HAUSDORFF DIMENSION OF SETS OF ESCAPING POINTS AND ESCAPING PARAMETERS FOR ELLIPTIC FUNCTIONS

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Abstract. Let $f: \mathbb{C} \to \mathbb{C}$ be a non-constant elliptic function. We prove that the Hausdorff dimension of the escaping set of $f$ equals $2q/(q + 1)$, where $q$ is the maximal multiplicity of poles of $f$. We also consider the escaping parameters in the family $f_\beta = \beta f$, i.e. the parameters $\beta$ for which the orbit of one critical value of $f_\beta$ escapes to infinity. Under additional assumptions on $f$ we prove that the Hausdorff dimension of the set of escaping parameters $E$ in the family $f_\beta$ is greater than or equal to the Hausdorff dimension of the escaping set in the dynamical space. It shows an analogy between dynamical plane and parameter space in the class of transcendental meromorphic functions. We also construct a subset of $E$ whose Hausdorff dimension equals $2q/(q + 1)$. We extend these results to simply-periodic meromorphic functions.

1. Introduction

The understanding of the dynamics and geometry of elliptic functions rapidly develops since the papers [5, 11, 12] have been published. Although these functions are relatively ‘regular’, they manifest such unexpected features as the fact that the Hausdorff dimension of their Julia set is always larger than 1 (see [11]) or, in the non-recurrent case, that the corresponding Hausdorff measure always vanishes whereas the packing measure, in the absence of parabolic points, is finite and positive (see [12]). A systematic exposition of the geometric measure theory and ergodic theory of regular pseudo-nonrecurrent elliptic functions is given in [13]. In spite of possible associations steaming from the name, this is not a narrow class of functions.

Let $f: \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. For $n \in \mathbb{N}$, denote by $f^n$ the $n$-th iterate of $f$. The Fatou set $F(f)$ of $f$ is the set of points $z \in \mathbb{C}$ such that all iterates $f^n(z)$ are well-defined and $\{f^n\}_{n \in \mathbb{N}}$ forms a normal family in some neighborhood of $z$. The complement $J(f)$ of $F(f)$ in $\overline{\mathbb{C}}$ is called the Julia set of $f$. P. Domínguez in [4] proved that for transcendental meromorphic functions with poles the escaping set

$$I(f) = \{z \in \mathbb{C} : \lim_{n \to \infty} f^n(z) = \infty\}$$

is non-empty and $J(f) = \partial I(f)$. Later, P. Rippon and G. Stallard [15] showed that if additionally a function $f$ is in the Eremenko-Lyubich class $B$, then $I(f) \subset J(f)$, which follows that $\text{Int } I(f) = \emptyset$. Several authors e.g. [1, 2, 3, 9, 16, 17] have studied properties of the escaping set of entire and meromorphic functions. The Hausdorff dimension $\text{dim}_H(I(f))$ of the escaping set for some class of meromorphic function was estimated

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from below by J. Kotus in [10]. Her result follows that if \( f \) is an elliptic function such that the closure of the postcritical set is disjoint from the set of poles, then

\[
\dim_H(I(f)) \geq \frac{2q}{q+1} ,
\]

(1.1)

where \( q \) is the maximal multiplicity of poles of \( f \). The upper bound on \( \dim_H(I(f)) \) for elliptic functions was proved by Bergweiler, Kotus and Urbański [3, 11] to be equal to the lower bound, i.e.

\[
\dim_H(I(f)) \leq \frac{2q}{q+1} .
\]

(1.2)

Our paper is divided into three parts. The first one (Sections 2 and 3) focuses on the generalization of (1.1) to the whole class of elliptic functions. Together with the estimate (1.2) it gives the following.

**Theorem 1.** Let \( f \) be a non-constant elliptic function. Then,

\[
\dim_H(I(f)) = \frac{2q}{q+1} ,
\]

where \( q \) is the maximal multiplicity of poles of \( f \).

The second part (Sections 4 and 5) is devoted to the estimates of the Hausdorff dimension of the escaping set in the parameter space for some families of elliptic functions. Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be an elliptic function such that one of its critical values, denoted by \( f(c_1) \neq 0 \), is a pole of the maximal multiplicity \( q \) and all the other critical values are attracted by attracting periodic points. We define a one-parameter family of functions \( f_\beta = \beta f, \beta \in \mathbb{C} \setminus \{0\} \). As a counterpart of escaping set \( I(f_\beta) \) we consider the set of escaping parameters in the family \( f_\beta \), i.e.

\[
\mathcal{E} := \{ \beta \in B(1, r) : \lim_{n \to \infty} f_\beta^n(c_1) = \infty \},
\]

(1.3)

where \( 0 < r < 1/4 - 1/(2\alpha + 4) \approx 0.04 \) with \( \alpha = \sin(\pi/8) = \sqrt{2 - \sqrt{2}}/2 \). The main result of the second part of the paper is the following theorem.

**Theorem 2.** Let \( f_\beta = \beta f, \beta \in \mathbb{C} \) be a one-parameter of family of elliptic functions such that one of the critical values of \( f \), denoted by \( f(c_1) \neq 0 \), is a pole of the maximal multiplicity \( q \) and all the other critical values of \( f \) are attracted by attracting periodic points. Then

\[
\dim_H(\mathcal{E}) \geq \frac{2q}{q+1} .
\]

**Corollary 1.** For \( q \searrow \infty \) we have \( \dim_H(\mathcal{E}) \geq \dim_H(I(f)) \searrow 2. \)

So far the equality \( \dim_H(\mathcal{E}) = \dim_H(I(f)) \) has been proved for the exponential family (see [14, 18]). However we would like to point out that in the exponential case it was enough to prove the lower bounds on \( \dim_H(I(f)) \) and \( \dim_H(\mathcal{E}) \).

We also consider a more general class of functions. Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be an elliptic function such that one of its critical values, denoted by \( f(c_1) \neq 0 \), is a pole of a given multiplicity \( p \) (not necessary maximal) and all the other critical values are attracted by attracting periodic points. We define the set
\[ G_R := \{ \beta \in B(1,r) : |f^\beta_n(c_1)| > R > 0 \quad \text{for} \quad n \geq 2 \}, \tag{1.4} \]

where \( r \) is chosen as above.

**Theorem 3.** Let \( f_\beta = \beta f, \beta \in \mathbb{C} \) be a one-parameter of family of elliptic functions such that one of the critical values of \( f \), denoted by \( f(c_1) \neq 0 \), is a pole of a given multiplicity \( p \leq q \) and all the other critical values of \( f \) are attracted by attracting periodic points. Then, for sufficiently large \( R \),

\[ \dim_H(G_R) \leq \frac{2q}{q+1}. \]

At the end of Section 5 we prove the following theorem.

**Theorem 4.** There is a subset \( E_0 \) of \( E \) such that

\[ \dim_H(E_0) = \frac{2q}{q+1}. \]

In the third part of the paper (Section 6) we give some comments on our results. It turns out that the presented methods show that the inequalities \( \dim_H(G_R) \leq \dim_H(I(f)) \leq \dim_H(E_0) = \dim_H(I(f)) \) can be also proved for a family \( \beta g, \) where \( g \) is a simply periodic meromorphic function satisfying the same assumptions as the elliptic function \( f \).

## 2. Notations and preliminary estimates

We fix a non-constant elliptic function \( f : \mathbb{C} \to \overline{\mathbb{C}} \). Then \( f \) is periodic with respect to some lattice \( \Lambda \subset \mathbb{C} \) defined by \( \Lambda = [\lambda_1, \lambda_2] = \{l\lambda_1 + m\lambda_2 : l, m \in \mathbb{Z}\} \), where \( \lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\} \) such that \( \text{Im}(\lambda_1/\lambda_2) \neq 0 \). Equivalently one may write \( \Lambda = [1, \lambda_1] \), where \( \text{Im}\lambda_1 \neq 0 \). Let

\[ \mathcal{R} = \{t_1 + t_2\lambda_2 : 0 \leq t_1, t_2 \leq 1\} \tag{2.1} \]

be the basic fundamental parallelogram of \( f \). It follows from periodicity of \( f \) that \( f^{-1}(\infty) = \bigcup_{l,m \in \mathbb{Z}}(\mathcal{R} \cap f^{-1}(\infty) + l + m\lambda_1) \). For each pole \( b \) of \( f \), let \( q_b \) denote its multiplicity defined as

\[ q := \sup\{q_b : b \in f^{-1}(\infty)\} = \max\{q_b : b \in f^{-1}(\infty) \cap \mathcal{R}\}. \tag{2.2} \]

Let \( \{c_i \in \mathcal{R} : f'(c_i) = 0, i = 1, \ldots, k\} \) be the set of critical points of \( f \) from the basic fundamental parallelogram \( \mathcal{R} \) defined in (2.1). Let \( \rho > 0 \) be such that

\[ \text{all critical values of } f \text{ are contained in } B(0,\rho - 1). \tag{2.3} \]

We fix a pole

\[ b_0 \in \mathbb{C} \setminus B(0,\rho) \tag{2.4} \]

of maximal multiplicity \( q \). In Section 2 and Section 3, we consider only poles of multiplicity \( q \) from the set

\[ f_q^{-1}(\infty) = \{b_0 + l + m\lambda_1 : l, m \in \mathbb{Z}\} \cap (\mathbb{C} \setminus B(0,\rho)). \tag{2.5} \]

Since \( f \) is periodic, there exist a constant

\[ 0 < \varepsilon_0 < \frac{1}{3} \min\{1, |\lambda_1|, |b_0| - \rho\} \tag{2.6} \]
and holomorphic functions $G, H$ such that for each pole $b \in f_q^{-1}(\infty)$ and all $z \in B(b, \varepsilon_0)$ we have

$$f(z) = \frac{G(z)}{(z - b)^q}, \quad f'(z) = \frac{H(z)}{(z - b)^{q+1}},$$

where $G(b) = a_{-q} \neq 0$, $H(b) = b_{-q-1} \neq 0$. Shrinking $\varepsilon_0$, if necessary, we may assume that $G(z) \neq 0$ and $H(z) \neq 0$ for $z \in B(b, \varepsilon_0)$. The periodicity of $f$ implies that there exist universal constants $C_1, C_2 > 0$ such that

$$C_1^{-1} \leq |G(z)| \leq C_1, \quad C_2^{-1} \leq |H(z)| \leq C_2$$

on all balls $B(b, \varepsilon_0)$. Hence,

$$\frac{C_1^{-1}}{|z - b|^q} \leq |f(z)| \leq \frac{C_1}{|z - b|^q} \quad (2.7)$$

and

$$\frac{C_2^{-1}}{|z - b|^{q+1}} \leq |f'(z)| \leq \frac{C_2}{|z - b|^{q+1}} \quad (2.8)$$

for all $b \in f_q^{-1}(\infty)$ and $z \in B(b, \varepsilon_0)$. Shrinking $\varepsilon_0$ if necessary we can choose constants $M_1, M_2, 0 < M_2 - M_1 < \pi/4$ such that

$$M_1 < \arg(G(z)) < M_2 \quad (2.9)$$

for all $b \in f_q^{-1}(\infty)$ and $z \in B(b, \varepsilon_0)$. Let

$$U(b, \varepsilon) := \{z \in \mathbb{C}: -\frac{3\pi}{4q} \leq \arg(z - z_0) \leq \frac{3\pi}{4q}, |z - z_0| \leq \varepsilon\}, \quad (2.10)$$

where $b \in f_q^{-1}(\infty)$ and $\varepsilon > 0$ is such that the following conditions are simultaneously satisfied

$$\varepsilon < \varepsilon_0, \quad \text{and } f \text{ is one-to-one in each of the segments defined in (2.10).} \quad (2.11)$$

