On the paper "Is Itô calculus oversold?" by A. Izmailov and B. Shay

M. Rutkowski and W. Szatzschneider

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The main message of the paper "Is Itô calculus oversold?" by A. Izmailov and B. Shay is, we quote: "However, when applied to the nonlinear interest rate models such as Cox-Ingersoll-Ross, Itô calculus only gives approximate results (...)" This surprising claim is first supported in the introduction by few words on the "diffusion approximation," in which the authors seem to suggest that the Stratonovich integral gives a more exact approximation than the Itô integral. We are, of course, aware about the difference in both kinds of integrals, however, it is unclear to us to what extent this is related to the main topic of the paper. Also, since the authors only mention vaguely the diffusion approximation, without clarifying this important issue, a serious discussion with their statements in this regard is not possible. Let us only mention that the Cox-Ingersoll-Ross (CIR) diffusion-type model is well known to be a good diffusion approximation for several branching processes. Since many other authors' statements are vague and frequently confusing, we shall comment on the purely mathematical aspects of the paper only.

Many references from physical literature are given in the list of references of the original paper; their relevance (if any) to the main topic of the paper is totally obscure, however. It is known that the Wiener process can model the motion of a particle (e.g., the Rayleigh gas) only in very particular cases. Therefore, the use of the Wiener process is a rather crude approximation of majority of real situations. For a detailed discussion of this point and further references, the interested reader is referred to Szatzschneider (1993).

A. Section "The CIR model and procedure"

In this section, the authors consider the short-term rate process $r(t)$, which is governed by the following stochastic differential equation

$$r(t) = a(t)\big(\theta(t) - r(t)\big)\,dt + \sqrt{r(t)}\sigma(t)\,dW_t. \tag{1}$$

where a, θ and σ are continuous functions (it is convenient to assume that $\sigma(t) > 0$ for every t). Typically, the stochastic integral is here understood in the Itô sense. It seems to us that the authors follow this convention – otherwise, they wouldn't be able to derive their equation (3) through an application of Itô's rule. In principle, it is possible to interpret the stochastic integral in (1) in the sense of Stratonovich; this would give rise to a different model of the short-term rate.

By definition, the T-maturity bond price is set to satisfy

$$B(t,T) = \mathbf{E_P}\Big(e^{-\int_t^T r(u)\,du}\,\Big|\,\mathcal{F}_t\Big), \tag{2}$$

where $\mathcal{F}_t$ is the filtration generated by the underlying Wiener process (standard Brownian motion) W. We shall thus write $\mathcal{F}_t = \mathcal{F}_t^W$. It should be observed that in the CIR model the filtration generated by W coincides with the filtration generated by the short-term rate process r; formally $\mathcal{F}_t^W = \mathcal{F}_t^r$ for every $t \in \mathrm{R}_+$. Actually this property holds in most short-term rate models put forward in existing literature. Therefore

$$B(t,T) = \mathbf{E_P}\Big(e^{-\int_t^T r(u)\,du}\,\Big|\,\mathcal{F}_t^r\Big). \tag{3}$$

It should be noticed that the probability measure $\mathbf{P}$ is "automatically" the martingale measure for the term structure model introduced above; indeed, it is obvious that for any maturity T the discounted bond price $B(t,T)/B_t$, where $B_t = \exp(\int_0^t r_u\,du)$ follows a martingale under $\mathbf{P}$.

A.1 The CIR Procedure

The so-called CIR procedure is presented incorrectly in the paper. Since the authors suggest that there are some approximations in this procedure, we shall describe it carefully.

Step 1. One does not assume that $R(t,T)$ is the function of the current value $r(t)$ of the short term rate. In fact, one shows that the bond price $B(t,T)$ is a function of time argument t and the short-term rate process $r(t)$. This is an almost immediate consequence of formula (2), combined with the Markov property of the process r. The authors' discussion about the "missed correlation" (see Page 53) is thus erroneous.

