T(\psi),$$

where we denote $\psi=\widehat{\phi}+\phi\in\Phi(\mathbb{F})$. Hence

$$\begin{aligned}\mathcal{V}_X(v-h) &= \sup_{\phi\in\Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V^{v-h}_T(\phi)+X\big)\big\}\\ &= \sup_{\psi\in\Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V^{v-h+\pi_0(X)}_T(\psi)\big)\big\} \geq \mathcal{V}(v),\end{aligned}$$

where the last inequality (which is a strict inequality) follows from $v<v-h+\pi_0(X)$ and the arbitrage principle. Therefore, the supremum over $\phi\in\Phi(\mathbb{F})$ of $\mathbb{E}_{\mathbb{P}}(u(V^{v-h}_T(\phi)+X))$ is greater than $\mathcal{V}(v)$. We conclude that the $\mathbb{F}$-Hodges buying price can not be smaller than the hedging price. Arguing in a similar way, one can show that the $\mathbb{F}$-Hodges selling price of an $\mathcal{F}_T$-measurable claim can not be smaller than the hedging price. Finally, almost identical arguments show that the $\mathbb{G}$-Hodges buying and selling price of an $\mathcal{F}_T$-measurable claim are equal to the hedging price of X (see Section 3.2).

Remark. It can be shown (see Rouge and El Karoui (2000), or Collin-Dufresne and Hugonnier (2002)) that in the general case of non-hedgeable contingent claim, the Hodges price belongs to the open interval

$$]\inf_{\widetilde{\mathbb{Q}}} \mathbb{E}_{\widetilde{\mathbb{Q}}}(Xe^{-rT}), \sup_{\widetilde{\mathbb{Q}}} \mathbb{E}_{\widetilde{\mathbb{Q}}}(Xe^{-rT})[,$$

where $\widetilde{\mathbb{Q}}$ runs over the set of all equivalent martingale measures, and thus it can not induce arbitrage opportunities.

2.2 Solution of Problem $(\mathcal{P})$

We briefly recall one of the solution methods for the problem $(\mathcal{P})$. Towards this end, we first observe that in view of (1) the discounted process $e^{-rt}V_t^{v-p}(\phi)$, $t \in \mathbb{R}_+$, is a martingale under any equivalent martingale measure, hence $\zeta_t V_t^{v-p}(\phi)$, $t \in \mathbb{R}_+$, is a $\mathbb{P}$-martingale and, in particular, $\mathbb{E}_{\mathbb{P}}(V_T^v(\phi)\zeta_T) = v$. It follows that in order to obtain a terminal wealth equal to, say V, the initial endowment v has to be greater or equal to $\mathbb{E}_{\mathbb{P}}(V\zeta_T)$; this condition is commonly referred to as the *budget constraint*.

Now, let us denote by I the inverse of the monotonic mapping u' (the first derivative of u). It is well known (see, e.g., Karatzas and Shreve (1998)) that the optimal terminal wealth in the problem $(\mathcal{P})$ is given by the formula

$$V_T^{v,*} = I(\mu\zeta_T), \quad \mathbb{P}\text{-a.s.}, \tag{2}$$

where μ is a real number such that the budget constraint is binding, that is,

$$v = \mathbb{E}_{\mathbb{P}}\big(\zeta_T V_T^{v,*}\big). \tag{3}$$

Consequently, the optimal value of the objective criterion for the problem $(\mathcal{P})$ is $\mathcal{V}(v) = \mathbb{E}_{\mathbb{P}}(u(V_T^{v,*}))$.

The above results are obtained by means of convex duality theory. The disadvantage of this approach, however, is the fact that it is typically very difficult to identify an optimal trading strategy. Thus, in general, using the convex duality approach we can only partially solve the problem $(\mathcal{P})$. Specifically, we can compute the optimal value of the objective criterion, but we can't identify the optimal strategy. Later, we shall use the BSDE approach in a more general setting. It will be seen that this approach will allow us to identify (at least in principle) an optimal trading strategy.

2.3 Solution of Problem $(\mathcal{P}_{\mathbb{F}}^X)$

In this subsection, we shall examine the problem $(\mathcal{P}_{\mathbb{F}}^X)$ for a defaultable claim of a particular form. First, we shall provide a solution $\mathcal{V}_X(v-p)$ to the related optimization problem. Next, we shall establish a quasi-explicit representation for the Hodges price of X in the case of exponential utility. Finally, we shall compare the spread obtained via the risk-neutral valuation with the spread determined by the Hodges price of a defaultable zero-coupon bond. The reader can refer to Bernis and Jeanblanc (2003) for other comments.

2.3.1 Particular Form of a Defaultable Claim

We restrict our attention to the case when X is of the form

$$X = X_1 1\!\!1_{\{\tau > T\}} + X_2 1\!\!1_{\{\tau \leq T\}}, \tag{4}$$

where $X_i, i = 1, 2$ are square-integrable under $\mathbb{P}$ and $\mathcal{F}_T$-measurable random variables. In this case, we have

$$V_T^{v-p,X}(\phi) = V_T^{v-p}(\phi) + X_1$$

if the default did not occur before maturity date T, that is, on the set $\{\tau > T\}$, and

$$V_T^{v-p,X}(\phi) = V_T^{v-p}(\phi) + X_2$$

otherwise. In other words,

$$V_T^{v-p,X}(\phi) = 1\!\!1_{\{\tau>T\}}(V_T^{v-p}(\phi) + X_1) + 1\!\!1_{\{\tau\le T\}}(V_T^{v-p}(\phi) + X_2).$$

Observe that the pay-off X_2 is not paid at time of default τ, but at the terminal time T.

Since the trading strategies are $\mathbb{F}$-adapted, the terminal wealth $V_T^{v-p}(\phi)$ is an $\mathcal{F}_T$-measurable random variable. Consequently, it holds that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p,X}(\phi)\big)\big\}\\
&= \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p}(\phi)+X_1\big)1\!\!1_{\{\tau>T\}} + u\big(V_T^{v-p}(\phi)+X_2\big)1\!\!1_{\{\tau\le T\}}\big\}\\
&= \mathbb{E}_{\mathbb{P}}\left\{\mathbb{E}_{\mathbb{P}}\left(u\left(V_T^{v-p}(\phi)+X_1\right)1\!\!1_{\{\tau>T\}} + u\left(V_T^{v-p}(\phi)+X_2\right)1\!\!1_{\{\tau\le T\}}|\mathcal{F}_T\right)\right\}\\
&= \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p}(\phi)+X_1\big)(1-F_T) + u\big(V_T^{v-p}(\phi)+X_2\big)F_T\big\},
\end{aligned}$$

where $F_T = \mathbb{P}\{\tau \le T \,|\, \mathcal{F}_T\}$. Define, for every $\omega \in \Omega$ and $y \in \mathbb{R}$,

$$J_X(y,\omega) = u(y + X_1(\omega))(1 - F_T(\omega)) + u(y + X_2(\omega))F_T(\omega).$$

Notice that under the present assumptions, the problem $(\mathcal{P}_{\mathbb{F}}^X)$ is equivalent to the following problem:

$$(\mathcal{P}_{\mathbb{F}}^X)\ :\ \mathcal{V}_X(v-p) := \sup_{\phi\in\Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{J_X\big(V_T^{v-p}(\phi),\omega\big)\big\}.$$

The mapping $J_X(\cdot,\omega)$ is a strictly concave and increasing real-valued mapping. Consequently, for any $\omega \in \Omega$ we can define the mapping $I_X(z,\omega)$ by setting $I_X(z,\omega) = \big(J'_X(\cdot,\omega)\big)^{-1}(z)$ for $z \in \mathbb{R}$, where $(J'_X(\cdot,\omega))^{-1}$ denotes the inverse mapping of the derivative of J_X with respect to the first variable. To simplify the notation, we shall usually suppress the second variable, and we shall write $I_X(\cdot)$ in place of $I_X(\cdot,\omega)$.

The following lemma provides the form of the optimal solution.

Lemma 2.1 *The optimal terminal wealth for the problem* $(\mathcal{P}_{\mathbb{F}}^X)$ *is given by* $V_T^{v-p,*} = I_X(\lambda^*\zeta_T)$, $\mathbb{P}$-*a.s., for some* λ^* *such that*

$$v - p = \mathbb{E}_{\mathbb{P}}\big(\zeta_T V_T^{v-p,*}\big). \tag{5}$$

Thus the optimal global wealth equals $V_T^{v-p,X,*} = V_T^{v-p,*} + X = I_X(\lambda^*\zeta_T) + X$ *and the optimal value of the objective criterion for the problem* $(\mathcal{P}_{\mathbb{F}}^X)$ *is*

$$\mathcal{V}_X(v-p) = \mathbb{E}_{\mathbb{P}}(u(V_T^{v-p,X,*})) = \mathbb{E}_{\mathbb{P}}(u(I_X(\lambda^*\zeta_T) + X)). \tag{6}$$

Proof. As a consequence of predictable representation property (see, e.g., Karatzas and Shreve (1998)), one knows that in order to find the optimal wealth it is enough to maximize $u(\Delta)$ over the set of square-integrable and $\mathcal{F}_T$-measurable random variables Δ, subject to the budget constraint, given by

$$\mathbb{E}_{\mathbb{P}}(\zeta_T\Delta) \le v - p.$$

The associated Lagrange multiplier, say λ^*, is non-negative. Moreover, by the strict monotonicity of u, we know that, at optimum, the constraint is binding, and thus $\lambda^* > 0$.

The mapping $J_X(\cdot)$ is strictly concave (for all ω). Hence, for every wealth process $V^{v-p}(\phi)$, starting from $v - p$, by tangent inequality, we have

$$\mathbb{E}_{\mathbb{P}}\big\{J_X(V_T^{v-p}(\phi)) - J_X(V_T^{v-p,*})\big\} \ \le\ \mathbb{E}_{\mathbb{P}}\big\{(V_T^{v-p}(\phi) - V_T^{v-p,*})J'_X(V_T^{v-p,*})\big\}.$$

Replacing $V^{v-p,*}$ by its expression given in Lemma 2.1 yields for any $\phi \in \Phi(\mathbb{F})$

$$\mathbb{E}_{\mathbb{P}}\big\{J_X(V_T^{v-p}(\phi)) - J_X(V_T^{v-p,*})\big\} \le \lambda^*\,\mathbb{E}_{\mathbb{P}}\big\{\zeta_T(V_T^{v-p}(\phi) - V_T^{v-p,*})\big\} \le 0,$$

where the last inequality follows from the budget constraint. To end the proof, it remains to observe that the first order conditions are also sufficient in the case of a concave criterion. Moreover, by virtue of strict concavity of the function J_X, the optimum is unique. □

2.3.2 Exponential Utility: Explicit Computation of the Hodges Price

For the sake of simplicity, we assume here that $r = 0$. Let us state the following result, the proof of which stems from Lemma 2.1, by direct computations.