We take $R_1$ such that

$$U(b_0, \varepsilon) \subset P(0, R_1, 2R_1) := \{z \in \mathbb{C}: R_1 < |z| < 2R_1\}.$$

For each $b \in f_q^{-1}(\infty)$ we get

$$\{z \in \mathbb{T}: |z| \geq \frac{C_1}{\varepsilon^q}, -\frac{3\pi}{4} + M_2 \leq \arg z \leq \frac{3\pi}{4} + M_1\} \subset f(U(b, \varepsilon))$$

$$\subset \{z \in \mathbb{T}: |z| \geq \frac{C_1^{-1}}{\varepsilon^q}, -\frac{3\pi}{4} + M_1 \leq \arg z \leq \frac{3\pi}{4} + M_2\}.$$

Since $0 < M_2 - M_1 < \pi/4$, there exists $\phi \in \mathbb{R}$ such that

$$\{z \in \mathbb{T}: |z| \geq \frac{C_1}{\varepsilon^q}, \phi - \frac{\pi}{8} \leq \arg z \leq \phi + \frac{9\pi}{8}\} \subset f(U(b, \varepsilon)). \quad (2.12)$$

We choose $\tilde{R}_2$ such that

$$\tilde{R}_2 > \max\left\{\frac{C_1}{\varepsilon^q}, \rho\right\}. \quad (2.13)$$
Let $a_1 = \tilde{R}_2/R_1 > \frac{C_1}{C_0 R_1}$. Now, we define a constant

$$a_0 = \max \left\{ 2, a_1, \frac{3}{R_1}, \frac{6^{2q} C_1^{2(q+1)}}{C_2 R_1^{2q+1}}, \left( \frac{2 \varepsilon C_1^{\frac{q}{2q+1}}}{C_2 R_1^{\frac{q}{2q+1}}} \right)^{\frac{q}{q+1}}, \frac{\sqrt{C_1}}{C_2^{\frac{q}{2q+1}} \sqrt{R_1}} \right\}. \quad (2.14)$$

Fix

$$a > a_0$$

and consider a sequence of radii

$$R_k := a^{k-1} R_1, \quad k \geq 2. \quad (2.15)$$

Let

$$P(0, R_k, 2R_k) := \{ z \in \mathbb{C} : R_k < |z| < 2R_k \}, \quad k \geq 2$$

and

$$P^+(0, R_k, 2R_k) := \{ z \in \mathbb{C} : R_k < |z| < 2R_k, \ \phi < \arg z < \phi + \pi \}, \quad k \geq 2. \quad (2.16)$$

The condition $a > a_0 \geq 2$ guarantees that the annuli $P(0, R_k, 2R_k)$ are pairwise disjoint. It follows from (2.12) that

$$\{ z \in \mathbb{C} : |z| > R_2 \geq \tilde{R}_2, \ \phi \leq \arg z \leq \phi + \pi \} \subset f(U(b, \varepsilon)) \quad (2.17)$$

for all poles $b \in f_q^{-1}(\infty)$. We consider the iterates $f^n, n \in \mathbb{N}$, which are defined outside a countable set of points.

**Definition 2.1.** We define the following family of sets

$$\mathcal{A}_1(a) = \{ A_1 = U(b_0, \varepsilon) \},$$

$$\mathcal{A}_2(a) = \{ A_2 \subset A_1 \mid \exists b^{(2)} \in f_q^{-1}(\infty) : U(b^{(2)}, \varepsilon) \subset P^+(0, R_2, 2R_2), \ A_2 \text{ is a component of } f^{-1}(U(b^{(2)}, \varepsilon)) \},$$

$$\ldots$$

$$\mathcal{A}_k(a) = \{ A_k \subset A_{k-1} \mid \exists b^{(k)} \in f_q^{-1}(\infty) : U(b^{(k)}, \varepsilon) \subset P^+(0, R_k, 2R_k), \ A_k \text{ is a component of } f^{-(k-1)}(U(b^{(k)}, \varepsilon)) \},$$

$$\ldots$$

Let

$$U_k(a) = \bigcup_{A_k \in \mathcal{A}_k(a)} A_k, \quad A(a) = \bigcap_{k=1}^{\infty} U_k(a).$$

**Proposition 2.2.** For each $n \in \mathbb{N}$, the set $\mathcal{A}_n(a)$ defined above is non-empty.

**Proof.** Observe that the function $f$ has a pole at $b_0 \in \partial A_1$. Since $A_1 = U(b_0, \varepsilon)$, it follows from (2.17) that

$$f(A_1) \supset P^+(0, R_2, 2R_2).$$

Take a pole $b^{(2)} \in f_q^{-1}(\infty) \cap P^+(0, R_2, 2R_2)$ with $U(b^{(2)}, \varepsilon) \subset P^+(0, R_2, 2R_2)$. Since $f(A_1) \supset P^+(0, R_2, 2R_2)$ there exists $z^{(2)} \in A_1$ such that $f(z^{(2)}) = b^{(2)}$. Thus, the set $\mathcal{A}_2(a)$ is non-empty. Now, we fix $n \geq 3$ and suppose that $\mathcal{A}_{n-1}(a) \neq \emptyset$. We will show
that \( A_n(a) \neq \emptyset \). Since 
\[
\begin{equation}
 f^{n-2}(A_{n-1}) = U(b^{(n-1)}, \varepsilon) \text{ for some } b^{(n-1)} \in f_q^{-1}(\infty) \cap P^+(0, R_{n-1}, 2R_{n-1}),
\end{equation}
\]
it follows from (2.17) that
\[
 f^{n-1}(A_{n-1}) \supset P^+(0, R_n, 2R_n),
\]
as \( R_n = a^{n-2}R_2 \) and \( a > a_0 \geq 2 \) in view of (2.14). Choosing \( z^{(n)} \in A_{n-1} \) such that
\[
 f^{n-1}(z^{(n)}) = b^{(n)} \in f_q^{-1}(\infty) \cap P^+(0, R_n, 2R_n) \text{ and } U(b^{(n)}, \varepsilon) \subset P^+(0, R_n, 2R_n),
\]
we obtain that \( A_n(a) \neq \emptyset \). By induction, the proposition is true for all \( n \in \mathbb{N} \). \( \square \)

Now we prove the following.

**Theorem 2.3.** Let \( f \) be a non-constant elliptic function and let \( a_0 \) be the constant given in (2.14). Then, for every \( a > a_0 \) there is a Cantor subset \( A(a) \) of \( I(f) \) and for this subset
\[
 \dim_H(A(a)) \geq \frac{2q}{q+1} - \frac{6 \log 2}{\log a}.
\]
Since \( \dim_H(A(a)) \geq 2q/(q+1) - 6 \log 2/\log a \not\nearrow 2q/(q+1) \) for \( a \not\nearrow \infty \), then Theorem 1 follows from Theorem 2.3.

In order to prove the lower bound on \( \dim_H(I(f)) \) we use the following theorem proved by C. McMullen in [14].

**Proposition 2.4.** For each \( n \in \mathbb{N} \), let \( A_n \) be a finite collection of disjoint compact subsets of \( \mathbb{R}^d \), each of which has positive \( d \)-dimensional Lebesgue measure. Define \( U_n = \bigcup_{A_n \in A_n} A_n \) and \( A = \bigcap_{n=1}^{\infty} U_n \). Suppose that for each \( A_n \in A_n \) there is \( A_{n+1} \in A_{n+1} \) and a unique \( A_{n-1} \in A_{n-1} \) such that \( A_{n+1} \subset A_n \subset A_{n-1} \). If \( \Delta_n, d_n \) are such that, for each \( A_n \in A_n \),
\[
 \frac{\text{vol}(U_{n+1} \cap A_n)}{\text{vol}(A_n)} \geq \Delta_n > 0, \quad \text{diam}(A_n) \leq d_n < 1 \quad \text{and} \quad d_n \not\nearrow 0,
\]
then \( \dim_H(A) \geq d - \lim_{n \rightarrow \infty} \sum_{j=1}^{n} \frac{|\log \Delta_j|}{|\log d_n|} \).
for some \( b \in f_q^{-1}(\infty) \cap P(0, R_j, 2R_j) \). The inequality (3.2) implies that \( \frac{C_1}{2R_{j+1}} \leq |z - b|^q \leq \frac{C_1}{R_{j+1}} \). Then,

\[
\frac{C_2}{\left( \frac{C_1}{R_{j+1}} \right)^{2+1 \over q}} \leq |f'(f^{j-1}(z))| \leq \frac{C_2}{\left( \frac{C_1}{2R_{j+1}} \right)^{2+1 \over q}}
\]

or, equivalently,

\[
\frac{C_2R_j^{q+1 \over q}}{C_1^{q+1 \over q}} \leq |f'(f^{j-1}(z))| \leq \frac{2^{q+1 \over q} C_2 R_j^{q+1 \over q}}{C_1^{q+1 \over q}} \tag{3.3}
\]

for \( z \in \mathcal{U}_n, n \geq 2 \) and \( j \in \{1, 2, \ldots, n-1\} \).

The first lemma is devoted to estimates of the derivatives \((f^n)'\), \( n \geq 2 \).

