Part II

Exponential distribution. Poisson process.

Lecture 3

Exponential distribution

3.1 Properties of exponential distribution

3.1.1 Geometric distribution:

We say random variable X has geometric distribution with parameter $p\in(0.1)$ if only

$$P(X=i)=p(1-p)^{i-1}; i=1,2,\ldots.$$

It is easy to notice supp$(X)=\mathbb{N}$.
Interpretation. Let us imagine sequence of uniform independent Bernoulli experiments . Let us assume that probability of success is $p>0$. Let us denote by X number of the first success. Of course an event $\{X=i\}$ means that before the first success (that happened at i) there were $i-1$ failures. That is $P(X=i)=p(1-p)^{i-1}$. Thus X has geometric distribution .
It turns out that geometric distribution is used in reliability theory. It has the following property called "lack of memory":

$$\forall n,k\in\mathbb{N}: P(X>n+k|X>n)=(1-p)^k \tag{3.1}$$

does not depend on n !!! .

1. The proof of this property is very simple. Let us note that $P(X>n+k|X>n)=\frac{P(X>n+k)}{P(X>n)}$. It remains hence to calculate $P(X>m)$ for any m . We have $P(X>m)=\sum_{i=m+1}^{\infty}p(1-p)^{i-1}=p(1-p)^m\sum_{i=0}^{\infty}(1-p)^i$ $=(1-p)^m$. Now it is easy to get formula (3.1) .

3.1.2 Exponential distribution:

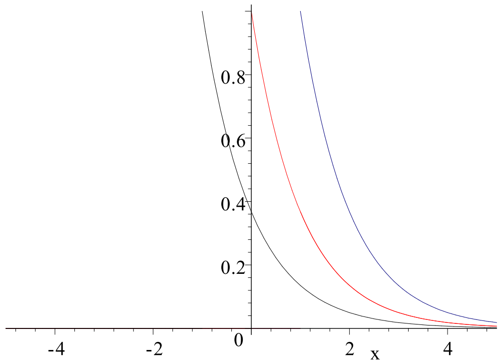
Continuous version of geometric distribution is so called. exponential distribution defined in the following way . Let $a\in\mathbb{R}$, $\lambda\in\mathbb{R}^+$. *Exponential distribution* : Let $a\in\mathbb{R}$, $\lambda\in\mathbb{R}^+$. Density of this distribution is given by the following formula:

$$f(x) = \begin{cases} \lambda \exp(-\lambda(x-a) & dla \quad x \geq a \\ 0 & dla \quad x < a \end{cases}$$

1. Its support is equal to $< a, \infty)$ and c.d.f.

$$F(x) = \begin{cases} 1 - \exp(-\lambda\,(x-a)) & dla \quad x \geq a \\ 0 & dla \quad x < 0 \end{cases}.$$

If random variable X has Exponential distribution with parameters a and λ, we write then: $X \sim Exp(a, \lambda)$. If $a = 0$ we simply write : $X \sim Exp(\lambda)$. Plots of the density of this distribution for $a = -1$, $a = 0$, $a = 2$ and $\lambda = 1$, $\lambda = 2$.



Moment generating function is:

$$\begin{aligned} g(t) &= E\left(\exp\left(tX\right)\right) = \lambda \int_0^\infty \exp(tx - \lambda x)dx \\ &= \frac{\lambda}{\lambda - t} \text{ for } t < \lambda. \end{aligned}$$

Hence we get first two moments of this distribution:

$$\begin{aligned} EX &= \left.\frac{dg(t)}{dt}\right|_{t=0} = \frac{1}{\lambda}, \\ EX^2 &= \left.\frac{d^2 g(t)}{dt^2}\right|_{t=0} = \frac{2}{\lambda^2}. \end{aligned}$$

And

$$\text{var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

'Lack of memory' property of exponential distribution: Let $X \sim Exp(\lambda)$ let us calculate: $P(X \geq x + y | X \geq x)$ for $x, y \geq 0$. Let us notice that $P(X \geq x+y|X \geq x) = \frac{P(X\geq x+y)}{P(X\geq x)}$. Hence it remains to calculate $P(X \geq x)$. We have : $P(X \geq x) = \int_x^\infty \lambda \exp(-\lambda y)dy = \exp(-\lambda x)$. Thus:

$$\forall x, y > 0 : P(X \geq x+y|X \geq x) = \exp(-\lambda y).$$

Again this quantity does not depend on x .
It turns out that these considerations one can generalize and indicate some applications of the exponential distribution in reliability theory. Let X be random variable of continuous type with the density $f(x)$. Let us assume that $P(X \geq 0) = 1$. (One can interpret this type of variables as so called life times).