Proposition 2.1 *Let $u(x) = 1 - \exp(-\varrho x)$ for some $\varrho > 0$. Assume that for $i = 1, 2$ the random variable $\zeta_T e^{-\varrho X^i}$ is $\mathbb{P}$-integrable. Then we have*

$$p^*_{\mathbb{F}}(v) = -\frac{1}{\varrho}\,\mathbb{E}_{\mathbb{P}}\left(\zeta_T \ln\big((1-F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}\big)\right) = \mathbb{E}_{\mathbb{P}}(\zeta_T \Psi),$$

where the $\mathcal{F}_T$-measurable random variable Ψ equals

$$\Psi = -\frac{1}{\varrho}\ln\big((1-F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}\big). \tag{7}$$

Thus, the $\mathbb{F}$-Hodges buying price $p^*_{\mathbb{F}}(v)$ is the arbitrage price of the associated claim Ψ. In addition, the claim Ψ enjoys the following meaningful property

$$\mathbb{E}_{\mathbb{P}}\big\{u\big(X - \Psi\big)\,\big|\,\mathcal{F}_T\big\} = 0. \tag{8}$$

Proof. In view of the form of the solution to the problem $(\mathcal{P})$, we obtain (cf. (2))

$$V_T^{v,*} = -\frac{1}{\varrho}\ln\left(\frac{\mu^*\zeta_T}{\varrho}\right).$$

The budget constraint $\mathbb{E}_{\mathbb{P}}(\zeta_T V_T^{v,*}) = v$ implies that the Lagrange multiplier μ^* satisfies

$$\frac{1}{\varrho}\ln\left(\frac{\mu^*}{\varrho}\right) = -\frac{1}{\varrho}\mathbb{E}_{\mathbb{P}}\big(\zeta_T \ln\zeta_T\big) - v. \tag{9}$$

In the case of an exponential utility, we have (recall that the variable ω is suppressed)

$$J_X(y) = (1 - e^{-\varrho(y+X_1)})(1-F_T) + (1 - e^{-\varrho(y+X_2)})F_T,$$

so that

$$J'_X(y) = \varrho e^{-\varrho y}(e^{-\varrho X_1}(1-F_T) + e^{-\varrho X_2}F_T).$$

Thus, setting

$$A = e^{-\varrho X_1}(1-F_T) + e^{-\varrho X_2}F_T = e^{-\varrho\Psi},$$

we obtain

$$I_X(z) = -\frac{1}{\varrho}\ln\left(\frac{z}{A\varrho}\right) = -\frac{1}{\varrho}\ln\left(\frac{z}{\varrho}\right) - \Psi.$$

It follows that the optimal terminal wealth for the initial endowment $v - p$ is

$$V_T^{v-p,*} = -\frac{1}{\varrho}\ln\left(\frac{\lambda^*\zeta_T}{A\varrho}\right) = -\frac{1}{\varrho}\ln\left(\frac{\lambda^*}{\varrho}\right) - \frac{1}{\varrho}\ln\zeta_T - \Psi,$$

where the Lagrange multiplier λ^* is chosen to satisfy the budget constraint $\mathbb{E}_{\mathbb{P}}(\zeta_T V_T^{v-p,*}) = v - p$, that is,

$$\frac{1}{\varrho}\ln\left(\frac{\lambda^*}{\varrho}\right) = -\frac{1}{\varrho}\,\mathbb{E}_{\mathbb{P}}\big(\zeta_T\ln\zeta_T\big) - \mathbb{E}_{\mathbb{P}}\big(\zeta_T\Psi\big) - v + p. \tag{10}$$

The $\mathbb{F}$-Hodges buying price is a real number $p^* = p^*_{\mathbb{F}}(v)$ such that

$$\mathbb{E}_{\mathbb{P}}\big(\exp(-\varrho V_T^{v,*})\big) = \mathbb{E}_{\mathbb{P}}\big(\exp(-\varrho(V_T^{v-p^*,*} + X))\big),$$

where μ^* and λ^* are given by (9) and (10), respectively. After substitution and simplifications, we arrive at the following equality

$$\mathbb{E}_{\mathbb{P}}\Big\{ \exp\Big(-\varrho\big(\mathbb{E}_{\mathbb{P}}(\zeta_T\Psi) - p^* + X - \Psi\big)\Big)\Big\} = 1. \tag{11}$$

Using (4), it is easy to check that

$$\mathbb{E}_{\mathbb{P}}\big(e^{-\varrho(X-\Psi)} \,\big|\, \mathcal{F}_T\big) = 1 \tag{12}$$

so that equality (8) holds, and $\mathbb{E}_{\mathbb{P}}\big(e^{-\varrho(X-\Psi)}\big) = 1$. Combining (11) and (12), we conclude that $p^*_{\mathbb{F}}(v) = \mathbb{E}_{\mathbb{P}}(\zeta_T\Psi)$. □

We briefly provide the analog of (7) for the $\mathbb{F}$-Hodges selling price of X . We have $p^{\mathbb{F}}_*(v) = \mathbb{E}_{\mathbb{P}}(\zeta_T\widetilde{\Psi})$, where

$$\widetilde{\Psi} = \frac{1}{\varrho}\ln\big((1-F_T)e^{\varrho X_1} + F_T e^{\varrho X_2}\big). \tag{13}$$

Remark. It is important to notice that the $\mathbb{F}$-Hodges prices $p^*_{\mathbb{F}}(v)$ and $p^{\mathbb{F}}_*(v)$ do not depend on the initial endowment v. This is an interesting property of the exponential utility function. In view of (8), the random variable Ψ will be called the *indifference conditional hedge.*

Comparison with the Davis price. Let us present the results derived from the marginal utility pricing approach. The *Davis price* (see Davis (1997)) is given by

$$d^*(v) = \frac{\mathbb{E}_{\mathbb{P}}\big\{u'\big(V_T^{v,*}\big)X\big\}}{\mathcal{V}'(v)}\,.$$

In our context, this yields

$$d^*(v) = \mathbb{E}_{\mathbb{P}}\big\{\zeta_T\big(X_1F_T + X_2(1-F_T)\big)\big\}.$$

In this case, the risk aversion ϱ has no influence on the pricing of the contingent claim. In particular, when F is deterministic, the Davis price reduces to the arbitrage price of each (default-free) financial asset X^i, $i = 1, 2$, weighted by the corresponding probabilities F_T and $1 - F_T$.

2.3.3 Risk-Neutral Spread Versus Hodges Spreads

Let us consider the case of a defaultable bond with zero recovery, so that $X_1 = 1$ and $X_2 = 0$. It follows from (13) that the $\mathbb{F}$-Hodges buying and selling prices of the bond are (it will be convenient here to indicate the dependence of the Hodges price on maturity T)

$$D^*_{\mathbb{F}}(0,T) = -\frac{1}{\varrho}\,\mathbb{E}_{\mathbb{P}}\big\{\zeta_T \ln(e^{-\varrho}(1-F_T) + F_T)\big\}$$

and

$$D^{\mathbb{F}}_*(0,T) = \frac{1}{\varrho}\,\mathbb{E}_{\mathbb{P}}\big\{\zeta_T \ln(e^{\varrho}(1-F_T) + F_T)\big\},$$

respectively. Let $\widetilde{\mathbb{Q}}$ be a risk-neutral probability for the filtration $\mathbb{G}$, that is, for the enlarged market. The "market" price at time $t = 0$ of defaultable bond, denoted as $D^0(0,T)$, is thus equal to the expectation under $\widetilde{\mathbb{Q}}$ of its discounted pay-off, that is,

$$D^0(0,T) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(\mathbb{1}_{\{\tau>T\}}R_T\big) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\big((1-\widetilde{F}_T)R_T\big),$$

where $\widetilde{F}_t = \widetilde{\mathbb{Q}}\{\tau \le t \,|\, \mathcal{F}_t\}$ for every $t \in [0,T]$. Let us emphasize that the risk-neutral probability $\widetilde{\mathbb{Q}}$ is chosen by the market, via the price of the defaultable asset. Hence, it should not be confused with the probability measure $\mathbb{Q}$, which combines, in a sense, the risk-neutral probability for the default-free market (Z^1, Z^2) with the real-life intensity of default.

Let us recall that in our setting the price process of the T-maturity unit discount Treasury (default-free) bond is $B(t,T) = e^{-r(T-t)}$. The Hodges buying and selling spreads at time $t = 0$ are defined as

$$S^*(0,T) = -\frac{1}{T} \ln \frac{D^*_{\mathbb{F}}(0,T)}{B(0,T)}$$

and

$$S_*(0,T) = -\frac{1}{T} \ln \frac{D^{\mathbb{F}}_*(0,T)}{B(0,T)}\,,$$

respectively. Likewise, the *risk-neutral spread* at time $t = 0$ is given as

$$S^0(0,T) = -\frac{1}{T} \ln \frac{D^0(0,T)}{B(0,T)}\,.$$

Since $D^*_{\mathbb{F}}(0,0) = D^{\mathbb{F}}_*(0,0) = D^0(0,0) = 1$, the respective *backward short spreads* at time $t = 0$ are given by the following limits (provided the limits exist)

$$s^*(0) = \lim_{T\downarrow 0} S^*(0,T) = -\frac{d^+ \ln D^*_{\mathbb{F}}(0,T)}{dT}\Big|_{T=0} - r$$

and

$$s_*(0) = \lim_{T\downarrow 0} S_*(0,T) = -\frac{d^+ \ln D^{\mathbb{F}}_*(0,T)}{dT}\Big|_{T=0} - r,$$

respectively. We also set

$$s^0(0) = \lim_{T\downarrow 0} S^0(0,T) = -\frac{d^+ \ln D^0(0,T)}{dT}\Big|_{T=0} - r.$$

Assuming, as we do, that the processes $\widetilde{F}_T$ and F_T are absolutely continuous with respect to the Lebesgue measure, and using the observation that the restriction of $\widetilde{\mathbb{Q}}$ to $\mathcal{F}_T$ is equal to $\mathbb{Q}$, we find out that

$$\begin{aligned}\frac{D^*_{\mathbb{F}}(0,T)}{B(0,T)} &= -\frac{1}{\varrho}\, \mathbb{E}_{\mathbb{Q}}\big\{ \ln\big(e^{-\varrho}(1-F_T) + F_T\big)\big\} \\ &= -\frac{1}{\varrho}\, \mathbb{E}_{\mathbb{Q}}\Big\{ \ln\Big(e^{-\varrho}\Big(1 - \int_0^T f_t\,dt\Big) + \int_0^T f_t\,dt\Big)\Big\},\end{aligned}$$

and

$$\begin{aligned}\frac{D^{\mathbb{F}}_*(0,T)}{B(0,T)} &= \frac{1}{\varrho}\, \mathbb{E}_{\mathbb{Q}}\big\{ \ln\big(e^{\varrho}(1-F_T) + F_T\big)\big\} \\ &= \frac{1}{\varrho}\, \mathbb{E}_{\mathbb{Q}}\Big\{ \ln\Big(e^{\varrho}\Big(1 - \int_0^T f_t\,dt\Big) + \int_0^T f_t\,dt\Big)\Big\}.\end{aligned}$$

Furthermore,

$$\frac{D^0(0,T)}{B(0,T)} = \mathbb{E}_{\mathbb{Q}}(1-\widetilde{F}_T) = \mathbb{E}_{\mathbb{Q}}\Big(1 - \int_0^T \widetilde{f}_t\,dt\Big).$$

Consequently,

$$s^*(0) = \frac{1}{\varrho}\,\big(e^{\varrho} - 1\big) f_0, \quad s_*(0) = \frac{1}{\varrho}\,\big(1 - e^{-\varrho}\big) f_0,$$

and $s^0(0) = \widetilde{f}_0$. Now, if we postulate, for instance, that $s_*(0) = s^0(0)$ (it would be the case if the market price is the Hodges price), then we must have

$$\widetilde{\gamma}_0 = \widetilde{f}_0 = \frac{1}{\varrho}\,\big(1 - e^{-\varrho}\big) f_0 = \frac{1}{\varrho}\,\big(1 - e^{-\varrho}\big)\gamma_0$$

so that $\widetilde{\gamma}_0 < \gamma_0$. Similar calculations can be made for any $t \in [0,T[$.