**Lemma 3.1.** Let \( A_n \in \mathcal{A}_n, n \geq 2 \). Then for every \( z \in A_n \)

\[
\left( \frac{C_2}{C_1^{q+1 \over q}} \right)^{n-1} a^{(q+1)(n-1) \over 2q} R_1^{(q+1) \over q} \leq |(f^{n-1})'(z)| \leq \left( \frac{2^{q+1 \over q} C_2}{C_1^{q+1 \over q}} \right)^{n-1} a^{(q+1)(n-1) \over 2q} R_1^{(q+1)(n-1) \over q}.
\]

**Proof.** We know that

\[
(f^{n-1})'(z) = \prod_{k=0}^{n-2} f'(f^k(z))
\]

for all \( n \geq 1 \) and every \( z \in A_n \). Since \( f^k(z) \in P^+(0, R_{k+1}, 2R_{k+1}) \), then using (3.3) we get

\[
\left| \prod_{k=0}^{n-2} f'(f^k(z)) \right| \leq \left( \frac{2^{q+1 \over q} C_2 R_2^{q+1 \over q}}{C_1^{q+1 \over q}} \right) \ldots \left( \frac{2^{q+1 \over q} C_2 R_n^{q+1 \over q}}{C_1^{q+1 \over q}} \right) = \left( \frac{2^{q+1 \over q} C_2}{C_1^{q+1 \over q}} \right)^{n-1} (aR_1)^{q+1 \over q} \ldots (a^{n-1} R_1)^{q+1 \over q}
\]

Analogously, we get the estimate from below

\[
\left| \prod_{k=0}^{n-2} f'(f^k(z)) \right| \geq \left( \frac{C_2}{C_1^{q+1 \over q}} \right)^{n-1} a^{(q+1)(n-1) \over 2q} R_1^{(q+1)(n-1) \over q}.
\]

Finally,

\[
\left( \frac{C_2}{C_1^{q+1 \over q}} \right)^{n-1} a^{(q+1)(n-1) \over 2q} R_1^{(q+1)(n-1) \over q} \leq |(f^{n-1})'(z)| \leq \left( \frac{2^{q+1 \over q} C_2}{C_1^{q+1 \over q}} \right)^{n-1} a^{(q+1)(n-1) \over 2q} R_1^{(q+1)(n-1) \over q} \tag{3.4}
\]

The next lemma follows immediately from Lemma 3.1.
**Lemma 3.2.** Let \( A_n \in \mathcal{A}_n, n \geq 2 \). Then the distortion \( L(f^{n^{-1}}, A_n) \leq 2^{\frac{(q+1)(n-1)}{q}} \) and
\[
\text{diam}(A_n) \leq \frac{2\varepsilon}{\left(\frac{C_2}{C_1} - 1\right)^{(n-1)} \cdot \frac{a}{a^q} \cdot \frac{R_1^{(q+1)(n-1)}}{(q+1)^q}} ,
\]
where \( \varepsilon \) is as in (2.11).

**Remark 3.3.** Observe that \( \text{diam}(A_n) \to 0 \) as \( n \to \infty \), since \( a > a_0 \geq 2 \). This proves that the set \( A \) from Definition 2.1 is a Cantor set of parameters.

By Lemma 3.2, the numbers \( d_n \) defined in Proposition 2.4 are equal to
\[
d_n = 2\varepsilon \left(\frac{C_2}{C_1} - 1\right)^{(n-1)} \cdot \frac{a}{a^q} \cdot \frac{R_1^{(q+1)(n-1)}}{(q+1)^q} , \quad n \geq 2 ,
\]
and \( d_1 = \text{diam}(A_1) \leq 2\varepsilon < 1 \) as \( \varepsilon < \varepsilon_0 < \frac{1}{3} \). A straightforward calculation shows that
\[
d_2 = \frac{2\varepsilon C_1^{q+1}}{C_2a^{q+1}R_1^{q+1}} < 1 \quad \text{if and only if} \quad a > \left(\frac{2\varepsilon C_1^{q+1}}{C_2R_1^{q+1}}\right)^{q+1}.
\]
Using (3.5), we get \( \frac{d_{n+1}}{d_n} = \frac{C_1^{q+1}}{C_2a^{q+1}R_1^{q+1}} \). Then \( d_3 < 1 \) if and only if \( a > \frac{\sqrt{C_1}}{C_2^{2(q+1)}R_1^{q+1}} \).

Since \( a > a_0 \geq \max \left\{ 2, \left(\frac{2\varepsilon C_1^{q+1}}{C_2R_1^{q+1}}\right)^{q+1}, \frac{\sqrt{C_1}}{C_2^{2(q+1)}R_1^{q+1}} \right\} \) and \( d_{n+1}/d_n < d_3/d_2 \) for \( n \geq 3 \), we get \( d_n < 1, n = 2, 3, \ldots \) as required in Proposition 2.4.

**Lemma 3.4.** There exists \( M > 0 \) such that
\[
\frac{\text{vol}(U_{n+1} \cap A_n)}{\text{vol}(A_n)} \geq \frac{M}{2^{-\frac{(q+1)(n-1)}{q}}R_{n+1}^{\frac{2}{q}-2}},
\]
for each \( A_n \in \mathcal{A}_n, n \geq 2 \). Moreover, \( \frac{\text{vol}(\partial f^{-1}(\infty))}{\text{vol}(A_n)} \geq M' R_2^{-\frac{2}{q}}, \) for some \( M' > 0 \).

**Proof.** First, we estimate the number \( N_k \) of parallelograms of the lattice \( \Lambda \) in the half-annulus \( P^+(0, R_k, 2R_k) \) for \( k \geq 2 \). We have
\[
N_k \simeq \frac{4\pi R_k^2 - \pi R_k^2}{2a^2(\Lambda)} = \frac{3\pi R_k^2}{2a^2(\Lambda)},
\]
where \( a^2(\Lambda) \) is the measure of each parallelogram of \( \Lambda \). Since we choose only one pole of multiplicity \( q \) in each parallelogram, we may write
\[
\# (f_q^{-1}(\infty) \cap P^+(0, R_k, 2R_k)) \simeq N_k.
\]
(3.6)

Recall that in Definition 2.1 we considered the segments
\[
U(b, \varepsilon) = \{ z \in \mathbb{C} : -\frac{3\pi}{4q} \leq \text{Arg}(z - b) \leq \frac{3\pi}{4q}, |z - b| \leq \varepsilon \},
\]
where \( b \in f_q^{-1}(\infty) \) and \( \varepsilon > 0 \) as in (2.11). Hence, \( \text{vol}(U(b, \varepsilon)) = 3\pi \varepsilon^2/(4q) \).

Take some \( n \geq 2 \) and \( A_n \in \mathcal{A}_n \). There exists \( b^{(n)} \in f_q^{-1}(\infty) \) such that \( A_n \) is a component of \( f^{-(n-1)}(U(b^{(n)}), \varepsilon)) \), where \( U(b^{(n)}, \varepsilon) \subset P^+(0, R_n, 2R_n) \). Moreover, for each \( A_k \in \mathcal{A}_{n+1} \) there is \( b^{(n+1)} \in f_q^{-1}(\infty) \) such that \( A_k \) is a component of \( f^{-n}(U(b^{(n+1)}), \varepsilon)) \), \( U(b^{(n+1)}, \varepsilon) \subset P^+(0, R_{n+1}, 2R_{n+1}) \). To simplify the formulas we denote \( b^{(n)} \) by \( b_n \). There are finitely many sets \( A_k \in \mathcal{A}_{n+1} \) contained in \( A_n \). We denote by \( b_k \) the pole corresponding to \( A_k \). Let \( z_n := f^{-(n-1)}(b_n) \in A_n \), \( z_k := f^{-n}(b_k) \in A_k \). Observe that \( f^{-n} \) is conformal on \( A_n \), as all critical values of \( f \) are contained in \( B(0, \rho^1 - 1) \), so all the branches of \( f^{-1} \) are well defined on \( \bigcup_{b \in f_q^{-1}(\infty)} (U(b, \varepsilon)) \). So \( L(f^{-n-1}, A_n) = L(f^{-(n-1)}, f^{-n}(A_n)) \). Hence,

\[
\text{vol}(A_n) = \text{vol}(f^{-(n-1)}(U(b_n, \varepsilon))) = \int \int_{f^{-(n-1)}(U(b_n, \varepsilon))} dz
\]

\[
= \int \int_{U(b_n, \varepsilon)} \left| (f^{-(n-1)})'(z) \right|^2 dz \leq \int \int_{U(b_n, \varepsilon)} \left( \sup_{z \in U(b_n, \varepsilon)} \left| (f^{-(n-1)})'(z) \right| \right)^2 dz
\]

\[
= \text{vol}(U(b_n, \varepsilon))\left( L(f^{-(n-1)}, U(b_n, \varepsilon)) \inf_{z \in U(b_n, \varepsilon)} \left| (f^{-(n-1)})'(z) \right| \right)^2 \leq \frac{3\pi \varepsilon^2}{4q} \left( L(f^{n-1}, A_n) \right)^2 = \frac{3\pi \varepsilon^2}{4q} \left( L(f^{n-1}, A_n) \right)^2
\]

Set \( P_{n+1} := P^+(0, R_{n+1}, 2R_{n+1}) \).

\[
\text{vol}(U_{n+1} \cap A_n) = \sum_{A_k \subset A_n} \text{vol}(A_k) = \sum_{b_k \in P_{n+1}} \text{vol}(f^{-n}(U(b_k, \varepsilon)))
\]

\[
= \sum_{b_k \in P_{n+1}} \int \int_{U(b_k, \varepsilon)} \left| (f^{-n})'(z) \right|^2 dz \geq \sum_{b_k \in P_{n+1}} \int \int_{U(b_k, \varepsilon)} \left( \inf_{z \in U(b_k, \varepsilon)} \left| (f^{-n})'(z) \right| \right)^2 dz
\]

\[
= \frac{3\pi \varepsilon^2}{4q} \sum_{b_k \in P_{n+1}} \left( \sup_{z \in U(b_k, \varepsilon)} \left| (f^{-n})'(z) \right| \right)^2 \geq \frac{3\pi \varepsilon^2}{4q} \sum_{b_k \in P_{n+1}} \left( \frac{\left| (f^{-n})'(b_k) \right|}{L(f^{n-1}, U(b_k, \varepsilon))} \right)^2 \geq \frac{3\pi \varepsilon^2}{4q} \sum_{z_k \in A_k \subset A_n} \left( L(f^n, A_k) \right)^2
\]

Now, combining (3.7) and (3.8), we estimate the density of the sets \( U_{n+1} \cap A_n \) in \( A_n \).

\[
\frac{\text{vol}(U_{n+1} \cap A_n)}{\text{vol}(A_n)} \geq \sum_{z_k \in A_k \subset A_n} \frac{\left( L(f^n, A_k) \right)^2 \left| (f^{-n})'(z_k) \right|^2}{\left( L(f^{n-1}, A_n) \right)^2 \left( \left| (f^{-n})'(z_k) \right| \right)^2} \geq \frac{\left| (f^{-n})'(z_n) \right|^2}{\left( L(f^{n-1}, A_n) \right)^2} \sum_{z_k \in A_k \subset A_n} \left( L(f^n, A_k) \right)^2 \left| (f^n)'(z_k) \right|^2 \left( \left| (f^n)'(z_k) \right| \right)^2
\]

We have

\[
\left| (f^{-1})'(z_n) \right| = \prod_{j=0}^{n-2} f'(f^j(z_n)) \quad \text{and} \quad \left| (f^n)'(z_k) \right| = \prod_{j=0}^{n-1} f'(f^j(z_k))
\]
It follows from Lemma 3.2 that
\[
(L(f^{n-1}, A_n))^2 \leq 2^{2(q+1)(n-1)} \quad \text{and} \quad (L(f^n, A_k))^2 \leq 2^{2(q+1)n}.
\] (3.11)