Step 2. We apply Itô's rule to derive the dynamics (that is, the semimartingale representation) of the process $v(t,r(t))$. In fact, we search here for the drift term rather than for the diffusion term, since the knowledge of the exact form of the diffusion term is not necessary for the derivation of the partial differential equation (PDE) which governs the value function $v(t,r)$. We obtain

$$dv(t,r(t)) = \Big(v_t'(t,r(t)) + a(t)\big(\theta(t) - r(t)\big)v_r'(t,r(t)) + \tfrac{1}{2}r(t)\sigma^2(t)v_{rr}''(t,r(t))\Big)\,dt$$
$$+v_r'(t,r(t))\sqrt{r(t)}\sigma(t)\,dW_t.$$

Step 3. In fact, this step is inexistent in the correct procedure.

Step 4. We simply observe that under $\mathbf{P}$ we have

$$dB(t,T) = r(t)B(t,T)\,dt + \zeta_t\,dW_t \tag{4}$$

for some predictable process ζ. Formula (4) is a consequence of the so-called predictable representation property of the Brownian filtration, applied to the martingale $B(t,T)/B_t$

and combined with the Itô formula. Combining formula (4) with the representation obtained in Step 2, we get for every $s \in \mathrm{R}_+$

$$\int_0^s \Big(v_t'(t,r(t)) + a(t)\big(\theta(t) - r(t)\big)v_r'(t,r(t)) + \tfrac{1}{2} r(t)\sigma^2(t)v_{rr}''(t,r(t)) - rv(t,r(t))\Big)\,dt$$

$$= \int_0^s \big(\zeta_t - v_r'(t,r(t))\sqrt{r(t)}\sigma(t)\big)\,dW_t.$$

Since the left-hand side in the last formula defines a process of finite variation, while the right-hand side manifestly follows a continuous (local) martingale, it is clear that both processes vanish. In particular, this yields for every $s \in \mathrm{R}_+$

$$\int_0^s \Big(v_t'(t,r(t)) + a(t)\big(\theta(t) - r(t)\big)v_r'(t,r(t)) + \tfrac{1}{2} r(t)\sigma^2(t)v_{rr}''(t,r(t)) - rv(t,r(t))\Big)\,dt = 0.$$

This leads to the partial differential equation for $v(t,r)$

$$v_t'(t,r) + a(t)\big(\theta(t) - r\big)v_r'(t,r) + \tfrac{1}{2} r\sigma^2(t)v_{rr}''(t,r) - rv(t,r) = 0 \tag{5}$$

with the terminal condition $v(r,T) = 1$ for $r \in \mathrm{R}_+$. It should be noticed that no assumptions other than dynamics (1) and formula (2) were used to derive the PDE (5). Therefore, there is no reason to suspect that the solution to this PDE would give an incorrect result for the bond price. Let us mention that the rigorous approach to the extended CIR model, as defined by (1), can be found in Rogers (1995).

B. Section "Analysis of the CIR Procedure"

To justify their conjecture, the authors consider the special case when the short-term rate equals

$$r(t) = \sigma^2 W_t^2, \quad \forall\, t \in [0,T], \tag{6}$$

where σ is a strictly positive constant. It is worthwhile to notice that formula (9) in the paper makes no sense; the differential $d\sqrt{r(t)}$ should have the form $d\sqrt{r(t)} = \sigma\, d|W_t|$. For obvious reasons, formula (6) above provides a much better representation of the process r introduced in this section. The authors insist that the direct calculations of the expected value

$$I_0 := \mathbf{E_P}\Big(e^{-\int_0^T r(u)\,du}\Big) = \mathbf{E_P}\Big(e^{-\int_0^T \sigma^2 W_u^2\,du}\Big)$$

gives a different result than the PDE approach. Let us mention here that the PDE approach should be correctly referred to as the Feynman-Kac procedure rather, than the CIR procedure. Cox, Ingersoll and Ross (1985) simply used in the context of bond valuation the well-known procedure, which is presented in most textbooks on stochastic calculus. Any person familiar with the topic easily recognizes that the discrepancy suggested by the authors is simply not possible (provided, of course, that all calculations are done properly).

It is noteworthy to mention that I_0 is simply the Laplace transform of the linear functional of the squared Wiener process. No stochastic integration whatsoever (whether of the Itô type or of any other kind) is directly involved in the definition of I_0.

B.1 Direct Approach.