3 Optimization Problems and BSDEs

The major distinction between this section and the previous one is that here we consider strategies ϕ that are predictable with respect to the full filtration $\mathbb{G}$. Unless explicitly stated otherwise, the underlying probability measure is the real-world probability $\mathbb{P}$. We consider the following dynamics for the risky asset Z^1

$$dZ^1_t = Z^1_{t-}(\nu dt + \sigma dW_t + \varphi dM_t), \tag{14}$$

where $M_t = H_t - \int_0^{t\wedge\tau} \gamma_s\, ds$, and where we impose the condition $\varphi > -1$, which ensures that the price Z^1_t remains strictly positive.

In order to simplify notation, we shall denote by ξ the process such that $dM_t = dH_t - \xi_t\, dt$ is a $\mathbb{G}$-martingale, i.e., $\xi_t = \gamma_t(1 - H_t)$. We assume that the so-called Hypothesis (H) holds, that is, any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale as well.

Throughout most of the section, we shall deal with the same market model as in the previous section, that is, we shall set $\varphi = 0$. Only in Section 5 we generalize the dynamics of the risky asset to the case when $\varphi \neq 0$, so that the dynamics of the risky asset Z^1 are sensitive to the default risk. In particular, the limit case $\varphi = -1$ corresponds to the case where the underlying risky asset has value 0 after the default.

We assume for simplicity that $r = 0$, and we change the notational convention for an admissible portfolio to the one that will be more suitable for problems considered here: instead of using the number of shares ϕ as before, we set $\pi = \phi Z^1$, so that π represents the value invested in the risky asset. The portfolio process π_t should not be confused with the arbitrage price process $\pi_t(X)$. In addition, we adopt here the following relaxed definition of admissibility of a self-financing trading strategy.

Definition 3.1 The class $\Pi(\mathbb{F})$ ($\Pi(\mathbb{G})$, respectively) of $\mathbb{F}$-*admissible* ($\mathbb{G}$-*admissible*, respectively) trading strategies is the set of all $\mathbb{F}$-predictable ($\mathbb{G}$-predictable, respectively) processes π such that $\int_0^T \pi_t^2\, dt < \infty$, $\mathbb{P}$-a.s.

The wealth process of a strategy π satisfies

$$dV_t(\pi) = \pi_{t-}\big(\nu dt + \sigma dW_t + \varphi dM_t\big). \tag{15}$$

Note that with the present definition of admissible strategies the "martingale part" of the wealth process is a local martingale, in general.

Let X be a given contingent claim, represented by a $\mathcal{G}_T$-measurable random variable. We shall study the following problem:

$$\sup_{\pi\in\Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^v(\pi) + X\big)\big\}.$$

3.1 Exponential Utility

In this section, we shall examine the problem introduced above in the case of the exponential utility, under the assumption that $\varphi = 0$ in dynamics (14). First, we examine the existence and the form of a solution to the optimization problem, under additional technical assumptions. Subsequently, we shall derive the expression for the Hodges selling price.

3.1.1 Optimization Problem

Let $X \in \mathcal{G}_T$ be a given non-negative contingent claim, and let v be the initial endowment of an agent. Our first goal is to solve an optimization problem for an agent who buys a claim X. To this end, it suffices to find a strategy $\pi \in \Pi(\mathbb{G})$ that maximizes $\mathbb{E}_{\mathbb{P}}(u(V^v_T(\pi)+X))$, where the wealth process $V_t = V^v_t(\pi)$ (for simplicity, we shall frequently skip v and π from the notation) satisfies

$$dV_t = \phi_t\, dZ^1_t = \pi_t(\nu dt + \sigma dW_t),\ V_0 = v.$$

We consider the exponential utility function $u(x) = 1 - e^{-\varrho x}$, with $\varrho > 0$. Therefore,

$$\sup_{\pi\in\Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big\{u(V^v_T(\pi)+X)\big\} = 1 - \inf_{\pi\in\Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V^v_T(\pi)}e^{-\varrho X}\big).$$

Let us describe the idea of a solution. Suppose that we can find a process Z with $Z_T = e^{-\varrho X}$, which depends only on the claim X and parameters ϱ, σ, ν, and such that the process $e^{-\varrho V^v_t(\pi)}Z_t$ is a $\mathbb{G}$-submartingale under $\mathbb{P}$ for any admissible strategy π and is a martingale under $\mathbb{P}$ for some admissible strategy $\pi^* \in \Pi(\mathbb{G})$. Then, we would have

$$\mathbb{E}_{\mathbb{P}}(e^{-\varrho V^v_T(\pi)}Z_T) \geq e^{-\varrho V^v_0(\pi)}Z_0 = e^{-\varrho v}Z_0$$

for any $\pi \in \Pi(\mathbb{G})$, with equality for some strategy $\pi^* \in \Pi(\mathbb{G})$. Consequently, we would obtain

$$\inf_{\pi\in\Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V^v_T(\pi)}e^{-\varrho X}\big) = \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V^v_T(\pi^*)}e^{-\varrho X}\big) = e^{-\varrho v}Z_0, \tag{16}$$

and thus we would be in a position to conclude that π^* is an optimal strategy. In fact, it will turn out that in order to implement the above idea we shall need to restrict further the class of $\mathbb{G}$-admissible trading strategies.

We shall search for an auxiliary process Z in the class of all processes satisfying the following BSDE

$$dZ_t = f_t\, dt + \widehat{z}_t\, dW_t + \widetilde{z}_t\, dM_t,\ t \in [0,T[,\ Z_T = e^{-\varrho X}, \tag{17}$$

where the process f will be determined later (see equation (19) below). By applying Itô's formula, we obtain

$$d(e^{-\varrho V_t}) = e^{-\varrho V_t}\left(\left(\tfrac{1}{2}\,\varrho^2\pi_t^2\sigma^2 - \varrho\pi_t\nu\right)dt - \varrho\pi_t\sigma\, dW_t\right),$$

so that

$$\begin{aligned} d(e^{-\varrho V_t}Z_t) &= e^{-\varrho V_t}\big(f_t + Z_t(\tfrac{1}{2}\varrho^2\pi_t^2\sigma^2 - \varrho\pi_t\nu) - \varrho\pi_t\sigma\widehat{z}_t\big)dt \\ &\quad + e^{-\varrho V_t}\big((\widehat{z}_t - \varrho\pi_t\sigma Z_t)\, dW_t + \widetilde{z}_t\, dM_t\big). \end{aligned}$$

Let us choose π^* such that it minimizes, for every t, the following expression

$$Z_t\big(\tfrac{1}{2}\,\varrho^2\pi_t^2\sigma^2 - \varrho\pi_t\nu\big) - \varrho\pi_t\sigma\widehat{z}_t = -\varrho\pi_t(\nu Z_t + \sigma\widehat{z}_t) + \tfrac{1}{2}\,\varrho^2\pi_t^2\sigma^2 Z_t.$$

It is easily seen that

$$\pi^*_t = \frac{\nu Z_t + \sigma\widehat{z}_t}{\varrho\sigma^2 Z_t} = \frac{1}{\varrho\sigma}\left(\theta + \frac{\widehat{z}_t}{Z_t}\right). \tag{18}$$

Now, let us choose the process f, by postulating that

$$\begin{aligned} f_t = f(Z_t, \widehat{z}_t) &= Z_t\left(\varrho\pi^*_t\nu - \tfrac{1}{2}\,\varrho^2(\pi^*_t)^2\sigma^2\right) + \varrho\pi^*_t\sigma\widehat{z}_t \\ &= \varrho\pi^*_t(Z_t\nu + \sigma\widehat{z}_t) - \tfrac{1}{2}\,\varrho^2(\pi^*_t)^2\sigma^2 Z_t = \frac{(\nu Z_t + \sigma\widehat{z}_t)^2}{2\sigma^2 Z_t}. \end{aligned} \tag{19}$$

We shall now study the BSDE (17) with the process f given by (19). In other words, we shall focus on the following BSDE:

$$dZ_t = \frac{(\nu Z_t + \sigma\widehat{z}_t)^2}{2\sigma^2 Z_t}\, dt + \widehat{z}_t\, dW_t + \widetilde{z}_t\, dM_t,\ t \in [0,T[,\ Z_T = e^{-\varrho X}. \tag{20}$$

Recall that W is a Brownian motion under $\mathbb{P}$, and that the risk-neutral probability $\mathbb{Q}$ is given by $d\mathbb{Q}|_{\mathcal{F}_t} = \eta_t \, d\mathbb{P}|_{\mathcal{F}_t}$, where $d\eta_t = -\eta_t \theta \, dW_t$ with $\theta = \nu/\sigma$ and $\eta_0 = 1$. Thus the process $W_t^{\mathbb{Q}} = W_t + \theta t$, $t \in [0,T]$, is a Brownian motion under $\mathbb{Q}$.

It will be convenient to write equation (20) as

$$dZ_t = \big(\tfrac{1}{2}\theta^2 Z_t + \theta \widehat{z}_t + \tfrac{1}{2} Z_t^{-1} \widehat{z}_t^2\big) dt + \widehat{z}_t \, dW_t + \widetilde{z}_t \, dM_t, \ t \in [0,T[, \ Z_T = e^{-\varrho X}.$$

Equivalently,

$$dZ_t = \big(\tfrac{1}{2}\theta^2 Z_t + \tfrac{1}{2} Z_t^{-1} \widehat{z}_t^2\big) dt + \widehat{z}_t \, dW_t^{\mathbb{Q}} + \widetilde{z}_t \, dM_t, \ t \in [0,T[, \ Z_T = e^{-\varrho X}. \tag{21}$$

Remark. To the best of out knowledge, no general theorem, which would establish the existence of a solution to equation (21), is available. The comparison theorem works for BSDEs driven by a jump process when the drift satisfies some Lipschitz condition (see Royer (2003)). Hence, the proofs of Lepeltier and San-Martin (1997) and Kobylanski (2000), which rely on comparison results, may not be directly carried to the case of quadratic BSDEs driven by a jump process. We shall solve the BSDE (21) under rather restrictive assumptions on X. Hence, the general case remains an open problem.