Plugging (3.10), (3.11) into (3.9), we have
\[
\frac{\text{vol}(U_{n+1} \cap A_n)}{\text{vol}(A_n)} \geq \left| \prod_{j=0}^{n-2} f'(f^j(z_n)) \right|^2 \sum_{z_k \in A_k \subset A_n} \frac{1}{2^{\frac{2(q+1)(n-1)}{q}} \left| \prod_{j=0}^{n-2} f'(f^j(z_k)) \right|^2} f'(f^j(z_n)) \sum_{z_k \in A_k \subset A_n} \frac{1}{2^{\frac{2(q+1)(n-1)}{q}} \left| \prod_{j=0}^{n-2} f'(f^j(z_k)) \right|^2} f'(f^j(z_k)) \]
\[
= \frac{1}{2^{\frac{2(q+1)(2n-1)}{q}}} \sum_{z_k \in A_k \subset A_n} \frac{1}{2^{\frac{2(q+1)(n-1)}{q}}} \left| \prod_{j=0}^{n-2} f'(f^j(z_n)) \right|^2 \cdot 1 \left| f'(f^n(z_k)) \right|^2.
\] (3.12)

For each \( j = 0, 1, \ldots, n-2 \)
\[
f(f^j(z_n)) = f^{j+1}(z_n) \in P^+(0, R_{j+2}, 2R_{j+2})
\]
and
\[
f(f^j(z_k)) = f^{j+1}(z_k) \in P^+(0, R_{j+2}, 2R_{j+2}),
\]
since \( z_n \in A_n \subset A_{j+1} \) and \( z_k \in A_k \subset A_n \subset A_{j+1} \). Thus, by (3.3), for \( j = 0, 1, \ldots, n-2 \) we have
\[
|f'(f^j(z_n))| \geq \frac{C_2 R_{j+2}}{C_1^{\frac{q+1}{q}}} \quad \text{and} \quad |f'(f^j(z_k))| \leq \frac{2^{\frac{q+1}{q}} C_2 R_{j+2}}{C_1^{\frac{q+1}{q}}}.
\]

This implies that
\[
\frac{|f'(f^j(z_n))|}{|f'(f^j(z_k))|} \geq \frac{1}{2^{\frac{q+1}{q}}}, \quad j = 1, 2, \ldots, n-2.
\] (3.13)

Analogously, \( f(f^{n-1}(z_k)) = f^n(z_k) \in P^+(0, R_{n+1}, 2R_{n+1}) \) as \( z_k \in A_k \subset A_{n+1} \). By applying this to (3.3), we get
\[
|f'(f^{n-1}(z_k))| \leq \frac{2^{\frac{q+1}{q}} C_2 R_{n+1}}{C_1^{\frac{q+1}{q}}}. \] (3.14)

Putting (3.13), (3.14) into (3.12) and by (3.6), we obtain
\[
\frac{\text{vol}(U_{n+1} \cap A_n)}{\text{vol}(A_n)} \geq \frac{1}{2^{\frac{2(q+1)(2n-1)}{q}} \left( \frac{1}{2^{\frac{q+1}{q}}} \right)^{2(n-1)} C_1^{2(q+1)} C_2^{2(q+1)} 2^{\frac{2(q+1)}{q}} C_2 R_{n+1}^{2(1+\frac{q}{q})} \sum_{z_k \in A_k \subset A_n} 1 2^{\frac{2(q+1)}{q}} C_2 R_{n+1}^{\frac{q}{q}} R_{n+1}^2 = M 2^{\frac{2(q+1)}{q}} C_2 R_{n+1}^2,
\]
where \( M = 2^{\frac{2(q+1)}{q}} C_2 R_{n+1}^2 \).

Similarly, we consider the case \( n = 1 \). By Definition 2.1, the set \( A_1 \) has only one element, i.e. \( A_1 \) and its Lebesgue measure \( \text{vol}(A_1) \leq \pi \varepsilon^2 \). The set \( A_1 \) contains finitely
many subsets $A_k \in A_2$. As for $n = 2$, we denote by $b_k$ the pole corresponding to $A_k$. Arguing as in (3.8), we get

$$\text{vol}(U_2 \cap A_1) \geq \frac{3\pi \varepsilon^2}{4q} \sum_{\Delta \in A_k \subset A_1} (L(f, A_k)|f(z_k)|)^{-2}.$$ 

Setting $n = 1$ in (3.11) we have $(L(f, A_k))^2 \leq 2^{-\frac{2(q+1)}{q}}$, which implies that

$$\text{vol}(U_2 \cap A_1) \geq \frac{3\pi \varepsilon^2}{4q} \sum_{\Delta \in A_k \subset A_1} \frac{1}{|f'(z_k)|^2 \cdot 2^{-\frac{2(q+1)}{q}}}.$$ 

Analogously as in (3.14), we obtain $|f'(z_k)| \leq \frac{2^{q+1}}{C_2 R_2^{q+1}}$ and, using (3.6), we conclude that

$$\frac{\text{vol}(U_2 \cap A_1)}{\text{vol}(A_1)} \geq \frac{3\pi \varepsilon^2 C_1^{\frac{2(q+1)}{q}}}{2^{\frac{2(q+1)}{q}} C_2^2 R_1^2 q} \sum_{\Delta \in A_k \subset A_1} \frac{1}{2^{\frac{2(q+1)}{q}}} \approx M' \frac{N_2}{R_2^{\frac{2(q+1)}{q}}} \approx M' \frac{R_2^2}{R_2^{\frac{2(q+1)}{q}}} = \frac{M'}{R_2^2},$$

where $M' = \frac{3\pi \varepsilon^2 C_1^{\frac{2(q+1)}{q}}}{2^{\frac{2(q+1)}{q}} C_2^2 R_1^2 q}$.

By Lemma 3.4, the numbers $\Delta_n$ from Proposition 2.4 are equal to

$$\Delta_1 = \frac{M'}{R_2^2}, \quad \Delta_n = \frac{M}{2^{\frac{q(q+1)n}{q}} R_{n+1}^{\frac{2}{q}}}, \quad n \geq 2.$$ 

Assembling the preceding lemmas, we may now prove Theorem 2.3.

Proof of Theorem 2.3. Lemma 3.4 implies that

$$\sum_{j=1}^{n} \log \Delta_j = \log \Delta_1 + \sum_{j=2}^{n} \log \Delta_j = \log \frac{M'}{R_2^2} + \sum_{j=2}^{n} \log \frac{M}{2^{\frac{q(q+1)n}{q}} R_{j+1}^{\frac{2}{q}}}$$

$$= \log \left( a R_1^{\frac{2}{q}} \right) - \log M' - \sum_{j=2}^{n} \log \left( 2^{\frac{q(q+1)n}{q}} a R_1^{\frac{2}{q}} \right) - (n - 1) \log M$$

$$= \log a + \log R_1 - \log M' - (n - 1) \log M + (n - 1) \log R_1 + 9 \log 2 \sum_{j=2}^{n} j + \log a \sum_{j=2}^{n} j$$

$$= \log M - \log M' + n \log \left( R_1^{\frac{2}{q}} \right) - n \log M + \frac{6(q+1)}{q} \log 2 \sum_{j=2}^{n} j + \frac{2}{q} \log a \sum_{j=1}^{n} j$$

$$= \log \frac{M}{M'} + n \log \frac{R_1^{\frac{2}{q}}}{M} + \frac{6(q+1)(n+2)(n-1)}{q} \log 2 + \frac{2n(n+1)}{2q} \log a$$

$$= \log \frac{M}{M'} + n \log \frac{R_1^{\frac{2}{q}}}{M} + \frac{3(q+1)(n+2)(n-1)}{q} \log 2 + \frac{n(n+1)}{q} \log a. \quad (3.15)$$

□
In view of Lemma 3.2, we have
\[ |\log d_n| = \log(4\varepsilon(1+r)) - \log \left( \frac{C_2/C_1^{q+1}}{a^{(q+1)n(1-1)/2q}R_1^{(q+1)n-1}} \right) \]
\[ = (n-1) \log \frac{C_2}{C_1^{q+1}} + \frac{(q+1)n(1-1)}{2q} \log a + \frac{(q+1)n-1}{q} \log R_1 - 4\varepsilon(1+r). \]  

The final estimate follows from (3.15) and (3.16).
\[ \dim_H(A(a)) \geq 2 - \limsup_{n \to \infty} \frac{\log \frac{M}{M'} + n \log R_1^{\frac{2}{M'}} + \frac{3(q+1)(n+2)(n-1)}{q} \log 2 + \frac{n(n+1)}{q} \log a}{(n-1) \log \frac{C_2}{C_1^{q+1}} + \frac{(q+1)n(n-1)}{2q} \log a + \frac{(q+1)n-1}{q} \log R_1 - 4\varepsilon(1+r)} \]
\[ = 2 - \frac{1}{q} \log a + \frac{3(q+1)}{q} \log 2 + \frac{(q+1)}{2q} \log a = 2 - \frac{2}{q+1} - \frac{6 \log 2}{q+1} - \frac{6 \log 2}{q} \log a. \]

\[ \square \]

4. The Proof of the Lower Estimate of \( \dim_H(\mathcal{E}) \)

As in Section 2 let \( \{c_i \in \mathcal{R} : f'(c_i) = 0, \ i = 1, \ldots, k\} \) be the set of critical points from the basic fundamental parallelogram \( \mathcal{R} \). Now, we assume that the critical value \( f(c_1) \neq 0 \) is a pole of the multiplicity \( q \) defined in (2.2) and all the other critical values \( f(c_i), i = 2, \ldots, k \), are attracted by attracting periodic points.

**Remark 4.1.** If \( f(c_1) = 0 \) is a pole, then \( f_\beta(c_1) = 0 \) and \( \beta = \infty \) for every \( \beta \in \mathbb{C}\setminus\{0\} \).

In this case \( \mathcal{E} = \emptyset \).