The authors make here a simple error; namely, they have miscalculated the expected value

$$\mathbf{E_P}\Big(\int_t^T \sigma^2 W_u^2\,du\Big)^3 = \tfrac{139}{120}(\sigma T)^6.$$

The right answer should thus read

$$I_0 = 1 - \tfrac{1}{2}(\sigma T)^2 + \tfrac{7}{24}(\sigma T)^4 - \tfrac{139}{720}(\sigma T)^6 + \ldots$$

and not

$$I_0 = 1 - \tfrac{1}{2}(\sigma T)^2 + \tfrac{7}{24}(\sigma T)^4 - \tfrac{152}{720}(\sigma T)^6 + \ldots$$

as is claimed in the paper (see formula (10) therein). In fact, standard calculations show that for any $0 \le u \le s$

$$\mathbf{E_P}\big(W_s^2 W_u^2\big) = su + 2u^2,$$

and

$$\mathbf{E_P}\big(W_t^2 W_s^2 W_u^2\big) = (t-s)\mathbf{E_P}\big(W_s^2 W_u^2\big) + \mathbf{E_P}\big(W_s^4 W_u^2\big) = tsu + 2tu^2 + 2s^2u + 10su^2$$

for any $0 \le u \le s \le t$. Consequently,

$$\mathbf{E_P}\Big(\int_0^T \sigma^2 W_v^2\,dv\Big)^2 = 2!\,\sigma^4 \int_0^T\int_0^s \mathbf{E_P}\big(W_s^2 W_u^2\big)\,duds = \tfrac{7}{12}(\sigma T)^4$$

and

$$\mathbf{E_P}\Big(\int_0^T \sigma^2 W_v^2\,dv\Big)^3 = 3!\,\sigma^6 \int_0^T\int_0^t\int_0^s \mathbf{E_P}\big(W_t^2 W_s^2 W_u^2\big)\,dudsdt = \tfrac{139}{120}(\sigma T)^6.$$

B.2 PDE Approach.

To derive the PDE for the bond price, one needs to find the stochastic differential equation which governs the short-term rate process. To this end, we apply Itô's formula to (7) and we get ($\operatorname{sgn}(x) = 1$ if $x \ge 0$, and equals -1 otherwise)

$$dr(t) = \sigma^2 dW_t^2 = 2\sigma^2 |W_t| \operatorname{sgn}(W_t)\,dW_t + \sigma^2\,dt,$$

or equivalently

$$dr(t) = \sigma^2 dW_t^2 = 2\sigma\sqrt{r(t)}\,d\tilde{W}_t + \sigma^2\,dt,$$

where the process $\tilde{W}$, which is given by the formula

$$\tilde{W}_t = \int_0^t \mathrm{sgn}\,(W_u)\,dW_u,$$

is another Brownian motion under **P**. Consequently, the PDE for the bond price has the form (cf. formula (5) above)

$$v'_t(t,r) + \sigma^2 v'_r(t,r) + 2r\sigma^2 \tfrac{1}{2} v''_{rr}(t,r) - rv(t,r) = 0. \tag{7}$$

As was already explained, no approximation of any kind is involved in the derivation of the PDE (7). Therefore, an exact solution of this PDE, subject to the terminal condition $v(r,T) = 1$, does necessarily satisfy the equality $v(0,T) = I_0$. The explicit solution provided in the paper (see formula (14) therein) only support this self-evident statement. Finally, let us mention that this formula is known since the forties; it is usually referred to as the Cameron-Martin formula. The interested reader may consult the original paper by Cameron and Martin (1945), or the monograph by Revuz and Yor (1994) (see Page 425 therein) in this regard.

B.3 Martingale Approach.

It should be noticed the Cameron-Martin formula can be easily derived with the use of the so-called exponential martingales. Once again, the reader is referred to Revuz and Yor (1994) (see Pages 424-425 therein). Let us reproduce here their arguments, adapted to the simple case analysed here. For $\sigma = 1$, we wish to evaluate the conditional expectation:

$$I_t := \mathbf{E}_{\mathbf{P}}\Big(e^{-\int_t^T W_u^2\,du}\,\Big|\,\mathcal{F}_t\Big).$$

Let $f : [0,T] \to \mathrm{R}$ be an auxiliary continuous function, continuously differentiable in $(0,T)$. It is well known that for any $t \le T$ we have

$$J_t(f) := \mathbf{E}_{\mathbf{P}}\Big\{\exp\Big(\int_t^T f(u)W_u\,dW_u - \frac{1}{2}\int_t^T f^2(u)W_u^2\,du\Big)\,\Big|\,\mathcal{F}_t\Big\} = 1.$$