Lemma 3.1 *Assume that there exists $\mathbb{G}$-predictable processes $\widehat{k}$, $\widetilde{k} > -1$ and a constant c such that*

$$\exp(K_T)\mathcal{E}_T(\widetilde{M}) = e^{-\varrho X}, \tag{22}$$

where

$$K_t = c + \int_0^t \widehat{k}_u \, dW_u^{\mathbb{Q}}, \quad \widetilde{M}_t = \int_0^t \widetilde{k}_u \, dM_u,$$

and $\mathcal{E}(\widetilde{M})$ is the Doléans exponential of $\widetilde{M}$. Then $U_t = \exp(K_t)\mathcal{E}_t(\widetilde{M})$ solves the following BSDE

$$dU_t = \tfrac{1}{2} U_t^{-1} \widehat{u}_t^2 \, dt + \widehat{u}_t \, dW_t^{\mathbb{Q}} + \widetilde{u}_t \, dM_t, \ t \in [0,T[, \ U_T = e^{-\varrho X}. \tag{23}$$

Proof. Since $d\mathcal{E}_t(\widetilde{M}) = \mathcal{E}_{t-}(\widetilde{M}) \, d\widetilde{M}_t$, the process U defined above satisfies

$$dU_t = \tfrac{1}{2} U_t \widehat{k}_t^2 \, dt + U_t \widehat{k}_t \, dW_t^{\mathbb{Q}} + U_{t-} \widetilde{k}_t \, dM_t$$

and thus

$$dU_t = \tfrac{1}{2} U_t^{-1} \widehat{u}_t^2 \, dt + \widehat{u}_t \, dW_t^{\mathbb{Q}} + \widetilde{u}_t \, dM_t$$

where we denote $\widehat{u}_t = U_t \widehat{k}_t$ and $\widetilde{u}_t = U_{t-} \widetilde{k}_t$. Since obviously $U_T = e^{-\varrho X}$, this ends the proof. □

Corollary 3.1 *Let X be a $\mathcal{G}_T$-measurable claim such that* (22) *holds for some $\mathbb{G}$-predictable processes $\widehat{k}$, $\widetilde{k} > -1$ and some constant c. Then there exists a solution $(Z, \widehat{z}, \widetilde{z})$ of the BSDE* (21)*. Moreover, the process Z is strictly positive.*

Proof. Let us set $Y_t = e^{-(T-t)\theta^2/2}$ and let U be the process introduced in Lemma 3.1. Then the process $Z_t = U_t Y_t$ satisfies

$$\begin{aligned} dZ_t &= Y_t \, dU_t + \tfrac{1}{2}\theta^2 Y_t U_t \, dt \\ &= \tfrac{1}{2}\theta^2 Y_t U_t \, dt + \tfrac{1}{2} Y_t U_t^{-1} \widehat{u}_t^2 \, dt + Y_t \widehat{u}_t \, dW_t^{\mathbb{Q}} + Y_t \widetilde{u}_t \, dM_t \\ &= \tfrac{1}{2}\theta^2 Z_t \, dt + \tfrac{1}{2} Z_t^{-1} Y_t^2 \widehat{u}_t^2 \, dt + Y_t \widehat{u}_t \, dW_t^{\mathbb{Q}} + Y_t \widetilde{u}_t \, dM_t \\ &= \tfrac{1}{2}\theta^2 Z_t \, dt + \tfrac{1}{2} Z_t^{-1} \widehat{z}_t^2 \, dt + \widehat{z}_t \, dW_t^{\mathbb{Q}} + \widetilde{z}_t \, dM_t \end{aligned}$$

where we set $\widehat{z}_t = Y_t \widehat{u}_t$ and $\widetilde{z}_t = Y_t \widetilde{u}_t$. It is also clear that $Z_T = U_T = e^{-\varrho X}$ and Z is strictly positive. □

Recall that the process Z depends on the choice of a contingent claim X, as well as on the model's parameters ϱ, σ and ν. The next lemma shows that the processes Z and π^* introduced above have indeed the desired properties that were described at the beginning of this section. To achieve our goal, we need to restrict the class of admissible trading strategies, however. We say that an admissible strategy π is *regular with respect to* X if the martingale part of the process $e^{-\varrho V_t^v(\pi)} Z_t$ is a martingale under $\mathbb{P}$, rather than a local martingale. We denote by $\Pi_X(\mathbb{G})$ the class of all admissible trading strategies, which are regular with respect to X.

Lemma 3.2 *Let X be a $\mathcal{G}_T$-measurable claim such that (22) holds for some $\mathbb{G}$-predictable processes $\widehat{k}$, $\widetilde{k}$ and some constant c. Assume that the default intensity γ and the processes $\widetilde{k}$, $\widehat{k}$ are bounded. Suppose that the process $Z = Z(X, \varrho, \sigma, \nu)$ is a solution to the BSDE (20) given in Corollary 3.1. Then:*
(i) *The process $e^{-\varrho V_t^v(\pi)} Z_t$ is a submartingale for any strategy $\pi \in \Pi_X(\mathbb{G})$.*
(ii) *The process $e^{-\varrho V_t^v(\pi^*)} Z_t$ is a martingale for the process π^* given by expression (18).*
(iii) *The process π^* belongs to the class $\Pi_X(\mathbb{G})$ of admissible trading strategies regular with respect to X.*

Proof. In view of the definition of π^* and the choice of the process f (see formula (19)), the validity of part (i) is rather clear. To establish (ii), we shall first check that the process $e^{-\varrho V_t^*} Z_t$ is a martingale (and not only a local martingale) under $\mathbb{P}$, where $V_t^* = V_t^v(\pi^*)$. From the choice of π^*, we obtain

$$\begin{aligned} d(e^{-\varrho V_t^*} Z_t) &= e^{-\varrho V_t^*} \big((\widehat{z}_t - \varrho \pi_t^* \sigma Z_t)\, dW_t + \widetilde{z}_t\, dM_t \big) \\ &= -\theta e^{-\varrho V_t^*} Z_t\, dW_t + e^{-\varrho V_t^*} \widetilde{z}_t\, dM_t. \end{aligned}$$

This means that

$$e^{-\varrho V_t^*} Z_t = e^{-\varrho v} Z_0 \exp\big(-\theta W_t - \tfrac{1}{2}\theta^2 t\big) \exp\Big(-\int_0^t \frac{\widetilde{z}_s}{Z_s}\, \xi_s\, ds\Big)\Big(1 + \frac{\widetilde{z}_{\tau-}}{Z_{\tau-}}\, H_t\Big).$$

The quantity $e^{-\varrho v} Z_0 \exp\big(-\theta W_t - \frac{1}{2}\theta^2 t\big)$ is clearly a continuous martingale under $\mathbb{P}$. Recall that

$$\widetilde{z}_t = Y_t \widetilde{u}_t = \widetilde{k}_t Z_t.$$

and thus $\widetilde{z}_t / Z_t = \widetilde{k}_t$ is a bounded process. We conclude that the process

$$\exp\Big(-\int_0^t \frac{\widetilde{z}_s}{Z_s}\, \xi_s\, ds\Big)\Big(1 + \frac{\widetilde{z}_{\tau-}}{Z_{\tau-}}\, H_t\Big)$$

is a bounded, purely discontinuous martingale under $\mathbb{P}$. To complete the proof, it remains to check that the process π^* given by (18) is $\mathbb{G}$-admissible, in the sense of Definition 3.1. To this end, it suffices to check that

$$\int_0^T \widehat{z}_t^2 Z_t^{-2}\, dt < \infty, \quad \mathbb{P}\text{-a.s.}$$

This is clear since the process $\widehat{z}_t / Z_t = \widehat{k}_t$ is bounded. We conclude that the strategy π^* belongs to the class $\Pi_X(\mathbb{G})$. □

Recall now that in this section we examine the following problem:

$$\sup_{\pi \in \Pi_X(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(u(V_T^v(\pi) + X)\big) = 1 - \inf_{\pi \in \Pi_X(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X}\big).$$

We are in a position to state the following result.

Proposition 3.1 *Let X be a $\mathcal{G}_T$-measurable claim such that* (22) *holds for some $\mathbb{G}$-predictable processes $\widehat{k}$, $\widetilde{k}$ and some constant c. Assume that the default intensity γ and the processes $\widetilde{k}$, $\widehat{k}$ are bounded. Then*

$$\inf_{\pi\in\Pi_X(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)}e^{-\varrho X}\big) = \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi^*)}e^{-\varrho X}\big) = e^{-\varrho v} Z_0^X,$$

where the optimal strategy $\pi^* \in \Pi_X(\mathbb{G})$ is given by the formula, for every $t \in [0,T]$,

$$\pi_t^* = \frac{1}{\varrho\sigma}\left(\theta + \frac{\widehat{z}_t^X}{Z_t^X}\right) = \frac{\theta + \widehat{k}_t}{\varrho\sigma},$$

where $Z_t^X = Z_t$ and $\widehat{z}_t^X = \widehat{z}_t$ are the two first components of a solution $(Z_t, \widehat{z}_t, \widetilde{z})$ of the BSDE

$$dZ_t = \frac{(\nu Z_t + \sigma\widehat{z}_t)^2}{2\sigma^2 Z_t}\,dt + \widehat{z}_t\,dW_t + \widetilde{z}_t\,dM_t,\ Z_T = e^{-\varrho X}. \tag{24}$$

More explicitly (see Corollary 3.1), we have $\widehat{z}_t = \widehat{k}_t Z_t$ and

$$Z_t = e^{-(T-t)\theta^2/2}\exp(K_t)\mathcal{E}_t(\widetilde{M}).$$

Proof. The proof is rather straightforward. We know that the process Z which solves (24) is such that: (i) the process $Z_t e^{-\varrho V_t^v(\pi^*)}$ is a martingale, and (ii) for any strategy $\pi \in \Pi_X(\mathbb{G})$ the process $Z_t e^{-\varrho V_t^v(\pi)}$ is equal to a martingale minus an increasing process (since the drift term is non-positive), and thus it is a submartingale. This shows that (16) holds with $\Pi(\mathbb{G})$ substituted with $\Pi_X(\mathbb{G})$. □

It should be acknowledged that the assumptions of Proposition 3.1 are restrictive, so that it covers only a very special case of a claim X. Let us now comment briefly on the case of a general claim; we do not pretend here to give strict results, our aim is merely to give some hints how one can deal with the general case.