We consider the one-parameter family of functions
\[ f_\beta(z) = \beta f(z), \ \beta \in B(1, r) \text{ for } 0 < r < \frac{1}{4} = \frac{1}{2\alpha + 4} \approx 0.04, \quad (4.1) \]
where \( \alpha = \sin(\pi/8) = \sqrt{2 - \sqrt{2}}/2 \). The functions \( f_\beta \) are periodic and their critical points are the same as for the elliptic function \( f \). As \( f \) has only finitely many periodic sinks and hyperbolicity (of periodic points) preserves under perturbation, without loss of generality we may assume that for every \( f_\beta \) defined in (4.1), its critical values, except of \( f_\beta(c_1) \), belong to basins of attraction of the attracting periodic cycles. We modify a definition of \( f_q^{-1}(\infty) \) given in (2.5). Now
\[ f_q^{-1}(\infty) := \{ f(c_1) + l + m\lambda_1 : l, m \in \mathbb{Z} \}. \quad (4.2) \]
The assumption on \( \varepsilon_0 \) are the same as in (2.6). Since \( \beta \in B(1, r) \), where \( r \) is defined in (4.1), we can clearly assume that the estimates (2.7) and (2.8) remain true for \( f_\beta \). Moreover, \( |\text{Arg}\beta| \leq \arcsin(1/4 - 1/(2\alpha + 4)) \approx 0.04 \) for \( \beta \in B(1, r) \), then we also assume that \( M_1 \leq \text{arg}(\beta G(z)) \leq M_2 \), for all \( b \in f_q^{-1}(\infty) \), where \( f_q^{-1}(\infty) \) is defined in (4.2) and \( M_1, M_2 \) are given in (2.9). A definition of \( U(b, \varepsilon) \) and \( \varepsilon \) are the same as in respectively (2.10) and (2.11). We additionally assume that \( \varepsilon \) satisfies the following conditions
\[ \varepsilon < |f(c_1)|/3, \quad B(f(c_1), \varepsilon) \subset g_1(B(1, r)), \quad (4.3) \]
where \( g_1 \) is the following function
\[
g_1 : B(1, r) \to \mathbb{C}, \quad g_1(\beta) = f_\beta(c_1),
\]
and \( c_1 \) is the critical point of \( f \) chosen above. In this chapter we modify a definition of \( R_1 \) given in Section 2. Now \( R_1 \) is a positive constant such that
\[
B(f(c_1), \varepsilon) \subset A(0, R_1, 2R_1).
\]
Note that the inclusion (2.12) is also true for \( f_\beta \). Let \( \tilde{R}_2 \) be such that
\[
\tilde{R}_2 > \frac{C_1}{(1 - \alpha) \varepsilon^q},
\]
where \( \alpha = \sin(\pi/8) \) and \( \hat{a}_1 = \tilde{R}_2/R_1 > \frac{C_1}{(1 - \alpha) \varepsilon^q R_1} \). Next, we define a constant
\[
\hat{a}_0 = \max \left\{ 2, \hat{a}_1, \frac{1}{R_1}, \frac{3C_1^2}{C_2 R_1}, \frac{6^2 \alpha C_1^{2(q+1)}}{C_2^2 R_1^{2q+1}}, \left( \frac{4\varepsilon(1 + r)C_1^{q+1}}{C_2^2 R_1^{2q+1}} \right)^{\frac{q}{q+1}}, \frac{\sqrt{C_1}}{C_2^{2(q+1)}} \right\}.
\]
For \( a > \hat{a}_0 \) we consider a sequence of radii \( \{R_k\}_{k \geq 2} \) defined in Section 2 (see (2.15)) with \( R_1 \) as in (4.4). Since (2.12) is true for \( f_\beta \), then
\[
\{ z \in \mathbb{C} : |z| > R_2 \geq R_2, \quad \phi \leq \arg z \leq \phi + \pi \} \subset f_\beta(U(b, \varepsilon))
\]
for all poles \( b \in f_q^{-1}(\infty) \) and \( \beta \in B(1, r) \).

We consider auxiliary functions \( g_n(\beta) = f_\beta(c_1), \quad n \in \mathbb{N}, \) which are defined outside a countable set of parameters.

**Definition 4.2.** We define the following family of sets
\[
\mathcal{D}_0(a) = \{D_0 = B(1, r)\},
\]
\[
\mathcal{D}_1(a) = \{D_1 = g_1^{-1}(U(f(c_1), \varepsilon)) \subset D_0\},
\]
\[
\mathcal{D}_2(a) = \{D_2 \subset D_1 \mid \exists b^{(2)} \in f_q^{-1}(\infty) : U(b^{(2)}, \varepsilon) \subset P^+(0, R_2, 2R_2), \quad D_2 \text{ is a component of } g_2^{-1}(U(b^{(2)}, \varepsilon))\},
\]
\[
\ldots
\]
\[
\mathcal{D}_k(a) = \{D_k \subset D_{k-1} \mid \exists b^{(k)} \in f_q^{-1}(\infty) : U(b^{(k)}, \varepsilon) \subset P^+(0, R_k, 2R_k), \quad D_k \text{ is a component of } g_k^{-1}(U(b^{(k)}, \varepsilon))\},
\]
\[
\ldots
\]
\[\text{Let } \quad \mathcal{V}_k(a) = \bigcup_{D_k \in \mathcal{D}_k(a)} D_k, \quad D(a) = \bigcap_{k=1}^{\infty} \mathcal{V}_k(a).\]

Figure 1 illustrates the sets defined above for \( q = 4 \).

**Proposition 4.3.** For each \( n \in \mathbb{N} \), the set \( \mathcal{D}_n(a) \) defined above is non-empty.
The proof of the above lemma is the same as Lemma 2.2.

**Theorem 4.4.** Let $f_\beta$ be the family of maps defined in (4.1) and let $\hat{a}_0$ be the constant given in (4.6). Then, for every $a > \hat{a}_0$ there is a Cantor subset $D(a)$ of $\mathcal{E}$ and for this subset

$$\dim_H(D(a)) \geq \frac{2q}{q + 1} - \frac{6 \log 2}{\log a}.$$  

Theorem 2 easily follows from Theorem 4.4.

In order to prove Theorem 4.4 we use again Proposition 2.4. We fix $a > \hat{a}_0$ and consider the sets $D_n(a)$, $n \geq 1$, given in Definition 4.2. As before, for simplicity, we suppress the explicit dependence on $a$ in the notation. Note that the estimates (2.7), (2.8) remain true for $f_\beta$, so

$$|f_\beta(z)| \asymp \frac{C_1}{|z - b|^q}, \quad |f_\beta'(z)| \asymp \frac{C_2}{|z - b|^{q + 1}}, \quad (4.8)$$

for each $b \in f_q^{-1}(\infty)$, every $z \in B(b, \varepsilon)$ and all $\beta \in B(1, r)$. Note that if $\beta \in \mathcal{V}_n$, $n \geq 2$ and $z = f_\beta^j(c_1)$ with $j \in \{1, 2, \ldots, n - 1\}$ we have $f_\beta(z) = f_\beta^{j+1}(c_1) = g_{j+1}(\beta) \in U(b^{j+1}, \varepsilon) \subset A^+(0; R_{j+1}, 2R_{j+1})$ and moreover, using (4.8),

$$R_{j+1} \leq |f_\beta(z)| \asymp \frac{C_1}{|z - b|^q} \leq 2R_{j+1}. \quad (4.9)$$

for some $b \in f_q^{-1}(\infty) \cap P(0, R_j, 2R_j)$. Thus,

$$\frac{C_2 R_{j+1}^{q+1}}{C_1^q} \leq |f_\beta'(z)| \leq \frac{2^{q+1} C_2 R_{j+1}^{q+1}}{C_1^{q+1}} \quad (4.10)$$

for $\beta \in \mathcal{V}_n$, $n \geq 2$ and $z = f_\beta^j(c_1)$ with $j \in \{1, 2, \ldots, n - 1\}$.

In the following lemma we estimate the derivative of $g_n$. We show that $g_n'$ is comparable to the product of the derivatives of $f_\beta$ over the trajectory of the critical value $f_\beta(c_1)$.

**Lemma 4.5.** Let $D_n \in \mathcal{D}_n$, $n \geq 2$. Then for every $\beta \in D_n$

$$\frac{1}{2(1 + r)} \left( \frac{C_2}{C_1^{q+1}} \right)^{-1} \left( \frac{2^{q+1} C_2}{C_1^{q+1}} \right)^{-1} a^{(q+1)(n-1)/q} R_1 \leq |g_n'(\beta)| \leq \frac{5}{2(1 - r)} \left( \frac{2^{q+1} C_2}{C_1^{q+1}} \right)^{-1} a^{(q+1)(n-1)/q} R_1.$$  

**Proof.** A simple calculation shows that

$$g_n'(\beta) = \frac{1}{\beta} \prod_{k=1}^{n-1} f_\beta^k(f_\beta^k(c_1)) \left[ f_\beta(c_1) + \sum_{k=2}^{n} \frac{f_\beta^k(c_1)}{\prod_{i=1}^{k-1} f_\beta^i(c_1)} \right] \quad (4.11)$$
for all \( n \geq 2 \) and every \( \beta \in D_n \). First, we estimate the product \( \prod_{k=1}^{n-1} f'_\beta(f^k_\beta(c_1)) \). Observe that \( f^k_\beta(f^k_\beta(c_1)) = f^k_{\beta+1}(c_1) = g_{k+1}(\beta), \ k = 1, 2, \ldots, n-1 \). The functions \( g_2, \ldots, g_n \) are well-defined for \( \beta \in D_n \), because \( D_n \subset D_k, \ k = 2, \ldots, n \).

Since \( g_{k+1}(\beta) \in P^+(0, R_{k+1}, 2R_{k+1}) \), then using (4.10) we get

\[
\prod_{k=1}^{n-1} f'_\beta(f^k_\beta(c_1)) \leq \left( \frac{2^{q+1}}{q+1} \frac{C_2 R_{q+1}^q}{C_1^q} \right)^{n-1} \prod_{k=1}^{n-1} f'_\beta(f^k_\beta(c_1)) \leq \left( \frac{2^{q+1}}{q+1} \frac{C_2}{C_1^q} \right)^{n-1} \prod_{k=1}^{n-1} f'_\beta(f^k_\beta(c_1)) \leq \left( \frac{2^{q+1}}{q+1} \frac{C_2}{C_1^q} \right)^{n-1} \prod_{k=1}^{n-1} f'_\beta(f^k_\beta(c_1)).
\]

Finally,

\[
\left( \frac{C_2}{C_1^q} \right)^{n-1} a^{(q+1)(n-1)} R_1^{\frac{(q+1)(n-1)}{q}} \leq \prod_{k=1}^{n-1} f'_\beta(f^k_\beta(c_1)) \leq \left( \frac{2^{q+1}}{q+1} \frac{C_2}{C_1^q} \right)^{n-1} a^{\frac{(q+1)(n-1)}{q}} R_1^{\frac{(q+1)(n-1)}{q}}.
\]

Now, applying (4.12), we estimate the sum \( \sum_{k=2}^{n} \prod_{i=1}^{k-1} f'_\beta(f^i_\beta(c_1)) \).

\[
\sum_{k=2}^{n} \prod_{i=1}^{k-1} f'_\beta(f^i_\beta(c_1)) \leq \sum_{k=2}^{n} \prod_{i=1}^{k-1} f'_\beta(f^i_\beta(c_1)) \leq \sum_{k=2}^{n} \left( \frac{C_2}{C_1^q} \right)^{k-1} \frac{2^{q+1}}{q+1} \frac{C_2}{C_1^q} \frac{a^{(q+1)(k-1)} R_1^{\frac{(q+1)(k-1)}{q}}}{\frac{(q+1)(k-1)}{q}}
\]

\[
= \frac{\sum_{k=2}^{n} \left( \frac{C_2}{C_1^q} \right)^{k-1} \frac{2^{q+1}}{q+1} \frac{a^{(q+1)(k-1)}}{2^{q+1}} R_1^{\frac{(q+1)(k-1)}{q}}}{\frac{1}{2^{q+1}} R_1^{\frac{(q+1)(k-1)}{q}}}
= \sum_{k=2}^{n} \frac{\left( \frac{C_2}{C_1^q} \right)^{k-1} \frac{2^{q+1}}{q+1} \frac{a^{(q+1)(k-1)}}{2^{q+1}} R_1^{\frac{(q+1)(k-1)}{q}}}{\frac{1}{2^{q+1}} R_1^{\frac{(q+1)(k-1)}{q}}}
\]

