Lecture 1

Generating Functions

1.1 General comments on the use of Generating Functions (GFs)

We mainly use three types of Generating Functions:

- the probability generating function (p.g.f.)

  $$g_X(\alpha) = E\alpha^X, \qquad 0 \le \alpha \le 1$$

  for $\mathbb{Z}$ -valued X,

- the Moment Generating Function (MGF)

  $$M_X(\alpha) = Ee^{\alpha X},$$

  for suitable X and suitable α;

- the Characteristic Function (CF)

  $$\varphi_X(t) = Ee^{i\alpha X}$$

  valid for all X and all real α.

As usual, F_X will denote the (cumulative) Distribution Function of X (c.d.f. of X). For any generating function G_X of X. we need the following

1. **A Uniqueness Theorem:** each of the functions F_X and G_X (on the appropriate domains) determines the other. Of course, F_X always determines G_X. The result in the other direction follows from the Convergence Theorem described below.

2. **An effective method of calculating from G_X the moments** $\mu_r(X) = EX^r$ $(r = 1, 2 \ldots)$ of X which exist.

3. **An 'Independence means Multiply' theorem**: if X and Y arc independent then
$$G_{X+Y}(\alpha) = G_X(\alpha)\, G_Y(\alpha).$$
Obviously, by combining results 1 and 3, we can in principle find the distribution of the sum of two independent RVs.

4. **A Convergence Theorem**: we have $G_{X_n}(\alpha) \to G_X(\alpha)$ for all relevant α if and only if $F_{X_n}(x)$ at every point x at which $F_X(x)$ is continuous Such continuity points of F are dense in $\mathbb{R}$ so this is good enough.

Remark 1 *We shall prove the relevant properties only for 'p.g.f.' case,. It requires too much Analysis to deal with the other cases.*

We shall use two other types of generating function: the **Laplace Transform:**
$$L_X(\alpha) = Ee^{-\alpha X}, \quad a \geq 0,$$
for RV's with values in $[0, \infty)$ and the **Cummulant Generating Function** (CGF)
$$C_X(\alpha) = \ln M_X(\alpha).$$
Of course, we have 'Independence means Add' for CGFs.

1.2 Probability generating functions (pgfs)

1.2.1 Probability generating function (p.g.f.) $g_X(.)$ of X.

Let X be a $\mathbb{Z}^+ \overset{df}{=} \{0, 1, 2, \ldots\}$ We define probability generating function (pgf) $g_X(.)$ of X on $< 0, 1 >$ via:
$$g_X(\alpha) = E\alpha^X = \sum_{k=0}^{\infty} \alpha^k P(X = k), \quad 0 \leq \alpha \leq 1.$$
Note that
$$g_X(1) = 1, \quad g_X(0) = P(X = 0).$$
In this section, it is understood that we deal only with $\mathbb{Z}^+$ RVs.

Example 2 *: p.g.f. of Poisson(λ). If $\lambda > 0$ and X has the Poisson(λ) distribution, then*
$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{Z}^+,$$
so that
$$g_X(\alpha) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\alpha\lambda)^k}{k!} = e^{-\lambda} e^{\alpha\lambda} = e^{\lambda(\alpha-1)}.$$
result of considerable importance.

1.2.2 The Uniqueness Theorem for pgfs.

If we know the values $g_X(\alpha)$, $\alpha \in< 0, 1 >$, then we can find $P(X = k)$ for $k \in \mathbb{Z}^+$. This follows from the Convergence Theorem below. Finding moments. We have - with discussion of rigor below

$$\begin{aligned} g_X(\alpha) &= E\alpha^X = \sum_{k=0}^{\infty} \alpha^k P(X = k), \\ g'_X(\alpha) &= E\left(X\alpha^{X-1}\right) = \sum_{k=1}^{\infty} k\alpha^{k-1} P(X = k), \\ g'_X(1) &= EX = \sum_{k=1}^{\infty} kP(X = k) \leq \infty, \\ g''_X(\alpha) &= E\left(X(X-1)\alpha^{X-2}\right), \\ g''(1) &= E(X(X-1)) \leq \infty, \end{aligned}$$

so we can find EX, EX^2, and hence, if these are finite. $\text{var}(X)$, from g_X. Rigor is provided by standard results on differentiation of power series together with monotonicity properties arising from the fact that the coefficients in the power series determining a p.g.f. are non-negative. Of course, the p.g.f. power series has radius of convergence at least 1. In a moment, we do some serious Analysis on the Convergence Theorem; and that's enough for this section.

Exercise 3

Poisson mean and variance. Use g_X, to show that if X has the Poisson(λ) distribution, then $E(X) = \lambda$ and $\text{var}(X) = \lambda$.

1.2.3 'Independence means Multiply' Lemma.

Lemma 4 *If X and Y are independent.: ($\mathbb{Z}^+$ valued) RVs, then*

$$g_{X+Y}(\alpha) = g_X(\alpha)\, g_Y(\alpha).$$

Proof. Using known properties of independent RV's, we have

$$g_{X+Y}(\alpha) = E\alpha^{X+Y} = E\alpha^X E\alpha^Y = g_X(\alpha)\, g_Y(\alpha).$$

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Exercise 5

Use above Lemma to show that if Y is the number of Heads in n tosses of a coin with probability p of Heads, then $g_Y(\alpha) = (q + p\alpha)^n$. This ties in the Binomial Theorem with the binomial distribution.