Recall that our aim is to find a solution $(Z, \widehat{z}, \widetilde{z})$ of the following BSDE

$$dZ_t = \big(\tfrac{1}{2}\theta^2 Z_t + \tfrac{1}{2}Z_t^{-1}\widehat{z}_t^2\big)dt + \widehat{z}_t\,dW_t^{\mathbb{Q}} + \widetilde{z}_t\,dM_t,\ t\in[0,T[,\ Z_T = e^{-\varrho X},$$

or equivalently, of the equation

$$dU_t = \tfrac{1}{2}U_t^{-1}\widehat{u}_t^2\,dt + \widehat{u}_t\,dW_t^{\mathbb{Q}} + \widetilde{u}_t\,dM_t,\ t\in[0,T[,\ U_T = e^{-\varrho X},$$

Assume that the process U is strictly positive and set $X_t = \ln U_t$. Then, denoting $\widehat{x}_t = \widehat{u}_t U_t^{-1}, \widetilde{x}_t = \widetilde{u}_t U_{t-}^{-1}$ and applying Itô's formula, we obtain (recall that we denote $\xi_t = \gamma_t \mathbb{1}_{\{\tau>t\}}$)

$$\begin{aligned} dX_t &= \widehat{x}_t\,dW_t^{\mathbb{Q}} + \widetilde{x}_t\,dM_t + (\ln(1+\widetilde{x}_t) - \widetilde{x}_t)\,dH_t \\ &= \widehat{x}_t\,dW_t^{\mathbb{Q}} + \widetilde{x}_t\,dM_t + (\ln(1+\widetilde{x}_t) - \widetilde{x}_t)(dM_t + \xi_t\,dt) \\ &= \widehat{x}_t\,dW_t^{\mathbb{Q}} + \ln(1+\widetilde{x}_t)\,dM_t + (\ln(1+\widetilde{x}_t) - \widetilde{x}_t)\xi_t\,dt \\ &= \widehat{x}_t\,dW_t^{\mathbb{Q}} + x_t^*\,dM_t + (1 - e^{x_t^*} + x_t^*)\xi_t\,dt \\ &= \widehat{x}_t\,dW_t^{\mathbb{Q}} + x_t^*\,dH_t + (1 - e^{x_t^*})\xi_t\,dt, \end{aligned}$$

where $x_t^* = \ln(1+\widetilde{x}_t)$ and the terminal condition is $X_T = -\varrho X$. It thus suffices to solve the following BSDE

$$dX_t = \widehat{x}_t\,dW_t^{\mathbb{Q}} + x_t^*\,dH_t + (1 - e^{x_t^*})\xi_t\,dt,\ t\in[0,T[,\ X_T = -\varrho X. \tag{25}$$

Assume first that $X \in \mathcal{F}_T$. In that case, it is obvious that we may take $\widehat{x} = \widetilde{x}^* = 0$ and thus $X_t = -\mathbb{E}_{\mathbb{Q}}(\varrho X \,|\, \mathcal{G}_t) = -\mathbb{E}_{\mathbb{Q}}(\varrho X \,|\, \mathcal{F}_t)$ is a solution. In the general case, we note that the continuous $\mathbb{G}$-martingales are stochastic integrals with respect to the Brownian motion $W^{\mathbb{Q}}$. We may thus transform the problem: it suffices to find a process x^* such that the process R, defined through the formula

$$R_t = \mathbb{E}_{\mathbb{Q}}\Big(-\varrho X + \int_0^T (e^{x^*_s} - 1)\xi_s\,ds - x^*_\tau \mathbb{1}_{\{\tau \le T\}} \,\Big|\, \mathcal{G}_t\Big),$$

is a continuous $\mathbb{G}$-martingale, so that $dR_t = \widehat{x}_t\,dW^{\mathbb{Q}}_t$ for some $\mathbb{G}$-predictable process $\widehat{x}$. Suppose that we can find a process x^* for which the last property is valid. Then, by setting

$$\begin{aligned} X_t &= R_t - \int_0^t (e^{x^*_s} - 1)\xi_s\,ds - x^*_\tau \mathbb{1}_{\{\tau \le t\}} \\ &= \mathbb{E}_{\mathbb{Q}}\Big(-\varrho X + \int_t^T (e^{x^*_s} - 1)\xi_s\,ds - x^*_\tau \mathbb{1}_{\{t<\tau \le T\}} \,\Big|\, \mathcal{G}_t\Big) \end{aligned}$$

we obtain a solution $(X, \widehat{x}, x^*)$ to (25).

Case of a survival claim. From now on, we shall focus on a survival claim $X = Y\mathbb{1}_{\{\tau>T\}}$, where Y is an $\mathcal{F}_T$-measurable random variable. Let us fix $t \in [0,T]$. On the set $\{t \le \tau\}$ we obtain

$$\mathbb{E}_{\mathbb{Q}}(\varrho Y \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t) = e^{\Gamma_t}\,\mathbb{E}_{\mathbb{Q}}(e^{-\Gamma_T}\varrho Y \,|\, \mathcal{F}_t)$$

and on the set $\{\tau < t\}$, we have $\mathbb{E}_{\mathbb{Q}}(\varrho Y \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t) = 0$. The jump of the term A_t, defined as

$$A_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T (e^{x^*_s} - 1)\xi_s\,ds - x^*_\tau \mathbb{1}_{\{\tau \le T\}} \,\Big|\, \mathcal{G}_t\Big),$$

can be computed as follows. On the set $\{t \le \tau\}$, we obtain

$$\begin{aligned} A_t &= \int_t^T \mathbb{E}_{\mathbb{Q}}\big((e^{x^*_s} - 1)\gamma_s \mathbb{1}_{\{\tau>s\}} \,|\, \mathcal{G}_t\big)\,ds - \mathbb{E}_{\mathbb{Q}}\big(x^*_\tau \mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{G}_t\big) \\ &= \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\,\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T (e^{x^*_s} - 1 - x^*_s) e^{-\Gamma_s}\gamma_s\,ds \,\Big|\, \mathcal{F}_t\Big). \end{aligned}$$

On the set $\{\tau < t\}$ for A_t we have

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T (e^{x^*_s} - 1)\gamma_s \mathbb{1}_{\{\tau>s\}}\,ds - x^*_\tau \mathbb{1}_{\{\tau \le T\}} \,\Big|\, \mathcal{G}_t\Big) = -\mathbb{E}_{\mathbb{Q}}\big(x^*_\tau \mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{G}_t\big) = -x^*_\tau.$$

We conclude that our problem is to find a process x^* such that

$$-\mathbb{E}_{\mathbb{Q}}\big(e^{-\Gamma_T}\varrho Y | \mathcal{F}_t\big) = -e^{-\Gamma_t} x^*_t - \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T (e^{x^*_s} - 1 - x^*_s) e^{-\Gamma_s}\gamma_s\,ds \,\Big|\, \mathcal{F}_t\Big).$$

In other words, we need to solve the following BSDE with $\mathbb{F}$-adapted processes x^* and κ

$$d\big(x^*_t e^{-\Gamma_t}\big) = \big(e^{x^*_t} - 1 - x^*_t\big) e^{-\Gamma_t}\gamma_t\,dt + \kappa_t\,dW^{\mathbb{Q}}_t,\ t \in [0,T[,\ x^*_T = \varrho Y.$$

From integration by parts, this BSDE can be written

$$dx^*_t = \big(e^{x^*_t} - 1\big) e^{-\Gamma_t}\gamma_t\,dt + \kappa_t\,dW^{\mathbb{Q}}_t,\ t \in [0,T[,\ x^*_T = \varrho X.$$

Unfortunately, the standard results for existence of solutions to BSDEs do not apply here because the drift term is not of a linear growth with respect to x^*.

3.2 Hodges Buying and Selling Prices

Particular case. Assume, as before, that $r = 0$ and let us check that the Hodges buying price is the hedging price in case of attainable claims. Assume that a claim X is $\mathcal{F}_T$-measurable. By virtue of the predictable representation theorem, there exists a pair $(x, \widehat{x})$, where x is a constant and $\widehat{x}_t$ is an $\mathbb{F}$-adapted process, such that $X = x + \int_0^T \widehat{x}_u \, dW_u^{\mathbb{Q}}$, where $W_t^{\mathbb{Q}} = W_t + \theta t$. Here $x = \mathbb{E}_{\mathbb{Q}} X$ is the arbitrage price $\pi_0(X)$ of X and the replicating portfolio is obtained through $\widehat{x}$. Hence, the time t value of X is $X_t = x + \int_0^t \widehat{x}_u \, dW_u^{\mathbb{Q}}$. Then $dX_t = \widehat{x}_t \, dW_t^{\mathbb{Q}}$ and the process

$$Z_t = e^{-\theta^2 (T-t)/2} e^{-\varrho X_t}$$

satisfies

$$\begin{aligned} dZ_t &= Z_t \Big(\big(\tfrac{1}{2}\theta^2 + \tfrac{1}{2}\varrho^2 \widehat{x}_t^2 \big) \, dt + \varrho \widehat{x}_t \, dW_t^{\mathbb{Q}} \Big) \\ &= \frac{1}{2\sigma^2 Z_t} (\nu Z_t + \sigma \varrho Z_t \widehat{x}_t)^2 \, dt + \varrho Z_t \widehat{x}_t \, dW_t. \end{aligned}$$

Hence $(Z_t, \varrho Z_t \widehat{x}_t, 0)$ is the solution of (24) with the terminal condition $e^{-\varrho X}$, and

$$Z_0 = e^{-\theta^2 T/2} e^{-\varrho x}$$

Note that, for $X = 0$, we get $Z_0 = e^{-\theta^2 T/2}$, therefore

$$\inf_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}(e^{-\varrho V_T^v(\pi)}) = e^{-\varrho v} e^{-\theta^2 T/2}.$$

The $\mathbb{G}$-Hodges buying price of X is the value of p such that

$$\inf_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)}\big) = \inf_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho (V_T^{v-p}(\pi) + X)}\big),$$

that is,

$$e^{-\varrho v} e^{-\theta^2 T/2} = e^{-\varrho (v - p + \pi_0(X))} e^{-\theta^2 T/2}.$$

We conclude easily that $p_*^{\mathbb{G}}(X) = \pi_0(X) = \mathbb{E}_{\mathbb{Q}} X$. Similar arguments show that $p_{\mathbb{G}}^*(X) = \pi_0(X)$.

General case. Assume now that a claim X is $\mathcal{G}_T$-measurable and the assumptions of Proposition 3.1 are satisfied. Since the process Z introduced in Corollary 3.1 is strictly positive, we can use its logarithm. Let us assume that the processes $\widehat{k}$ and $\widetilde{k}$ are strictly positive, and let us denote $\widehat{\psi}_t = Z_t / \widehat{z}_t = \widehat{k}_t^{-1}$, $\widetilde{\psi}_t = Z_t / \widetilde{z}_t = \widetilde{k}_t^{-1}$ and

$$\kappa_t = \frac{\widetilde{\psi}_t}{\ln(1 + \widetilde{\psi}_t)} \geq 0.$$

Then we get

$$d(\ln Z_t) = \tfrac{1}{2} \theta^2 dt + \widehat{\psi}_t \, dW_t^{\mathbb{Q}} + \ln(1 + \widetilde{\psi}_t) \big(dM_t + \xi_t (1 - \kappa_t) \, dt \big),$$

and thus

$$d(\ln Z_t) = \tfrac{1}{2} \theta^2 dt + \widehat{\psi}_t \, dW_t^{\mathbb{Q}} + \ln(1 + \widetilde{\psi}_t) \, d\widehat{M}_t,$$

where

$$d\widehat{M}_t = dM_t + \xi_t (1 - \kappa_t) \, dt = dH_t - \xi_t \kappa_t \, dt.$$

The process $\widehat{M}$ is a martingale under the probability measure $\widehat{\mathbb{Q}}$ defined as $d\widehat{\mathbb{Q}}|_{\mathcal{G}_t} = \widehat{\eta}_t \, d\mathbb{P}|_{\mathcal{G}_t}$, where $\widehat{\eta}$ satisfies

$$d\widehat{\eta}_t = -\widehat{\eta}_{t-} \big(\theta \, dW_t + \xi_t (1 - \kappa_t) \, dM_t \big)$$

with $\widehat{\eta}_0 = 1$.