Since \( a > \hat{a}_0 \geq 2 \) and \((q+1)(k-2q)(k-1) \geq 2(q+1)(k-1) - 2q \) for \( q \in \mathbb{N}, \ k = 2, 3, \ldots \), then

\[
\sum_{k=2}^{n} \left( \frac{C_2}{C_1^q} \right)^{k-2} \frac{1}{\frac{(q+1)(k-2q)(k-1)-1}{2^{q+1}} R_1^{\frac{(q+1)(k-1)-2q-1}{2^{q+1}}}} \leq \sum_{k=2}^{n} \left( \frac{C_2}{C_1^q} \right)^{k-2} \frac{1}{(aR_1)^{\frac{(q+1)(k-1)-2q-1}{2^{q+1}}}}.
\]
Using the inequality $(2(q + 1)(k - 1) - 2q - 1)/(2q) \geq k - 2$, $q \in \mathbb{N}$, $k = 2, 3, \ldots$ and the fact that $a > \hat{a}_0 \geq \max\{\frac{1}{R_1}, \frac{3C_1^q}{C_2R_1}\}$ we get

$$
\sum_{k=2}^{n} \left( \frac{C_1^q}{C_2} \right)^{k-2} \leq \frac{1}{(aR_1)^{2(q+1)(k-1)-2q-1}} \leq \sum_{k=2}^{n} \left( \frac{C_1^q}{aR_1C_2} \right)^{k-2},
$$

Hence,

$$
\left| \sum_{k=2}^{n} \frac{f^k_{\beta}(c_1)}{\prod_{i=1}^{k-1} f^i_{\beta}(f^i_{\beta}(c_1))} \right| \leq \frac{2C_1^{q+1}}{C_2(aR_1)^{q+1}} \frac{3}{2} \leq \frac{R_1}{2}, \tag{4.13}
$$

because $a > \hat{a}_0 \geq \frac{2q(2q+1)}{C_2C_1^{q+1}}$. Therefore,

$$
\frac{R_1}{2} = R_1 - \frac{R_1}{2} \leq \left| f_{\beta}(c_1) + \sum_{k=2}^{n} \frac{f^k_{\beta}(c_1)}{\prod_{i=1}^{k-1} f^i_{\beta}(f^i_{\beta}(c_1))} \right| \leq 2R_1 + \frac{R_1}{2} = \frac{5R_1}{2}. \tag{4.14}
$$

Plugging (4.12), (4.14) into the formula (4.11) for $g'_n$ we prove the lemma. \(\square\)

In the next part of this section, we estimate the diameters of $D_n$ and the ratios $\text{vol}(V_{n+1} \cap D_n)/\text{vol}(D_n)$ and in order to do that we prove that the functions $g_n$, $n \geq 2$, are holomorphic outside a countable set of points and have poles at $\beta_{n-1} \in \partial D_{n-1}$.

**Lemma 4.6.** For each $D_n \in \mathcal{D}_n$, $n \geq 1$, the map $g_n$ is conformal on $D_n$.

**Proof.** The map $g_1$ is one-to-one and holomorphic on $D_1$. By induction, we show that the maps $g_n$, $n \geq 2$, are conformal. Suppose that $g_n$, $n \geq 1$, is conformal on $D_n$, we prove that $g_{n+1}$ is conformal on $D_{n+1} \subset D_n$. If $n = 1$ then we take the segment $U(b^{(1)}, \varepsilon) \subset P(0, R_1, 2R_1)$ with $b^{(1)} = f(c_1)$ and if $n \geq 2$ we consider a segment $U(b^{(n)}, \varepsilon) \subset P^+(0, R_n, 2R_n)$. We know that $D_n$ is a component of $g_n^{-1}(U(b^{(n)}, \varepsilon))$, $n \geq 1$. Let $b^{(n)} = b_n$, $\beta_n = g_n^{-1}(b_n) \in \partial D_n$ and $b^{(n+1)} = b_{n+1}$. If $U(b_{n+1}, \varepsilon) \subset P^+(0, R_{n+1}, 2R_{n+1})$, then $D_{n+1} \subset D_n$ is a component of $g_n^{-1}(U(b_{n+1}, \varepsilon))$. We define a map $\hat{g}_{n+1}(\beta) = \beta_n f(g_n(\beta))$. Since (2.12) holds also for $f_{\beta}$ we have

$$
\hat{g}_{n+1}(D_n) \supset \{ z \in \mathbb{C} : |z| \geq \frac{C_1}{3q}, \phi - \frac{\pi}{8} \leq \arg z \leq \phi + \frac{9\pi}{8} \}.
$$

We show that $\hat{g}_{n+1}$ is one-to-one in $D_n$. Take $\beta', \beta'' \in D_n$ such that $\hat{g}_{n+1}(\beta') = \hat{g}_{n+1}(\beta'')$. By definition of the map $\hat{g}_{n+1}$, we have $f(g_n(\beta')) = f(g_n(\beta''))$, where $g_n(\beta'), g_n(\beta'') \in g_n(D_n) = U(b_n, \varepsilon)$. Since $f$ is one-to-one in $U(b_n, \varepsilon)$, then $g_n(\beta') = g_n(\beta'')$ and this implies that $\beta' = \beta''$. This follows from the injectivity of the map $g_n$. There is a set $\hat{D}_{n+1} \subset D_n$ such that

$$
\hat{g}_{n+1}(\hat{D}_{n+1}) = \{ z \in \mathbb{C} : (1 - \alpha)R_{n+1} < |z| < (2 + \alpha)R_{n+1}, \phi - \frac{\pi}{8} < \arg z < \phi + \frac{9\pi}{8} \} \tag{4.15}
$$
for $\alpha = \sin(\pi/8)$ and $\phi$ as in (2.12).

Now, we show that $D_{n+1} \subset \hat{D}_{n+1}$. Note that $\hat{g}_{n+1}(\beta) = (\beta/\alpha)g_{n+1}(\beta)$. Since $g_{n+1}(D_{n+1}) = U(b_{n+1}, \varepsilon) \subset P^+(0, R_{n+1}, 2R_{n+1})$ and $0 < r < 1/4 - 1/(2\alpha + 4)$, then for $\beta \in D_{n+1}$ we have

$$|\hat{g}_{n+1}(\beta)| > \frac{1 - r}{1 + r} R_{n+1} > \frac{3\alpha + 8}{5\alpha + 8} R_{n+1} \approx 0, 92R_{n+1} > (1 - \alpha)R_{n+1} \approx 0, 62R_{n+1}$$

$$|\hat{g}_{n+1}(\beta)| < \frac{1 + r}{1 - r} 2R_{n+1} < \frac{2(5\alpha + 8)}{3\alpha + 8} R_{n+1} \approx 2, 17R_{n+1} < (2 + \alpha)R_{n+1} \approx 2, 38R_{n+1},$$

$$\arg \hat{g}_{n+1}(\beta) < \phi + \pi + 2 \max_{\beta \in B(1, r)} \arg \beta < \phi + \pi + 2 \arcsin \left(\frac{1}{4} - \frac{1}{2\alpha + 4}\right) \approx \phi + 0, 08 < \phi + \frac{9\pi}{8},$$

$$\arg \hat{g}_{n+1}(\beta) > \phi - 2 \max_{\beta \in B(1, r)} \arg \beta > \phi - 2 \arcsin \left(\frac{1}{4} - \frac{1}{2\alpha + 4}\right) \approx \phi - 0, 08 > \phi - \frac{\pi}{8}.$$
does the latter. Since $D_{n+1} \subset \hat{D}_{n+1}$, then $g_{n+1}$ is one-to-one in $D_{n+1}$. The map $g_{n+1}$ is holomorphic on $\text{int}D_n$, then is conformal on $D_{n+1}$. □

**Remark 4.7.** In Lemma 4.6 we showed in fact that there is a unique set $D_1 = g_1^{-1}(U(f(c_1), \varepsilon))$ and the segments $U(b_n, \varepsilon) \subset P^+(0, R_n, 2R_n)$, $n \geq 2$, are in one-to-one correspondence with the sets $D_n \subset \mathcal{V}_n \cap D_{n-1}$ for each $D_{n-1} \in \mathcal{D}_{n-1}$. Hence, each $D_n$, $n \geq 1$, is a finite collection of the sets $D_n$.

The next lemma follows immediately from Lemma 4.5.

**Lemma 4.8.** Let $D_n \in \mathcal{D}_n$, $n \geq 2$. Then the distortion $L(g_n, D_n) \leq \frac{5(1+r)}{1-r} \cdot 2^{\frac{(q+1)(n-1)}{q}}$ and

$$\text{diam}(D_n) \leq \frac{4\varepsilon(1+r)}{\left(\frac{C_2}{C_1}\right)^{\frac{q+1}{q}}} \frac{a^{-\frac{(q+1)(n-1)}{2q}} R_1^{-\frac{(q+1)(n-1)}{q}}}{n^{-1}},$$

where $\varepsilon$ is as in (4.3).

**Remark 4.9.** Observe that $\text{diam}(D_n) \to 0$ as $n \to \infty$, since $a > a_0 \geq 2$. This proves that the set $D$ from Definition 4.2 is a Cantor set of parameters.

By Lemma 4.8, the numbers $d_n$ defined in Proposition 2.4 are equal to

$$d_n = 4\varepsilon(1+r) \left(\frac{C_2}{C_1}\right)^{\frac{q+1}{q}} a^{-\frac{(q+1)(n-1)}{2q}} R_1^{-\frac{(q+1)(n-1)}{q}} , \hspace{1em} n \geq 2, \hspace{1em} (4.16)$$
and \( d_1 = \text{diam}(A_1) \leq 2r < 1 \) by (4.1). Making a similar calculation as the one following Remark 3.3, it is easy to show that \( d_n < 1, n = 2, 3, \ldots \).

In the next lemma, we estimate from below the density of the sets \( V_{n+1} \cap D_n \) in the set \( D_n \in \mathcal{D}_n, n \geq 1 \). Since the proof is very similar to the proof of Lemma 3.4, we leave the details to the reader.

**Lemma 4.10.** There exists \( M > 0 \) such that

\[
\frac{\text{vol}(V_{n+1} \cap D_n)}{\text{vol}(D_n)} \geq M \frac{M}{2^{\frac{q(q+1)n}{q} R_{n+1}^2}},
\]

for each \( D_n \in \mathcal{D}_n, n \geq 2 \). Moreover, \( \frac{\text{vol}(V_2 \cap D_1)}{\text{vol}(D_1)} \geq M'R_{2}^{-\frac{2}{q}}, \) for some \( M' > 0 \).