Let us calculate:

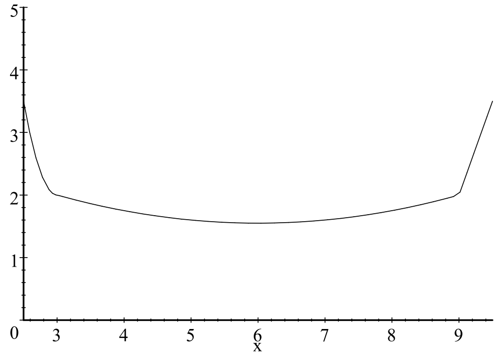
$$\lim_{\delta\to 0} P(X\in <x, x+\delta)|X \geq x)/\delta.$$

Notice that $P(X \in < x, x+\delta)|X \geq x)$ is the probability of the event that the device will be out of order soon after (i.e. during the next δ time units) after x, given that up to the moment x there were no failures. It is easy to notice that, if only the above limit existed, then it would be equal to $\frac{f(x)}{P(X\geq x)}$ since $P(X \in (x, x+\delta)|X \geq x) = \frac{P(X\in <x,x+\delta))}{P(X\geq x)}$ and $P(X \in < x, x+\delta)) \approx f(x)\delta$.

Let us denote this limit by $r(x)$ (in reliability theory it is called hazard rate or failure rate, and $P(X \geq x) = \int_x^\infty f(y)dy = N(x)$ is called reliability). Let us notice that the function $r(x)$ defines $N(x)$, since we have:

$$N(x) = \exp(-\int_x^\infty r(y)dy). \tag{3.2}$$

Typical plot of the function $r(x)$ is presented below:



Notice that one can distinguish three typical intervals of variability of the function $r(x)$. Introductory when $r(x)$ is decreasing. Interval when $r(x)$ is approximately constant and the last period when it is increasing. Simple comparison of this plot with one own's experience confirms the above remark about typical shape of the plot of the function $r(x)$.

Remark 9 *It is easy to notice (of course using formula (3.2)) that when $r(x)$ is a constant function then X has Exponential distribution and conversely!*

Proposition 10 *Let $X_1, \ldots, X_n$ be a sequence of independent random variables with identical distributions exponential $Exp(\lambda)$. Then density f the sum $X_1 + \ldots + X_n$ is equal:*

$$f_{X_1+\ldots+X_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}\exp(-\lambda t),$$

for $t \geq 0$.

Proof. By induction we have for $n = 1$.

$$f_{X_1}(t) = \lambda \exp(-\lambda t),$$

hence formula is true. Let us assume that formula is true for $n = k - 1$. Making use of the formula for density of the sum of independent random variables, we get:

$$\begin{aligned} f_{X_1+\ldots+X_k}(t) &= \int_0^\infty f_{X_n}(t-s)\, f_{X_1+\ldots+X_{k-1}}(s)ds \\ &= \int_0^t \lambda \exp(-\lambda(t-s))\frac{\lambda^{k-1}s^{k-2}}{(k-2)!}\exp(-\lambda s)ds \\ &= \lambda^k \exp(-\lambda t)\int_0^t \frac{s^{k-2}}{(k-2)!}ds = \frac{\lambda^k t^{k-1}}{(k-1)!}\exp(-\lambda t). \end{aligned}$$

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Remark 11 *Jest to density of the distribution Γ with parameters n, λ. Introducing new parameter $l = \lambda/n$ we get so called. Erlang distribution appearing in queuing theory.*

Proposition 12 *Let X_1 and X_2 be independent random variables with distributions respectively $Exp(\lambda_1)$ and $Exp(\lambda_2)$. Then:*

$$\begin{aligned} P(X_1 &< X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}, \\ f_{X_1+X_2}(t) &= \frac{\lambda_1}{\lambda_1-\lambda_2}\lambda_2\exp(-\lambda_2 t) + \frac{\lambda_2}{\lambda_2-\lambda_1}\lambda_1\exp(-\lambda_1 t), \end{aligned}$$

for $t \geq 0$.

Proof. We have

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\begin{aligned}
P\left(X_1 < X_2\right) &= \int_0^\infty \int_x^\infty \lambda_2 \exp(-\lambda_2 y)\lambda_1 \exp(-\lambda_1 x) dy dx \\
&= \int_0^\infty \lambda_1 \exp(-\lambda_1 x)\exp(-\lambda_2 x) dx = \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}
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\begin{aligned}
f_{X_1+X_2}(t) &= \int_0^t f_{X_1}(s) f_{X_2}(t-s) ds \\
&= \lambda_1\lambda_2 \int_0^t \exp\left(-\lambda_1 s\right)\exp\left(-\lambda_2\left(t-s\right)\right) ds \\
&= \lambda_1\lambda_2 \exp\left(-\lambda_2 t\right) \int_0^t \exp\left(-\left(\lambda_1-\lambda_2\right) s\right) ds \\
&= \frac{\lambda_1}{\lambda_1-\lambda_2}\lambda_2 \exp\left(-\lambda_2 t\right)\left(1-\exp\left(-\left(\lambda_1-\lambda_2\right) t\right)\right) \\
&= \frac{\lambda_1}{\lambda_1-\lambda_2}\lambda_2 \exp\left(-\lambda_2 t\right) + \frac{\lambda_2}{\lambda_2-\lambda_1}\lambda_1 \exp\left(-\lambda_1 t\right).
\end{aligned}
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