Proposition 3.2 *The $\mathbb{G}$-Hodges buying price of X with respect to the exponential utility is the real number p such that $e^{-\varrho(v-p)} Z_0^X = e^{-\varrho v} Z_0^0$, that is, $p^*_{\mathbb{G}}(X) = \varrho^{-1} \ln(Z_0^0/Z_0^X)$ or, equivalently, $p^*_{\mathbb{G}}(X) = \mathbb{E}_{\widehat{\mathbb{Q}}} X$.*

Our previous study establishes that the dynamic hedging price of a claim X is the process $X_t = \mathbb{E}_{\widehat{\mathbb{Q}}}(X \,|\, \mathcal{G}_t)$. This price is the expectation of the payoff, under some martingale measure, as is any price in the range of no-arbitrage prices.

4 Quadratic Hedging

We assume here that the wealth process follows

$$dV_t^v(\pi) = \pi_t(\nu dt + \sigma dW_t), \quad V_0^v(\pi) = v.$$

The more general case

$$dV_t^v(\pi) = \pi_t(\nu dt + \sigma dW_t + \varphi dM_t), \quad V_0^v(\pi) = v$$

is too long to be presented here. In this section, we examine the issue of the quadratic pricing and hedging, specifically, for a given $\mathbb{P}$-square-integrable claim X we solve the following minimization problems:

• For a given initial endowment v, solve the minimization problem:

$$\min_{\pi} \mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi) - X)^2\big).$$

A solution to this problem provides the portfolio which, among the portfolios with a given initial wealth, has the closest terminal wealth to a given claim X, in sense of L^2-norm under $\mathbb{P}$.

• Solve the minimization problem:

$$\min_{\pi, v} \mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi) - X)^2\big).$$

The minimal value of v is called the *quadratic hedging price* and the optimal π the *quadratic hedging strategy.*

The mean-variance hedging problem was examined in a fairly general framework of incomplete markets by means of BSDEs in several papers; see, for example, Mania (2000), Mania and Tevzadze (2003), Bobrovnytska and Schweizer (2004), Hu and Zhou (2004) or Lim (2004). Since this list is by no means exhaustive, the interested reader is referred to the references quoted in the above-mentioned papers.

4.1 Quadratic Hedging with $\mathbb{F}$-Adapted Strategies

We shall first solve, for a given initial endowment v, the following minimization problem

$$\min_{\pi \in \Pi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi) - X)^2\big),$$

where X is given as

$$X = X_1 \mathbb{1}_{\{\tau > T\}} + X_2 \mathbb{1}_{\{\tau \le T\}}$$

for some $\mathcal{F}_T$-measurable, $\mathbb{P}$-square-integrable random variables X_1 and X_2. Using the same approach as in the previous section, we define the auxiliary function J_X by setting

$$J_X(y) = (y - X_1)^2(1 - F_T) + (y - X_2)^2 F_T,$$

so that its derivative equals

$$J'_X(y) = 2\left[(y - X_1)(1 - F_T) + (y - X_2)F_T\right] = 2\left[y - X_1(1 - F_T) - X_2F_T\right].$$

Hence

$$I_X(z) = \tfrac{1}{2}z + X_1(1 - F_T) + X_2F_T,$$

and thus the optimal terminal wealth equals

$$V_T^{v,*} = \tfrac{1}{2}\,\lambda^*\zeta_T + X_1(1 - F_T) + X_2F_T,$$

where λ^* is specified through the budget constraint:

$$\mathbb{E}_{\mathbb{P}}(\zeta_T V_T^{v,*}) = \tfrac{1}{2}\lambda^*\,\mathbb{E}_{\mathbb{P}}(\zeta_T^2) + \mathbb{E}_{\mathbb{P}}(\zeta_T X_1(1 - F_T)) + \mathbb{E}_{\mathbb{P}}(\zeta_T X_2 F_T) = v\,.$$

We deduce that

$$\begin{aligned}
&\min_{\pi} \mathbb{E}_{\mathbb{P}}((V_T^v - X)^2) \\
&= \mathbb{E}_{\mathbb{P}}\left[\left(\tfrac{1}{2}\lambda^*\zeta_T + X_1(1 - F_T) + X_2F_T - X_1\right)^2(1 - F_T)\right] \\
&\quad + \mathbb{E}_{\mathbb{P}}\left[\left(\tfrac{1}{2}\lambda^*\zeta_T + X_1(1 - F_T) + X_2F_T) - X_2\right)^2 F_T\right] \\
&= \tfrac{1}{4}(\lambda^*)^2\,\mathbb{E}_{\mathbb{P}}(\zeta_T^2) + \mathbb{E}_{\mathbb{P}}\left((X_1 - X_2)^2F_T(1 - F_T)\right) \\
&= \frac{1}{2\mathbb{E}_{\mathbb{P}}(\zeta_T^2)}\Big(v - \mathbb{E}_{\mathbb{P}}(\zeta_T(X_1 + F_T(X_2 - X_1))\Big)^2 \\
&\quad + \mathbb{E}_{\mathbb{P}}((X_1 - X_2)^2F_T(1 - F_T)).
\end{aligned}$$

Therefore, we obtain the following result.

Proposition 4.1 *If we restrict our attention to $\mathbb{F}$-adapted strategies, the quadratic hedging price of the claim $X = X_1\mathbb{1}_{\{\tau>T\}} + X_2\mathbb{1}_{\{\tau\le T\}}$ equals*

$$\mathbb{E}_{\mathbb{P}}(\zeta_T(X_1 + F_T(X_2 - X_1)) = \mathbb{E}_{\mathbb{Q}}(X_1(1 - F_T) + F_TX_2)\,.$$

The optimal quadratic hedging of X is the strategy which duplicates the $\mathcal{F}_T$-measurable contingent claim $X_1(1 - F_T) + F_TX_2$.

Let us now examine the case of a generic $\mathcal{G}_T$-measurable random variable X. Here, we shall only examine the solution of the second problem introduced above, that is,

$$\min_{v,\pi}\,\mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi) - X)^2\big).$$

As we have explained in Bielecki et al. (2004b), this problem is essentially equivalent to a problem where we restrict our attention to the terminal wealth. From the properties of conditional expectations, we have

$$\min_{V\in\mathcal{F}_T}\,\mathbb{E}_{\mathbb{P}}((V - X)^2) = \mathbb{E}_{\mathbb{P}}((\mathbb{E}_{\mathbb{P}}(X\,|\,\mathcal{F}_T) - X)^2)$$

and the initial value of the strategy with terminal value $\mathbb{E}_{\mathbb{P}}(X\,|\,\mathcal{F}_T)$ is

$$\mathbb{E}_{\mathbb{P}}(\zeta_T\mathbb{E}_{\mathbb{P}}(X\,|\,\mathcal{F}_T)) = \mathbb{E}_{\mathbb{P}}(\zeta_TX).$$

In essence, the latter statement is a consequence of the completeness of the default-free market model. In conclusion, the quadratic hedging price equals $\mathbb{E}_{\mathbb{P}}(\zeta_TX) = \mathbb{E}_{\mathbb{Q}}X$ and the quadratic hedging strategy is the replicating strategy of the attainable claim $\mathbb{E}_{\mathbb{P}}(X\,|\,\mathcal{F}_T)$ associated with X.

4.2 Quadratic Hedging with $\mathbb{G}$-Adapted Strategies

Out next goal is to solve, for a given initial endowment v, the following minimization problem

$$\min_{\pi\in\Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi)-X)^2\big).$$

We have seen in Bielecki et al. (2004b) that one way of solving this problem is to project the random variable X on the set of stochastic integrals. Here, we present an alternative approach. We are looking for $\mathbb{G}$-adapted processes X, Θ and Ψ such that the process

$$J_t(\pi) = \big(V_t^v(\pi) - X_t\big)^2\Theta_t + \Psi_t, \quad \forall\, t\in[0,T], \tag{26}$$

is a $\mathbb{G}$-submartingale for any $\mathbb{G}$adapted trading strategy π and a $\mathbb{G}$-martingale for some strategy π^*. In addition, we require that $X_T = X$, $\Theta_T = 1$, $\Phi_T = 0$. Let us assume that the dynamics of these processes are of the form

$$\begin{aligned} dX_t &= x_t\,dt + \widehat{x}_t\,dW_t + \widetilde{x}_t\,dM_t, && (27)\\ d\Theta_t &= \Theta_{t-}\big(\vartheta_t\,dt + \widehat{\vartheta}_t\,dW_t + \widetilde{\vartheta}_t\,dM_t\big), && (28)\\ d\Psi_t &= \psi_t\,dt + \widehat{\psi}_t\,dW_t + \widetilde{\psi}_t\,dM_t, && (29)\end{aligned}$$

where the drifts x_t, ϑ_t and ψ_t are yet to be determined. From Itô's formula, we obtain (recall that $\xi_t = \gamma_t 1\!\!1_{\{\tau>t\}}$)

$$\begin{aligned} d(V_t-X_t)^2 &= 2(V_t-X_t)(\pi_t\sigma-\widehat{x}_t)\,dW_t - 2(V_t-X_{t-})\widetilde{x}_t\,dM_t\\ &\quad + \big[(V_t-X_{t-}-\widetilde{x}_t)^2 - (V_t-X_{t-})^2\big]dM_t\\ &\quad + \Big(2(V_t-X_t)(\pi_t\nu - x_t) + (\pi_t\sigma-\widehat{x}_t)^2\\ &\quad + \xi_t\big[(V_t-X_t-\widetilde{x}_t)^2 - (V_t-X_t)^2\big]\Big)dt,\end{aligned}$$

where we denote $V_t = V_t^v(\pi)$. The process $J(\pi)$ is a martingale if and only if its drift term $k(t,\pi_t,x_t,\vartheta_t,\psi_t)=0$ for every $t\in[0,T]$. Straightforward calculations show that

$$\begin{aligned} k(t,\pi_t,\vartheta_t,x_t,\psi_t) &= \psi_t + \Theta_t\Big[\vartheta_t(V_t-X_t)^2\\ &\quad + 2(V_t-X_t)\big[(\pi_t\nu-x_t)+\widehat{\vartheta}_t(\pi_t\sigma-\widehat{x}_t)+\xi_t\widetilde{x}_t\big]\\ &\quad + (\pi_t\sigma-\widehat{x}_t)^2 + \xi_t(\widetilde{\vartheta}_t+1)\big[(V_t-X_t-\widetilde{x}_t)^2-(V_t-X_t)^2\big]\Big].\end{aligned}$$

In the first step, for any $t\in[0,T]$ we shall find π_t^* such that the minimum of $k(t,\pi_t,x_t,\vartheta_t,\psi_t)$ is attained.

Subsequently, we shall choose the processes $x=x^*, \vartheta=\vartheta^*$ and $\psi=\psi^*$ in such a way that $k(t,\pi_t^*,x_t^*,\vartheta_t^*,\psi_t^*)=0$. This choice will imply that $k(t,\pi_t,x_t^*,\vartheta_t^*,\psi_t^*)\ge 0$ for any trading strategy π and any $t\in[0,T]$.