**Proof.** Take some \( n \geq 2 \) and \( D_n \in \mathcal{D}_n \). There exists \( b^{(n)} \in f_q^{-1}(\infty) \) such that \( D_n \) is a component of \( g_n^{-1}(U(b^{(n)}, \varepsilon)) \), where \( U(b^{(n)}, \varepsilon) \subset P^+(0, R_n, 2R_n) \). Moreover, for each \( D_k \in \mathcal{D}_{n+1} \) there is \( b^{(n+1)} \in f_q^{-1}(\infty) \) such that \( D_k \) is a component of \( g_n^{-1}(U(b^{(n+1)}, \varepsilon)) \), \( U(b^{(n+1)}, \varepsilon) \subset P^+(0, R_{n+1}, 2R_{n+1}) \). To simplify the formulas we denote \( b^{(n)} \) by \( b_n \). There are finitely many sets \( D_k \in \mathcal{D}_{n+1} \) contained in \( D_n \). We denote by \( b_k \) the pole corresponding to \( D_k \). Let \( \beta_n := g_n^{-1}(b_n) \in D_n, \beta_k := g_n^{-1}(b_k) \in D_k \). Lemma 4.6 implies that \( g_n \) is conformal on \( D_n \), so \( L(g_n, D_n) = L(g_n^{-1}, g_n(D_n)) \). Hence,

\[
\text{vol}(D_n) \leq \frac{3\pi\varepsilon^2}{4q} \left( \frac{L(g_n, D_n)}{|g'_n(\beta_n)|} \right)^2 \quad (4.17)
\]

and

\[
\text{vol}(V_{n+1} \cap D_n) = \sum_{D_k \subset D_n} \text{vol}(D_k) \geq \frac{3\pi\varepsilon^2}{4q} \sum_{\beta_k \in D_k \subset D_n} (L(g_{n+1}, D_k)|g'_{n+1}(\beta_k)|)^{-2}. \quad (4.18)
\]

It follows from (4.17) and (4.18) that

\[
\frac{\text{vol}(V_{n+1} \cap D_n)}{\text{vol}(D_n)} \geq \frac{|g'_n(\beta_n)|^2}{(L(g_n, D_n))^2} \sum_{\beta_k \in D_k \subset D_n} (L(g_{n+1}, D_k)|g'_{n+1}(\beta_k)|)^{-2}.
\]

Then, by (4.11), (4.14) and Lemma 4.8, we get

\[
\frac{\text{vol}(V_{n+1} \cap D_n)}{\text{vol}(D_n)} \geq \frac{(1-r)^6}{5^{2q+1}} \left( \prod_{j=1}^{n-1} \frac{|f'_{\beta_n}(f_{\beta_n}(c_1))|}{|f'_{\beta_k}(f_{\beta_k}(c_1))|} \right)^2 \cdot \frac{1}{|f'_{\beta_k}(f_{\beta_k}(c_1))|^2}. \quad (4.19)
\]

By the estimates (4.10) we have

\[
|f'_{\beta_k}(f_{\beta_k}(c_1))| \leq \frac{2^\frac{q+1}{q} C_2 R_{n+1}^\frac{q+1}{q}}{C_1^\frac{q+1}{q}} \quad \text{and} \quad \frac{|f'_{\beta_n}(f_{\beta_n}(c_1))|}{|f'_{\beta_k}(f_{\beta_k}(c_1))|} \geq \frac{1}{2^\frac{q+1}{q}}. \quad (4.20)
\]
for \( j = 1, 2, \ldots, n - 1 \). Putting (4.20) into (4.19) and using Remark 4.7 and (3.6), we get

\[
\frac{\text{vol}(\mathcal{V}_{n+1} \cap D_n)}{\text{vol}(D_n)} \geq \left( \frac{1 - r}{1 + r} \right)^6 \frac{1}{5^6} \frac{1}{2^{(q+1)(2n-1)}} \left( \frac{1}{2^{q+1}} \right)^{2(n-1)} \frac{C_1}{2\frac{q}{R}} \frac{C_2}{2\frac{q}{R}} \sum_{\beta \in D_k \cap D_n} 1
\]

\[
\simeq \left( \frac{1 - r}{1 + r} \right)^6 \frac{2^{2(q+1)}}{5^6} \frac{C_1}{2^{q+1}} \frac{C_2}{2^{q+1}} \cdot \frac{1}{5^6} \frac{1}{2^{(q+1)n}} \cdot \frac{1}{2^{q+1}} \sum_{\beta \in D_k \cap D_n} 1
\]

\[
= \frac{M}{2^{q+1} R_n + 1}
\]

where \( M = \frac{(1 - r)^6 2^{q+1}}{5^6 (1 + r)^6 C_2} \).

Similarly, we consider the case \( n = 1 \). By Definition 4.2, the set \( D_1 \) has only one element, i.e. \( D_1 \) and its Lebesgue measure \( \text{vol}(D_1) \leq \pi r^2 \). The set \( D_1 \) contains finitely many subsets \( D_k \in D_2 \). We leave to the reader to show that

\[
\frac{\text{vol}(\mathcal{V}_2 \cap D_1)}{\text{vol}(D_1)} \geq \frac{M'}{R_2^{\frac{q}{2}}},
\]

for some \( M' > 0 \) depending on \( \varepsilon, r, q, C_1, C_2, R_1 \). \( \square \)

By Lemma 4.10, the numbers \( \Delta_n \) from Proposition 2.4 are equal to

\[
\Delta_1 = \frac{M'}{R_2^{\frac{q}{2}}}, \quad \Delta_n = \frac{M}{2\frac{q}{n^q} R_n^{\frac{q}{n^q}}}, \quad n \geq 2.
\]

Theorem 4.4 is a consequence of McMullen’s result (Proposition 2.4) and Lemmas 4.8 and 4.10. We omit the proof since it is analogous to the proof of Theorem 2.3.

5. The proof of the upper estimate of \( \dim_H(G_R) \)

Let \( f_\beta = \beta f, \beta \in \mathbb{C} \) be a one-parameter of family of elliptic functions such that one of the critical values of \( f \), denoted by \( f(c_1) \neq 0 \), is a pole of multiplicity \( p \leq q \) and all the other critical values of \( f \) are attracted by attracting periodic points. Let \( B_R(\infty) = \{ z \in \mathbb{C}: |z| > R \}, R > 0 \). Since in the estimate (2.7) we can replace \( f \) by \( f_\beta \), then

\[
|f_\beta(z)| \leq \frac{C_1(b)}{|z - b|^{q_b}},
\]

where \( q_b \) denotes the order of the pole \( b \) and

\[
\left\{ z \in \mathbb{C}: |z| \geq \frac{C_1(b)}{\varepsilon^{q_b}} \right\} \subset f_\beta(B(b, \varepsilon)) \quad (5.1)
\]

for \( \beta \in B(1, r), z \in B(b, \varepsilon_0), b \in f^{-1}(\infty) \cap B_R(\infty) \).

As the series

\[
\sum_{b \in f^{-1}(\infty) \cap B_R(\infty)} \frac{1}{|b|^m}
\]

(5.2)
converges for all \( m > 2 \), then there exists \( R' > 0 \) such that
\[
\sum_{b \in f^{-1}(\infty) \cap B_R(\infty)} \frac{q}{|b|(q+1)s/q} < 1
\]
for all \( R \geq R' \) and \( s > 2q/(q+1) \).

Set
\[
R'' := \max\{R', C/\varepsilon^q, 1\},
\]
where \( C = \max\{|C_1(b)| : b \in \mathcal{R} \cap f^{-1}(\infty)\} \). We shrink \( \varepsilon \) such that \( \bigcup_{b \in f^{-1}(\infty) \cap R} B(b, 2\varepsilon) \subset \mathcal{R} \) and \( B(b_i, 2\varepsilon) \cap B(b_j, 2\varepsilon) = \emptyset \), \( b_i \neq b_j \in \mathcal{R} \cap f^{-1}(\infty) \). As in Section 4, we consider the auxiliary functions \( g_n(\beta) = f_\beta^n(c_1), \ n \geq 1 \). Fix \( R > R'' \) and define
\[
\mathcal{I}(\infty) := \{b: b \in f^{-1}(\infty) \cap B_R(\infty)\}.
\]

**Definition 5.1.** We define the following family of sets

\[
\mathcal{F}_0 = \{F_0 = B(1, r)\},
\]
\[
\mathcal{F}_1 = \{F_1 = g_1^{-1}(B(f(c_1), \varepsilon)) \subset F_0\},
\]
\[
\mathcal{F}_2 = \{F_2 \subset F_1 | \exists b^{(2)} \in \mathcal{I}(\infty): B(b^{(2)}, 2\varepsilon) \subset B_R(\infty), \ F_2 \text{ is a component of } g_2^{-1}(B(b^{(2)}, \varepsilon))\},
\]
\[
\cdots
\]
\[
\mathcal{F}_n = \{F_n \subset F_{n-1} | \exists b^{(n)} \in \mathcal{I}(\infty): B(b^{(n)}, 2\varepsilon) \subset B_R(\infty), \ F_n \text{ is a component of } g_n^{-1}(B(b^{(n)}, \varepsilon))\},
\]
\[
\cdots
\]

Let
\[
\mathcal{G}_k = \bigcup_{F_k \in \mathcal{F}_k} F_k, \quad \mathcal{G}_R := \bigcap_{k=1}^{\infty} \mathcal{G}_k.
\]

Moreover, let \( F^*_n \) denote components of \( g_n^{-1}(B(b^{(n)}, 2\varepsilon)) \) and \( \mathcal{F}^*_n \) be the family of the sets \( F^*_n \subset F_{n-1} \).

The definition with \( q = 3 \) is illustrated in Figure 2. Parameters \( \beta^{(n)} \) denote preimages of poles \( b^{(n)} \).

We skip the proof of non-emptiness of \( \mathcal{F}_n \), since it follows directly from (5.1) and it is similar to the proof of Proposition 4.3.

**Lemma 5.2.** For each \( F^*_n \subset \mathcal{F}^*_n, \ n \geq 1 \), the map \( g_n \) is conformal on \( F^*_n \).

**Proof.** This proof is a modification of the proof of Lemma 4.6, where one of the key points consisted in showing that \( \hat{g}_{n+1} \) is injective in \( D_n \). It is not true that \( \hat{g}_{n+1} \) is one-to-one in \( F^*_n \) since \( g_n(F^*_n) = B(b^{(n)}, 2\varepsilon) \) and the argument used in the proof of Lemma 4.6 fails. Nevertheless we can construct a subset \( \hat{F}_{n+1} \) of \( F^*_n \) (which will be an analogue of \( \hat{D}_{n+1} \)) on which \( \hat{g}_{n+1} \) is univalent and repeat the previous arguing.