Summing independent Poisson variables. Use above Lemma to show that if X and Y are independent, $X \sim$ Poisson(λ), $Y \sim$ Poisson(μ), then $X + Y \sim$ Poisson($\lambda + \mu$).

1.2.4 The Convergence Theorem for pgfs.

Theorem 6 *Suppose that X and X_n (where $n = 1, 2, 3, \ldots$) are $\mathbb{Z}^+$ valued RV's, and that*

$$g_{X_n}(\alpha) -> g_X(\alpha), \text{ for } 0 \leq \alpha \leq 1,$$

then

$$P(X_n = k) -> P(X = k), \quad k \in \mathbb{Z}^+.$$

for $n -> \infty$.

Proof. We need the following lemma.

Lemma 7 *Suppose that $a_{k,n}$, $k \in \mathbb{Z}^+$, $n \in \mathbb{N}$ are non negative constants with $\sum_{k\geq 0} a_{k,n} \leq 1$. Suppose that for $0 \leq \alpha \leq 1$, we have*

$$a_n(\alpha) = \sum_{k=0}^{\infty} a_{k,n}\alpha^k \to a(\alpha) = \sum_{k=0}^{\infty} a_k \alpha^k.$$

Then

$$a_{0,n} \to a_0.$$

Proof. Note that for $0 \leq \alpha \leq 1$,

$$a_n(\alpha) = a_{0,n} + A_n(\alpha), \qquad a(\alpha) = a_0 + A(\alpha),$$

where $0 \leq A_n(\alpha) \leq \alpha \sum_{k=1}^{\infty} a_{k,n}\alpha^{k-1} = \alpha \sum_{k=1}^{\infty} a_{k,n} \leq \alpha$, similarly for $A(\alpha)$. Let $\varepsilon \in (03 >$. Choose N so that for $n \geq N$,

$$\left| a_n\left(\frac{1}{3}\varepsilon\right) - a\left(\frac{1}{3}\varepsilon\right) \right| < \frac{1}{3}\varepsilon.$$

Moreover for $n \geq N$,

$$\begin{aligned}
|a_{0,n} - a_0| &= \left| \left(a_n\left(\frac{1}{3}\varepsilon\right) - A_n\left(\frac{1}{3}\varepsilon\right)\right) - \left(a\left(\frac{1}{3}\varepsilon\right) - A\left(\frac{1}{3}\varepsilon\right)\right) \right| \\
&\leq \left| a_n\left(\frac{1}{3}\varepsilon\right) - a\left(\frac{1}{3}\varepsilon\right) \right| + \left| A_n\left(\frac{1}{3}\varepsilon\right) \right| + \left| A\left(\frac{1}{3}\varepsilon\right) \right| \\
&\leq \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.
\end{aligned}$$

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Immediately from the lemma

$$P(X_n = 0) \to P(X = 0), \quad g_{X_n}(0) \to g_X(0).$$

Next for $\alpha \in (0, 1 >$,

$$\frac{g_{X_n}(\alpha) - P(X_n = 0)}{\alpha} \to \frac{g_X(\alpha) - P(X = 0)}{\alpha}$$

as $n \to \infty$. Thus

$$\sum_{k=1}^{\infty} P(X_n = k)\,\alpha^{k-1} \to \sum_{k=1}^{\infty} P(X = k)\,\alpha^{k-1}, \quad 0 < \alpha \leq 1.$$

By lemma we have $P(X_n = 1) \to P(X = 1)$. The whole result follows now by induction. ■

1.2.5 Composition of pgf's

Suppose that $\{X_i\}_{i\geq 1}^{\infty}$ is a sequence of i.i.d. $\mathbb{Z}^+$ valued random variable with common p.g.f. $g_X(s)$. Suppose also that M is another $\mathbb{Z}^+$ valued random variable with p.g.f. $g_M(s)$ .that is independent of $\{X_i\}_{i\geq 1}^{\infty}$. Let

$$Y = \sum_{i=0}^{M} X_i,$$

with $X_0 = 0$ convention. We have

Proposition 8 *P.g.f. g_Y of Y is given by:*

$$g_Y(s) = g_M(g_X(s)),$$

for $s \in <0, 1>$.

Proof. We have by the definition of g_Y

$$\begin{aligned}
g_Y(s) &= Es^{\sum_{i=0}^{M} X_i} = E\left(E\left(s^{\sum_{i=0}^{M} X_i} | M\right)\right) \\
&= \sum_{j=0}^{\infty} E\left(s^{\sum_{i=0}^{M} X_i} | M = j\right) P(M = j) \\
&= P(M = 0) + \sum_{j=1}^{\infty} E\left(s^{\sum_{i=0}^{M} X_i} | M = j\right) P(M = j) \\
&= P(M = 0) + \sum_{j=0}^{\infty} g_X^j(s)\, P(M = j) = g_M(g_X(s)).
\end{aligned}$$

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