The strategy π^* which minimizes $k(t,\pi_t,x_t,\vartheta_t,\psi_t)$ is the solution of the following equation:

$$(V_t^v(\pi)-X_t)(\nu+\widehat{\vartheta}_t\sigma) + \sigma(\pi_t\sigma-\widehat{x}_t) = 0, \quad \forall\, t\in[0,T].$$

Hence, the strategy π^* is implicitly given by

$$\pi_t^* = \sigma^{-1}\widehat{x}_t - \sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma)(V_t^v(\pi^*)-X_t) = A_t - B_t(V_t^v(\pi^*)-X_t),$$

where we denote

$$A_t = \sigma^{-1}\widehat{x}_t, \quad B_t = \sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma).$$

After some computations, we see that the drift term of the process J admits the following representation:

$$\begin{aligned}k(t,\pi_t,\vartheta_t,x_t,\psi_t) &= \psi_t + \Theta_t(V_t - X_t)^2(\vartheta_t - \sigma^2 B_t^2)\\ &+ 2\Theta_t(V_t - X_t)\big(\sigma^2 A_t B_t - \widehat{\vartheta}_t\widehat{x}_t - \widetilde{\vartheta}_t\widetilde{x}_t\xi_t - x_t\big) + \Theta_t\xi_t(\widetilde{\vartheta}_t + 1)\widetilde{x}_t^2.\end{aligned}$$

From now on, we shall assume that the auxiliary processes ϑ, x and ψ are chosen as follows:

$$\begin{aligned}\vartheta_t &= \vartheta_t^* = \sigma^2 B_t^2,\\ x_t &= x_t^* = \sigma^2 A_t B_t - \widehat{\vartheta}_t\widehat{x}_t - \widetilde{\vartheta}_t\widetilde{x}_t\xi_t,\\ \psi_t &= \psi_t^* = -\Theta_t\xi_t(\widetilde{\vartheta}_t + 1)\widetilde{x}_t^2.\end{aligned}$$

It is rather clear that if the drift coefficients ϑ, x, ψ in (27)-(29) are chosen as above, then the drift term in dynamics of J is always non-negative, and it is equal to 0 for $\pi_t^* = A_t - B_t(V_t^v(\pi^*) - X_t)$.

Our next goal is to solve equations (27)-(29). Since $\vartheta_t = \sigma^2 B_t^2$, the three-dimensional process $(\Theta, \widehat{\vartheta}, \widetilde{\vartheta})$ is the unique solution to the linear BSDE (28)

$$d\Theta_t = \Theta_t\big(\sigma^{-2}(\nu + \widehat{\vartheta}_t\sigma)^2\,dt + \widehat{\vartheta}_t\,dW_t + \widetilde{\vartheta}_t\,dM_t\big),\ \Theta_T = 1.$$

It is obvious that $\widehat{\vartheta} = 0, \widetilde{\vartheta} = 0$ and

$$\Theta_t = \exp(-\theta^2(T-t)), \quad \forall\, t \in [0,T]. \tag{30}$$

The three-dimensional process $(X, \widehat{x}, \widetilde{x})$ solves equation (27) with $x_t = x_t^* = \sigma^2 A_t(\nu/\sigma^2) = \theta\widehat{x}_t$. This means that $(X, \widehat{x}, \widetilde{x})$ is the unique solution to the linear BSDE

$$dX_t = \theta\widehat{x}_t\,dt + \widehat{x}_t\,dW_t + \widetilde{x}_t\,dM_t,\ X_T = X.$$

The unique solution to the last equation is $X_t = \mathbb{E}_{\mathbb{Q}}(X\,|\,\mathcal{G}_t)$, where $\mathbb{Q}$ is the risk-neutral probability measure, so that $d\mathbb{Q} = \eta_t\,d\mathbb{P}$, where

$$d\eta_t = -\theta\eta_t\,dW_t, \quad \eta_0 = 1.$$

The components $\widehat{x}$ and $\widetilde{x}$ are given by the integral representation of the $\mathbb{G}$-martingale X with respect to $W^{\mathbb{Q}}$ and M. Notice also that since $\widehat{\vartheta} = 0$, the optimal portfolio π^* is given by the feedback formula

$$\pi_t^* = \sigma^{-1}\big(\widehat{x}_t - \theta(V_t^v(\pi^*) - X_t)\big).$$

Finally, since $\widetilde{\vartheta} = 0$, we have $\psi_t = -\xi_t\widetilde{x}_t^2\Theta_t$. Therefore, we can solve explicitly the BSDE (29) for the process Ψ. Indeed, we are now looking for a three-dimensional process $(\Psi, \widehat{\psi}, \widetilde{\psi})$, which is the unique solution of the BSDE

$$d\Psi_t = -\Theta_t\xi_t\widetilde{x}_t^2\,dt + \widehat{\psi}_t\,dW_t + \widetilde{\psi}_t\,dM_t,\ \Psi_T = 0.$$

Noting that the process

$$\Psi_t + \int_0^t \Theta_s\xi_s\widetilde{x}_s^2\,ds$$

is a $\mathbb{G}$-martingale under $\mathbb{P}$, we obtain the value of Ψ in a closed form:

$$\Psi_t = \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s\xi_s\widetilde{x}_s^2\,ds\,\Big|\,\mathcal{G}_t\Big). \tag{31}$$

Substituting (30) and (31) in (26), we conclude that the value function for our problem is $J_t^* = J_t(\pi^*)$, where in turn

$$\begin{aligned} J_t(\pi^*) &= (V_t^v(\pi^*) - X_t)^2 e^{-\theta^2(T-t)} + \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds \, \Big| \, \mathcal{G}_t \Big) \\ &= (V_t^v(\pi^*) - X_t)^2 e^{-\theta^2(T-t)} + \int_t^T e^{-\theta^2(T-s)} \, \mathbb{E}_{\mathbb{P}}\big(\gamma_s \widetilde{x}_s^2 \mathbb{1}_{\{\tau > s\}} \, \big| \, \mathcal{G}_t \big) \, ds \\ &= (V_t^v(\pi^*) - X_t)^2 e^{-\theta^2(T-t)} + \mathbb{1}_{\{\tau > t\}} \int_t^T e^{-\theta^2(T-s)} \, \mathbb{E}_{\mathbb{P}}\big(\gamma_s \widetilde{x}_s^2 e^{\Gamma_t - \Gamma_s} \, \big| \, \mathcal{F}_t \big) \, ds, \end{aligned}$$

where we have identified the process $\widetilde{x}$ with its $\mathbb{F}$-adapted version (recall that any $\mathbb{G}$-predictable process is equal, prior to default, to an $\mathbb{F}$-predictable process). In particular,

$$J_0^* = e^{-\theta^2 T} \Big((v - X_0)^2 + \mathbb{E}_{\mathbb{P}} \Big(\int_0^T e^{\theta^2 s} \gamma_s \widetilde{x}_s^2 e^{-\Gamma_s} \, ds \Big) \Big).$$

From the last formula, it is obvious that the quadratic hedging price is $X_0 = \mathbb{E}_{\mathbb{Q}} X$. We are in the position to formulate the main result of this section. A corresponding theorem for a default-free financial model was established by Kohlmann and Zhou (2000).

Proposition 4.2 *Let a claim X be $\mathcal{G}_T$-measurable and square-integrable under $\mathbb{P}$. The optimal trading strategy π^*, which solves the quadratic problem*

$$\min_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}((V_T^v(\pi) - X)^2),$$

is given by the feedback formula

$$\pi_t^* = \sigma^{-1} \big(\widehat{x}_t - \theta (V_t^v(\pi^*) - X_t) \big),$$

where $X_t = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}_t)$ for every $t \in [0, T]$, and the process $\widehat{x}_t$ is specified by

$$dX_t = \widehat{x}_t \, dW_t^Q + \widetilde{x}_t \, dM_t.$$

The quadratic hedging price of X is equal to $\mathbb{E}_{\mathbb{Q}} X$.

4.2.1 Survival Claim

Let us consider a simple survival claim $X = \mathbb{1}_{\{\tau > T\}}$, and let us assume that Γ is deterministic, specifically, $\Gamma(t) = \int_0^t \gamma(s) \, ds$. In that case, from the well-known representation theorem (see Bielecki and Rutkowski (2002), Page 159), we have $dX_t = \widetilde{x}_t \, dM_t$ with $\widetilde{x}_t = -e^{\Gamma(t) - \Gamma(T)}$. Hence

$$\begin{aligned} \Psi_t &= \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds \, \Big| \, \mathcal{G}_t \Big) \\ &= \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \gamma(s) \mathbb{1}_{\{\tau > s\}} e^{2\Gamma(s) - 2\Gamma(T)} \, ds \, \Big| \, \mathcal{G}_t \Big) \\ &= \mathbb{1}_{\{\tau > t\}} \, e^{\Gamma(t) - 2\Gamma(T)} \, \mathbb{E}_{\mathbb{P}}\Big(\int_t^T e^{-\theta^2(T-s)} \gamma(s) e^{\Gamma(s)} \, ds \, \Big| \, \mathcal{F}_t \Big) \\ &= \mathbb{1}_{\{\tau > t\}} \, e^{\Gamma(t) - 2\Gamma(T)} \int_t^T e^{-\theta^2(T-s)} \gamma(s) e^{\Gamma(s)} \, ds. \end{aligned}$$

One can check that, at time 0, the value function is indeed smaller that the one obtained with $\mathbb{F}$-adapted portfolios.

4.2.2 Case of an Attainable Claim

Assume now that a claim X is $\mathcal{F}_T$-measurable. Then $X_t = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}_t)$ is the price of X, and it satisfies $dX_t = \widehat{x}_t \, dW_t^{\mathbb{Q}}$. The optimal strategy is, in a feedback form,

$$\pi_t^* = \sigma^{-1}\big(\widehat{x}_t - \theta(V_t - X_t)\big)$$

and the associated wealth process satisfies

$$dV_t = \pi_t^*(\nu dt + \sigma dW_t) = \pi_t^* \sigma \, dW_t^{\mathbb{Q}} = \sigma^{-1}\big(\sigma \widehat{x}_t - \nu(V_t - X_t)\big)\, dW_t^{\mathbb{Q}}.$$

Therefore,

$$d(V_t - X_t) = -\theta(V_t - X_t)\, dW_t^{\mathbb{Q}}.$$

Hence, if we start with an initial wealth equal to the arbitrage price $\pi_0(X)$ of X, then we obtain that $V_t = X_t$ for every $t \in [0, T]$, as expected.