The map \( g_1 \) is one-to-one and holomorphic on \( F^*_1 \). By induction, we show that the maps \( g_n, \ n \geq 2 \), are conformal. Suppose that \( g_n, \ n \geq 1 \), is conformal on \( F^*_n \subset \mathcal{F}^*_n \), we
prove that $g_{n+1}$ is conformal on $F^*_{n+1} \subset F^*_n$, where $F^*_{n+1} \subset F^*_{n+1}, F^*_n \in F^*_n$. If $n = 1$ then we take the disc $B(b^{(1)}, 2\varepsilon)$ with $b^{(1)} = f(c_1) \in \mathcal{I}(\infty)$ and if $n \geq 2$ we consider a disc $B(b^{(n)}, 2\varepsilon) \subset B_R(\infty)$, where $b^{(n)} \in \mathcal{I}(\infty)$. We know that $F^*_n$ is a component of $g_n^{-1}(B(b^{(n)}, 2\varepsilon)), n \geq 1$. Let $\beta_n = g_n^{-1}(b^{(n)}) \in F^*_n$. Analogously $F^*_{n+1} \subset F^*_n$ is a component of $g_n^{-1}(B(b^{(n+1)}, 2\varepsilon))$ for some $B(b^{(n+1)}, 2\varepsilon) \subset B_R(\infty), b^{(n+1)} \in \mathcal{I}(\infty)$. There is $\bar{R} > R$ such that $B(b^{(n+1)}, 2\varepsilon) \subset P(0, \bar{R}, 2\bar{R})$ and

$$P(0, (1 - \alpha)\bar{R}, (2 + \alpha)\bar{R}) := \{z \in \mathbb{C}: (1 - \alpha)\bar{R} < |z| < (2 + \alpha)\bar{R}\} \subset B_R(\infty).$$

Observe that

$$f_\beta(B(b^{(n)}, 2\varepsilon) \supset B_R(\infty) \supset P(0, (1 - \alpha)\bar{R}, (2 + \alpha)\bar{R})$$

for $\beta \in B(1, r)$. There exist $\psi \in \mathbb{R}$ such that $B(b^{(n+1)}, 2\varepsilon) \subset P^*(0, (1 - \alpha)\bar{R}, (2 + \alpha)\bar{R})$, where

$$P^*(0, (1 - \alpha)\bar{R}, (2 + \alpha)\bar{R}) = \{z \in \mathbb{C}: \psi - \pi/8 \leq \arg z \leq \psi + 9/8\pi, (1 - \alpha)\bar{R} < |z| < (2 + \alpha)\bar{R}\}.$$

We define a map $\hat{g}_{n+1}(\beta) = \beta_n f(g_n(\beta))$. Let $\hat{F}_{n+1}$ be a component of $\hat{g}_{n+1}^{-1}(P^*(0, (1 - \alpha)\bar{R}, (2 + \alpha)\bar{R}))$ contained in $F^*_n$, i.e.

$$\hat{g}_{n+1}(\hat{F}_{n+1}) = \{z \in \mathbb{C}: (1 - \alpha)\bar{R} < |z| < (2 + \alpha)\bar{R}, \psi - \pi/8 < \arg z < \psi + \pi\},$$

for $\alpha = \sin(\pi/8)$ and $\psi \in \mathbb{R}$. We prove that $\hat{g}_{n+1}$ is univalent on $\hat{F}_{n+1} \subset F^*_n$. Let $\hat{F}_n$ be a component of $h(P^*(0, (1 - \alpha)\bar{R}, (2 + \alpha)\bar{R}))$, where $h$ is the branch of $f_\beta^{-1}$ such that $h(\infty) = b^{(n)}$. As $\hat{F}_n$ we choose the component of $h(P^*(0, (1 - \alpha)\bar{R}, (2 + \alpha)\bar{R}))$ which is equal to $g_n(\hat{F}_{n+1})$. Observe that $\hat{F}_n \subset \text{Rot}(U(b^{(n)}, \varepsilon))$, where $\text{Rot}$ denotes rotation about $b^{(n)}$ by some angle (for definition of $U(b^{(n)}, \varepsilon)$ see (2.10)). Thus, $f$ is univalent on $\hat{F}_n$.

We show that $\hat{g}_{n+1}$ is one-to-one in $\hat{F}_{n+1} \subset F^*_n$. Take $\beta', \beta'' \in \hat{F}_{n+1}$ such that $\hat{g}_{n+1}(\beta') = \hat{g}_{n+1}(\beta'')$. By definition of the map $\hat{g}_{n+1}$, we have $f(g_\beta(\beta')) = f(g_\beta(\beta''))$, where $g_\beta(\beta'), g_\beta(\beta'') \in \hat{F}_n$. Since $f$ is one-to-one in $\hat{F}_n$, so $g_{\beta}(\beta') = g_{\beta}(\beta'')$ and this implies that $\beta' = \beta''$. This follows from the injectivity of the map $g_{\beta}$ in $F^*_n$.

To end the proof, it is enough to follow the arguing from the proof of Lemma 4.6 replacing the semi-annulus $P^+(0, R_{n+1}, 2R_{n+1})$ with $P^+(0, \bar{R}, 2\bar{R}) = \{z \in \mathbb{C}: (1 - \alpha)\bar{R} < |z| < (2 + \alpha)\bar{R}, \psi < \arg z < \psi + \pi\}$. 

Now we show the main result of this section.

**Proof of Theorem 3.** For $n \geq 2$, consider a sequence of sets $F_n \subset F_{n-1} \subset \ldots \subset F_2 \subset F_1, F_k \in F_k$, $k = 1, \ldots, n$, and the corresponding sequence of poles $b^{(n)}, b^{(n-1)}, \ldots, b^{(2)}, b^{(1)} = f(c_1)$ (see Definition 5.1). Taking $\beta \in F_n$ and $z = g_{n-1}(\beta) = f_{\beta}^{-1}(c_1)$, we have

$$|b^{(n)}| \approx |f(z)| \approx \frac{1}{|z - b^{(n-1)}| q_\beta^{(n-1)}} \quad \text{and} \quad |f'(z)| \approx |b^{(n)}|^{\frac{q_\beta^{(n-1)} + 1}{q_\beta^{(n-1)}}}.$$ 

Since $|b^{(n)}| > R$ (enlarging $R$ if necessary), it follows that

$$|g'_\beta(\beta)| = |f(g_{n-1}(\beta)) + \beta f'(g_{n-1}(\beta)) g'_{n-1}(\beta)| \approx |b^{(n)}| \frac{q_\beta^{(n-1)} + 1}{q_\beta^{(n-1)}} \left(|b^{(n)}|^{\frac{1}{q_\beta^{(n-1)}}} + |g'_{n-1}(\beta)|\right) \approx |b^{(n)}| \frac{q_\beta^{(n-1)} + 1}{q_\beta^{(n-1)}} |g'_{n-1}(\beta)|.$$
and, by induction,
\[ |g'_n(\beta)| \approx |f(c_1)|, \quad (5.6) \]
as \(|g'_1(\beta)| = |f(c_1)|\). As \(g_n\) is conformal on \(g_{n-1}^{-1}(B(b^{(n)}, 2\varepsilon))\), so each of \(q_{b^{(n)}}\) inverse branches \(g_{n-1}\) is conformal on \(B(b^{(n)}, 2\varepsilon)\). Let the branch \(g_{n-1}^{-1}, i \in \{1, 2, \ldots, q_{b^{(n)}}\}\) send \(B(b^{(n)}, \varepsilon)\) to \(F_n\). Using Koebe Lemma, we obtain that \(L(g_{n-1}^{-1}, B(b^{(n)}, \varepsilon)) \leq K = 3^4\). Therefore, by (5.6),
\[ \text{diam}(F_n) \leq \varepsilon K |(g_{n-1}^{-1})'(b^{(n)})| \]
\[ = \varepsilon K \left| |b^{(n)}|^{\frac{q_{b^{(n-1)}}^{(n-1)}+1}{q_{b^{(n-2)}}^{(n-2)} \cdot q_{b^{(n-2)}}^{(n-2)} \cdot \ldots \cdot |b^{(2)}|^{\frac{q_{b^{(1)}}^{(1)}}{q_{b^{(1)}}^{(1)}}}|f(c_1)|} \right|, \quad (5.7) \]

The family \(F_n\) provided in Definition 5.1 covers \(G_R\). In the following part we use the notation \(|b^{(k)}| > R\), considering poles \(b^{(k)} \in \mathcal{I}(\infty)\). Taking \(s > 0\) and using (5.7), we get
\[ \sum_{F_n \in F_n} \text{diam}^s(F_n) = \sum_{F_2 \in F_2} \sum_{F_3 \in F_2} \ldots \sum_{F_n \in F_{n-1}} \text{diam}^s(F_n) \]
\[ \leq q \sum_{|b^{(2)}| > R} q \sum_{|b^{(3)}| > R} \ldots q \sum_{|b^{(n)}| > R} \text{diam}^s(F_n) \]
Proof of Theorem 4. Consider the family $f \in F_n$ given in Definition 5.1 assuming that $f(c_1) \neq 0$ is a pole of maximal multiplicity $q$. Recall that $D(a)$ is a set of parameters such that $f_{\beta}^{n}(c_1) \in P^+(0, R_n, 2R_n)$ for all $n \geq 2$ (see Definition 4.2). Taking $a$ such that $R_2 \geq R$, we have $D(a) \subset \mathcal{E}_0 := \mathcal{E} \cap \mathcal{G}_R$. It follows from Theorem 4.4 that $\dim_H(\mathcal{E}_0) \geq 2q/(q + 1)$. The converse inequality is a consequence of Theorem 3.

6. Final remarks

An example of a function satisfying the assumptions of Theorem 2 with $q = 2$ is given in [7]. For the Weierstrass function $\wp_\Lambda$ generated by the lattice $\Lambda$ with the invariants $g_2(\Lambda) \approx 26.5626$ and $g_3(\Lambda) \approx -26.2672$, Hawkins and Koss proved that $\wp_\Lambda$ has an attracting fixed point $p \approx 1.5566$, which attracts two critical values $e_1 \approx 1.5539$ and $e_2 \approx 1.4206$, while the third critical value $e_3 \approx -2.9746$ is a pole. Thus, in this case $\dim_H(\mathcal{E}) \geq 4/3$.

Observe that analogues of Theorems 1, 2, 3, 4 can be proved if we assume that $f$ is a simply periodic meromorphic function from the Speiser class $\mathcal{S}$ such that $\infty$ is not an asymptotic value of $f$ (so $f$ has infinitely many poles). Then,

$$\dim_H(I(f)) = \frac{q}{q + 1},$$

(6.1)

where $q$ is the maximal multiplicity of poles of $f$. In order to obtain the upper bound, it is enough to repeat the proof for elliptic functions given in [11], while the lower bound can be proved by applying the methods from the proof of Theorem 2.3.

Assuming additionally that one of the singular values $e_1 \neq 0$ of $f$ is a pole of the multiplicity $q$ and the other singular values $e_2, \ldots, e_k$ belong to basins of attraction of the periodic sinks, we define a one-parameter family $f_\beta(z) = \beta f(z)$, $\beta \in \mathbb{C} \setminus \{0\}$ and the set of escaping parameters $\mathcal{E} = \{\beta \approx 1: \lim_{n \to \infty} f_\beta^n(e_1) = \infty\}$. Then, we can prove that

$$\dim_H(\mathcal{E}) \geq \frac{q}{q + 1}.$$  

(6.2)

The estimate follows from the proof of Theorem 4.4. To get the upper estimate of $\dim_H(\mathcal{G}_R)$, where $\mathcal{G}_R$ can be defined for this family similarly as in Definition 5.1, we
adopt the techniques used in the proof of Theorem 3, replacing the condition $m > 2$ for convergence of the series (5.2) by $m > 1$. Putting (6.1) and (6.2) together, we get
\[
\dim_H(\mathcal{G}_R) \leq \dim_H(I(f)) = \frac{q}{q+1} \leq \dim_H(\mathcal{E})
\]
for some $R > 0$. Examples of simply periodic functions satisfying above assumptions can be find in the family $\lambda \tan^q z$, $q \geq 2$.

References

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