Since $dW_t^2 = 2W_t\,dW_t + t$, we deduce that

$$J_t(f) = \mathbf{E}_{\mathbf{P}}\Big\{\exp\Big(\frac{1}{2}\int_t^T f(u)\,dW_u^2 - \frac{1}{2}\int_t^T f(u)\,du - \frac{1}{2}\int_t^T f^2(u)W_u^2\,du\Big)\,\Big|\,\mathcal{F}_t\Big\}. \tag{8}$$

The Itô integration by parts formula yields:

$$\int_t^T f(u)\,dW_u^2 = f(T)W_T^2 - f(t)W_t^2 - \int_t^T f'(u)W_u^2\,du.$$

Plugging the last expression into (8) and rearranging, we obtain

$$J_t(f) = e^{-\frac{1}{2}\int_t^T f(u)\,du}\,\mathbf{E_P}\Big\{\exp\Big(\frac{f(T)W_T^2 - f(t)W_t^2}{2} - \frac{1}{2}\int_t^T \big(f'(u)+f^2(u)\big)W_u^2\,du\Big)\,\Big|\,\mathcal{F}_t\Big\}.$$

In the next step, we choose the function f in such a way that

$$f'(u) + f^2(u) = 2, \quad f(T) = 0. \tag{9}$$

Then we get

$$J_t(f) = e^{-\frac{1}{2}\int_t^T f(u)\,du}\,\mathbf{E_P}\Big\{\exp\Big(-\frac{1}{2}f(t)W_t^2 - \int_t^T W_u^2\,du\Big)\,\Big|\,\mathcal{F}_t\Big\} = 1,$$

and finally

$$I_t = \exp\Big(\frac{1}{2}\int_t^T f(u)\,du + \frac{1}{2}f(t)W_t^2\Big).$$

It remains to determine explicitly the function f that solves equation (9). To this end, we set $f(u) = h'(u)/h(u)$ for some twice continuously differentiable function h, where by convention we set $h(t) = 1$. In view of (9), it is easily seen that h satisfies the ordinary differential equation $h''(u) = 2h(u)$, $u \in (t,T)$, with suitable boundary conditions.

Solving the last equation, we obtain $h(u) = Ae^{\sqrt{2}u} + Be^{-\sqrt{2}u}$, and thus

$$f(u) = \frac{h'(u)}{h(u)} = \frac{\sqrt{2}\big(Ae^{\sqrt{2}u} - Be^{-\sqrt{2}u}\big)}{Ae^{\sqrt{2}u} + Be^{-\sqrt{2}u}}\,,$$

where the values of constants A and B are found from the equalities (recall that $h(t) = 1$ and $f(T) = 0$)

$$Ae^{\sqrt{2}t} + Be^{-\sqrt{2}t} = 1, \quad Ae^{\sqrt{2}T} - Be^{-\sqrt{2}T} = 0.$$

Observe now that

$$\int_t^T f(u)\,du = \int_t^T h'(u)h^{-1}(u)\,du = \int_t^T d\ln h(u) = \ln h(T) - \ln h(t) = \ln h(T)$$

and $f(t) = h'(t)$. We thus finally obtain

$$I_t = \exp\Big(\tfrac{1}{2}\ln h(T) + \tfrac{1}{2}h'(t)W_t^2\Big) = \sqrt{h(T)}\exp\big(\tfrac{1}{2}h'(t)W_t^2\big).$$

Elementary calculations now lead to the well-known explicit formula for I_t (cf. equality (14) in the paper). It is worthwhile to notice that the same approach can be applied to the extended CIR model, as defined by the dynamics (1) (see Szatzschneider and Flores-López (1997)). The only difference is that the explicit solution for the corresponding Sturm-Liouville ordinary differential equation is usually not available (one needs thus to employ approximate solution which can be obtained through an appropriate numerical procedure).

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Marek Rutkowski
Warsaw University of Technology, 00-661 Warszawa, Poland
e-mail: markrut@alpha.mini.pw.edu.pl

Wojciech Szatzschneider
Universidad Anáhuac, Huixquilucan, Mexico, C.P. 52760
e-mail: wojciech@anahuac.mx