Lecture 2

Galtons branching processes

In this section we consider a class of Markov chains, known as *branching* processes, which have a wide variety of applications in the biological, sociological, and engineering sciences.

Consider a population consisting of individuals able to produce offspring of the same kind. Suppose that each individual will, by the end of its lifetime, have produced j new offspring with probability P_j, $j \geq 0$, independently of the number produced by any other individual. We suppose that $P_j < 1$ for all $j \geq 0$. The number of individuals initially present, denoted by X_0, is called the size of the *zeroth* generation. All offspring of the zeroth generation constitute the *first* generation and their number is denoted byX_1. In general, let X_n denote the size of the n th generation. It follows that $\{X_n,\ n = 0, 1, ...\}$ is a Markov chain having as its state space the set of nonnegative integers.

Note that state 0 is a *recurrent* state, since clearly $P_{00} = 1$. Also, if $P_0 > 0$, all other states are *transient*. This follows since $P_{i0} = P_0^i$, which implies that starting with i individuals there is a positive probability of at least P_0^i that no later generation will ever consist of i individuals. Moreover, since any finite set of transient states $\{1, 2, ..., n\}$ will be visited only finitely often, this leads to the important conclusion that, if $P_0 > 0$, then the population will either *die out* or its size will converge to *infinity*.

Let

$$\mu = \sum_{i=0}^{\infty} j P_j$$

denote the *mean number* of offspring of a single individual, and let

$$\sigma^2 = \sum_{j=0}^{\infty} (j - \mu)^2 P_j$$

be the *variance* of the number of offspring produced by a single individual.

Let us suppose that $X_0 = 1$, that is, initially there is a *single* individual present. We calculate EX_n and $\mathrm{var}\,(X_n)$ by first noting that we may write

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i \tag{2.1}$$

where Z_i represents the number of offspring of the ith individual of the $(n-1)$st generation.

2.1 Elementary approach

By conditioning on X_{n-1}, we obtain

$$\begin{aligned} EX_n &= E\left[E\left[X_n|X_{n-1}\right]\right] \\ &= E\left[E\left[\sum_{i=1}^{X_{n-1}} Z_i|X_{n-1}\right]\right] \\ &= E\left[\mu X_{n-1}\right] \\ &= \mu E X_{n-1} \end{aligned} \tag{2.2}$$

where we have used the fact that $EZ_i = \mu$. Since $EX_0 = 1$, Equation (2.2) yields

$$\begin{aligned} EX_1 &= \mu, \\ EX_2 &= \mu E X_1 = \mu^2, \\ &\vdots \\ EX_n &= \mu E X_{n-1} = \mu^n. \end{aligned}$$

Similarly, $\mathrm{var}\,(X_n)$ may be obtained by using the conditional variance formula

$$\mathrm{var}\,(X_n) = E\left[\mathrm{var}\,(X_n|X_{n-1})\right] + \mathrm{var}\,(E\,(X_n|X_{n-1}))\,.$$

Now, given X_{n-1}, X_n is just the sum of X_{n-1} independent random variables each having the distribution $\{P_j, j > 0\}$. Hence,

$$\mathrm{var}\,(X_n|X_{n-1}) = \sigma^2 X_{n-1}.$$

Thus, the conditional variance formula yields

$$\begin{aligned} \mathrm{var}\,(X_n) &= E\left[\sigma^2 X_{n-1}\right] + \mathrm{var}\,(\mu X_{n-1}) \\ &= \sigma^2 \mu^{n-1} + \mu^2\,\mathrm{var}\,(X_{n-1})\,. \end{aligned}$$

Using the fact that $\mathrm{var}\,(X_0) = 0$ we can show by mathematical induction that the preceding implies

$$\mathrm{var}\,(X_n) = \left\{ \begin{array}{ccc} \sigma^2\mu^{n-1}\left(\frac{\mu^n-1}{\mu-1}\right) & if & \mu \neq 1 \\ n\sigma^2 & if & \mu = 1 \end{array} \right. . \tag{2.3}$$

Let π_0 denote the probability that the population will eventually *die out* (under the assumption that $X_0 = 1$). More formally,

$$\pi_0 = \lim_{n\to\infty} P\{X_n = 0|X_0 = 1\}$$

The problem of determining the value of π_0 was first raised in connection with the extinction of family surnames by Galton in 1889. We first note that $\pi_0 = 1$ if $\mu < 1$. This follows since

$$\begin{aligned} \mu^n &= EX_n = \sum_{i=1}^{\infty} iP(X_n = i) \\ &\geq \sum_{i=1}^{\infty} 1P(X_n = i) \\ &= P(X_n \geq 1) = 1 - P(X_n = 0). \end{aligned}$$