4.2.3 Hodges Price

Let us emphasize that the Hodges price has no real meaning here, since the problem $\min \mathbb{E}_{\mathbb{P}}((V_T^v)^2)$ has no financial interpretation. We have studied in Bielecki et al. (2004b) a more pertinent problem, with a constraint on the expected value of V_T^v under $\mathbb{P}$. Nevertheless, from a mathematical point of view, the Hodges price would be the value of p such that

$$(v^2 - (v-p)^2) = \int_0^T e^{\theta^2 s} \mathbb{E}_{\mathbb{P}}(\gamma_s \widetilde{x}_s^2 e^{-\Gamma_s}) \mathbb{1}_{\{\tau > t\}} ds$$

In the case of the example studied in Section 4.2.1, the Hodges price would be the non-negative value of p such that

$$2vp - p^2 = e^{-2\Gamma_T} \int_0^T e^{\theta^2 s} \gamma_s e^{\Gamma_s} \, ds.$$

Let us also mention that our results are different from results of Lim (2004). Indeed, Lim studies a model with Poisson component, and thus in his approach the intensity of this process does not vanish after the first jump.

5 Optimization in Incomplete Markets

In this last section, we shall briefly examine a specific optimization problem associated with a defaultable claim. The interested reader is referred to Lukas (2001) for more details on the approach examined in this section.

We now assume that the only risky asset available in the market is

$$dZ_t^1 = Z_t^1(\nu dt + \sigma \, dW_t + \varphi \, dM_t),$$

and we assume that $r = 0$. We deal with the following problem:

$$\sup_{\pi} \mathbb{E}_{\mathbb{P}}\big(u(V_{\tau \wedge T}^v(\pi) + X)\big)$$

for the claim X of the form

$$X = \mathbb{1}_{\{\tau > T\}} g(Z_T^1) + \mathbb{1}_{\{\tau \le T\}} h(Z_\tau^1)$$

for some functions $g, h : \mathbb{R} \to \mathbb{R}$. Note that here the recovery payment is paid at hit, that is, at the time of default. In addition, we assume that the default intensity γ under $\mathbb{P}$ is constant (hence, it is constant under any equivalent martingale measure). After time τ, the market reduces to a standard Black-Scholes model, and thus in this case the solution to our optimization problem is well-known.

In the particular case of the exponential utility function $u(x) = 1 - \exp(-\varrho x)$, $\varrho > 0$, we are in a position to use the duality theory. This problem was studied by, among others, Rouge and El Karoui (2000), Delbaen et al. (2002) and Collin-Dufresne and Hugonnier (2002). In the particular case of the exponential utility function $u(x) = 1 - \exp(-\varrho x)$, $\varrho > 0$, we are in a position to use the duality theory. This problem was studied by, among others, Rouge and El Karoui (2000), Delbaen et al. (2002) and Collin-Dufresne and Hugonnier (2002).

By $H(\mathbb{Q}\,|\,\mathbb{P})$ we denote the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$. Recall that if $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ then

$$H(\mathbb{Q}\,|\,\mathbb{P}) = \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\ln\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = \mathbb{E}_{\mathbb{Q}}\left(\ln\frac{d\mathbb{Q}}{d\mathbb{P}}\right).$$

Otherwise, the relative entropy $H(\mathbb{Q}\,|\,\mathbb{P})$ equals ∞.

It is well known that, under suitable technical assumptions (see Rouge and El Karoui (2000) or Delbaen et al. (2002) for details), we have

$$\sup_{\pi} \mathbb{E}_{\mathbb{P}}\Big(1 - e^{-\varrho(V_T^v(\pi)+X)}\Big) = 1 - \exp\Big(-\inf_{\pi}\inf_{\mathbb{Q}\in\mathcal{Q}_T}\big(H(\mathbb{Q}\,|\,\mathbb{P}) + \varrho\,\mathbb{E}_{\mathbb{Q}}(V_T^v(\pi)+X)\big)\Big)$$

where π runs over a suitable class of admissible trading strategies, and $\mathcal{Q}_T$ stands for the set of equivalent martingale measures on the σ-field $\mathcal{G}_T$.

Since for any admissible trading strategy π the expected value under $\mathbb{Q}$ of the terminal wealth $V_T^v(\pi)$ equals v , we obtain

$$\sup_{\pi} \mathbb{E}_{\mathbb{P}}\Big(1 - e^{-\varrho(V_T^v(\pi)+X)}\Big) = 1 - \exp\Big(-\inf_{\mathbb{Q}\in\mathcal{Q}_T}\big(H(\mathbb{Q}\,|\,\mathbb{P}) + \varrho\,\mathbb{E}_{\mathbb{Q}}X + \varrho v\big)\Big).$$

Furthermore, since, without loss of generality, we may stop all the processes considered here at the default time τ, we end up with the following equality

$$\inf_{\pi} \mathbb{E}_{\mathbb{P}}\Big(e^{-\varrho(V_{T\wedge\tau}^v(\pi)+X)}\Big) = \exp\Big(-\inf_{\mathbb{Q}\in\mathcal{Q}_{T\wedge\tau}}\big(H(\mathbb{Q}\,|\,\mathbb{P}) + \varrho\,\mathbb{E}_{\mathbb{Q}}X + \varrho v\big)\Big),$$

where $\mathcal{Q}_{T\wedge\tau}$ is the set of equivalent martingale measures on $\mathcal{G}_{T\wedge\tau}$.

The following result provides a description of the class $\mathcal{Q}_{T\wedge\tau}$.

Lemma 5.1 *The class $\mathcal{Q}_{T\wedge\tau}$ of all equivalent martingale measures on $(\Omega, \mathcal{G}_{T\wedge\tau})$ is the set of all probability measures $\mathbb{Q}_{k,h}$ of the form*

$$d\mathbb{Q}_{k,h}|_{\mathcal{G}_{T\wedge\tau}} = \eta_{T\wedge\tau}(k,h)\,d\mathbb{P},$$

where the Radon-Nikodym density process $\eta(k,h)$ is given by the formula

$$\eta_t(k,h) = \mathcal{E}_t(kM)\mathcal{E}_t(hW), \quad \forall\, t\in[0,T],$$

for some $\mathbb{F}$-adapted process k such that the inequality $k_t > -1$ holds for every $t \in [0,T]$, and for the associated process $h_t = -\theta - \varphi\gamma\sigma^{-1}(1+k_t)$, where $\theta = \nu/\sigma$. Under the martingale measure $\mathbb{Q} = \mathbb{Q}_{k,h}$, the process

$$W_{t\wedge\tau}^h = W_{t\wedge\tau} - \int_0^{t\wedge\tau} h_s\,ds, \quad \forall\, t\in[0,T],$$

is a stopped Brownian motion, and the process

$$M_{t\wedge\tau}^k = M_{t\wedge\tau} - \int_0^{t\wedge\tau} \gamma k_s\,ds, \quad \forall\, t\in[0,T],$$

is a martingale stopped at τ.

Straightforward calculations show that the relative entropy of an arbitrary martingale measure $\mathbb{Q} = \mathbb{Q}_{k,h} \in \mathcal{Q}_{T\wedge\tau}$ with respect to $\mathbb{P}$ equals

$$\begin{aligned} H(\mathbb{Q}\,|\,\mathbb{P}) &= \mathbb{E}_{\mathbb{Q}}\Big(\int_0^{\tau\wedge T} h_s\, dW_s^h + \int_0^{\tau\wedge T} \big(\tfrac{1}{2}h_s^2 - \gamma k_s + \gamma(1+k_s)\ln(1+k_s)\big)\, ds \Big) \\ &\quad + \mathbb{E}_{\mathbb{Q}}\Big(\int_0^{\tau\wedge T} \ln(1+k_s)\, dM_s^k \Big). \end{aligned}$$

Consequently, the optimization problem

$$\inf_{\mathbb{Q}\in\mathcal{Q}_{T\wedge\tau}} \big(H(\mathbb{Q}\,|\,\mathbb{P}) + \varrho\, \mathbb{E}_{\mathbb{Q}} X \big)$$

can be reduced to the following problem

$$\inf_{k,h} \mathbb{E}_{\mathbb{Q}}\Big(\int_0^{\tau\wedge T} \big(\tfrac{1}{2}h_s^2 - \gamma k_s + \gamma(1+k_s)\ln(1+k_s)\big)\, ds + \varrho X \Big), \tag{32}$$

where the processes k and h are as specified in the statement of Lemma 5.1. Let us set

$$\ell(k_s) = \tfrac{1}{2}h_s^2 - \gamma k_s + \gamma(1+k_s)\ln(1+k_s)$$

so that

$$l(k) = \tfrac{1}{2}\big(\theta + \varphi\gamma(1+k)\big)^2 - \gamma k + \gamma(1+k)\ln(1+k). \tag{33}$$

Consider a dynamic version of the minimization problem (32)

$$\inf_{k,h} \mathbb{E}_{\mathbb{Q}}\Big(\int_t^{\tau\wedge T} \ell(k_s)\, ds + \varrho \mathbb{1}_{\{\tau>T\}} g(Z_T^1) + \varrho \mathbb{1}_{\{\tau\le T\}} h(Z_\tau^1) \,\Big|\, \mathcal{G}_t \Big).$$

Let us denote $K_s^t = e^{-\int_t^s \gamma(1+k_u)\, du}$ for $t \le s$. Then, on the pre-default event $\{\tau > t\}$, we obtain the following problem:

$$\inf_{k,h} \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T K_s^t\big(\ell(k_s) + \varrho\gamma(1+k_s)h(Z_s^1(1+\varphi))\big)\, ds + \varrho K_T^t g(Z_T^1) \,\Big|\, \mathcal{F}_t \Big).$$

The value function $J(t,x)$ of the latter problem satisfies the HJB equation

$$\partial_t J(t,x) + \tfrac{1}{2}\sigma^2 x^2 \partial_{xx} J(t,x) + \inf_{k>-1} \big(-\varphi\gamma(1+k)x\partial_x J(t,x) - \gamma(1+k)J(t,x) + \psi(k,x) \big) = 0$$

with the terminal condition $J(T,x) = \varrho g(x)$, where we denote

$$\psi(k,x) = \ell(k) + \varrho\gamma(1+k)h(x(1+\varphi))$$

and where the function ℓ is given by (33). The minimizer is given by $k = k^*(t,x)$, which is the unique root of the following equation:

$$\frac{\varphi}{\sigma^2}(\nu + \varphi\gamma(1+k)) + \ln(1+k) = J(t,x) + \varphi x \partial_x J(t,x) - \varrho h(x(1+\varphi)),$$

and the optimal strategy π^* is given by the formula

$$\pi_t^* = (\varrho\sigma^2)^{-1}\big(\nu + \varphi\gamma(1 + k^*(t, Z_t^1)) - \sigma^2 Z_{t-}^1 \partial_x J(t, Z_{t-}^1)\big).$$

Remark. Note that in the case $\varphi = 0$ this result is consistent with our result established in Section 3.1. When $\varphi = 0$, the process Z^1 is continuous, and thus we obtain

$$\pi_t^* = (\varrho\sigma)^{-1}\big(\theta - \sigma Z_t^1 \partial_x J(t, Z_t^1)\big),$$

where the value function $J(t,x)$ satisfies the simplified HJB equation

$$\partial_t J(t,x) + \tfrac{1}{2}\sigma^2 x^2 \partial_{xx} J(t,x) + \inf_{k>-1} \big(\ell(k) - \gamma(1+k)J(t,x) + \varrho\gamma(1+k)h(x)\big) = 0,$$

where in turn

$$\ell(k) = \tfrac{1}{2}\theta^2 - \gamma k + \gamma(1+k)\ln(1+k).$$

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