In fact, it can be shown that $\pi_0 = 1$ even when $\mu = 1$. When $\mu > 1$, it turns out that $\pi_0 < 1$, and an equation determining π_0 may be derived by conditioning on the number of offspring of the initial individual, as follows:

$$\begin{aligned} \pi_0 &= P[\textit{population dies out}] \\ &= \sum_{j=0}^{\infty} P[\textit{population dies out}|X_1 = j]\, P_j \end{aligned}$$

Now, given that $X_1 = j$, the population will eventually die out if and only if each of the j families started by the members of the first generation eventually dies out. Since each family is assumed to act independently, and since the probability that any particular family dies out is just π_0, this yields

$$P[\textit{population dies out}|X_1 = j] = \pi_0^j$$

and thus π_0 satisfies

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j \tag{2.4}$$

in fact when $\mu_j > 1$, it can be shown that π_0 is the smallest positive number satisfying Equation (2.4).

2.2 Approach based on pgfs

Let $g_n(s)$ denote the p.g.f. of X_n, $n = 1, 2, \ldots$. Let $g_1(s)$ be denoted by $g(s)$. Then based on the Proposition 8 we get

$$g_{n+1}(s) = g_n(g(s)). \tag{2.5}$$

More precisely $g_1(s) = g(s)$, $g_2(s) = g(g(s)) = g^{(2)}(s), \ldots, g_n(s) = g^{(n)}(s)$, where $g^{(k)}$ denotes $k-$ fold convolution of function g with itself. Now formula (2.5) can be rewritten in the following way:

$$g_{n+1}(s) = g(g_n(s)).$$

Notice that probability π_0 of population dying out can be obtained from continuity of probability property, namely

$$\pi_0 = P\left(\bigcup_{n=1}^{\infty} \{X_n = 0\}\right) = \lim_{n\to\infty} P(X_n = 0).$$

But $P(X_n = 0) = g_n(0) \stackrel{df}{=} \alpha_n$. Hence we have

$$\alpha_{n+1} = g(\alpha_n), \quad n \geq 1.$$

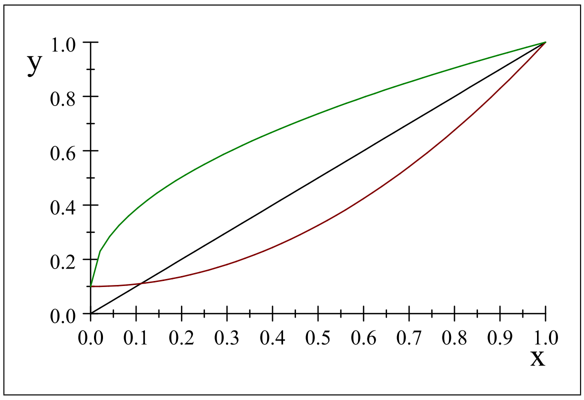
Now notice that :

1. $g(1) = 1$, $g(0) = P(X_1 = 0)$, and $g(s)$ increases from $g(0)$ to 1.
2. $g'(1) = EX_1$.

Hence if a sequence $\{\alpha_n\}_{n\geq 1}$ is convergent and tends to a solution of an equation

$$\alpha = g(\alpha). \tag{2.6}$$

Depending on value of $g'(1)$ two possible situations are possible. One is pictured in 'red' while the other in 'green'. It is easily seen that if $g(0) = 0$ then there are two solutions of the equation (2.6) 0 and 1. But it is obvious that then $\pi_0 = 0$. If $g(0) > 0$, then one can see that if $g'(1) \leq 0$ then (green line) the only solution of the equation (2.6) is 1 if $g'(1) > 1$ then there are two solutions and π_0 is the smaller one.



Examples

1. If $P_0 = \frac{1}{2}$, $P_1 = \frac{1}{4}$, $P_2 = \frac{1}{4}$, then determine π_0.
   Solution: Since $\mu = \frac{3}{4} < 1$, it follows that $\pi_0 = 1$.

2. If $P_0 = \frac{1}{4}$, $P_1 = \frac{1}{4}$, $P_2 = \frac{1}{2}$, then determine π_0.
   Solution: π_0 satisfies
   $$\pi_0 = \frac{1}{4} + \frac{1}{4}\pi_0 + \frac{1}{2}\pi_0^2.$$
   or
   $$2\pi_0^2 - 3\pi_0 + 1 = 0.$$
   The smallest positive solution of this quadratic equation is $\pi_0 = \frac{1}{2}$.

3. In Examples 1 and 2, what is the probability that the population will die out if it initially consists of n individuals?

Solution: Since the population will die out if and only if the families of each of the members of the initial generation die out, the desired probability is π_0^n. For Example 1 this yields $\pi_0^n = 1$, and for Example 2, $\pi_0^n = \left(\frac{1}{2}\right)^n$.