Lecture 4

Introduction to Poisson processes

4.1 Counting process

Random process $\{N_t; t \geq 0\}$ is called a counting *process*, if N_t is equal total number of 'events' that have happened up to moment t. Examples of counting processes are:

1. number of people, that entered some magazine before or at the moment t. N_t denotes here this number of people,

2. number of people that were born before or at the moment t, N_t denotes here this number of people.

3. number of goals that was collected by a football player up to the moment t..

Let us notice that every counting process has to satisfy the following conditions

(i) $N_t \geq 0$,

(ii) values of N_t are integers,

(iii) for $s < t$, $N_t - N_s$ is equal to the number of 'events' that happened in time interval $(s, t>$.

Remark 13 *Counting processes are sometimes called streams of calls.*

Counting process is called with independent increments, if in disjoint time intervals increments of the process are independent.

Remark 14 *In example no* 1 *probably assumption of independence of increments is satisfied. On the other hand this assumption is not satisfied in example no.* 2*.. If*

Counting process has stationary increments if number of events, that have happened in the time interval $(s, t>$ depends only on the length of the interval i.e. on the quantity $t - s$.

Counting process $\{N_t; t \geq 0\}$ is called single if $a)$ $\forall t > s \geq 0 : P(N_t - N_s = 1) = \lambda(t-s) + o(|t-s|)$, $b)$ $\forall t > s \geq 0 : P(N_t - N_s \geq 2) = o(|t-s|)$.

Definition 15 *Counting process* $\{N_t; t \geq 0\}$ *is called, Poisson process with intensity* λ*, if:and*
i) $N_0 = 0$,
ii) has independent and stationary increments
iii) number of 'events' in the interval of the length t has Poisson distribution with parameterλt*, i.e.*

$$\forall t, s \geq 0 : P(N_{t+s} - N_s = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

Theorem 16 *The only counting process, single with stationary, independent increments is Poisson process.*

Proof. Let us denote

$$g(t) = E \exp(-vN_t).$$

We have

$$\begin{aligned} g(t+h) &= E \exp(-vN_{t+h}) \\ &= E \exp(-vN_t) E \exp(-v(N_{t+t} - N_t)) \\ &= g(t) E \exp(-vN_h). \end{aligned}$$

We have mad use of independence of increments and their stationarity.

Further we have:

$$\begin{aligned} E \exp(-vN_h) &= E(\exp(-vN_h) | N_h = 0) P(N_h = 0) \\ &\quad + E(\exp(-vN_h) | N_h = 1) P(N_h = 1) \\ &\quad + E(\exp(-vN_h) | N_h \geq 2) P(N_h \geq 2) \\ &= 1 - \lambda h + o_1(h) + e^{-v}(\lambda h + o_2(h)) + o_3(h) \\ &= 1 - \lambda h + \lambda h e^{-v} + o(h). \end{aligned}$$

Hence

$$\frac{g(t+h) - g(t)}{h} = g(t) \lambda \left(e^{-v} - 1\right) + \frac{o(h)}{h}.$$

Let $h \to 0$. We'll get then:

$$g'(t) = g(t) \lambda \left(e^{-v} - 1\right).$$

and consequently:

$$g(t) = \exp\left(\lambda t \left(e^{-v} - 1\right)\right).$$

This is a Laplace transform of Poisson distribution with parameterλt. ∎

4.2 Time intervals between calls

Let be given Poisson process $\{N_t; t \geq 0\}$ and let T_1 a moment of the occurrence of the first event. Further let for $n > 0$, T_n denotes time interval between $n-1$ th and n -th event. Sequence $\{T_i\}_{i\geq 1}$ is called a sequence of inter-arrival times. Let us notice that $\{T_1 > t\} = \{N_t = 0\}$

$$P(T_1 > t) = P(N_t = 0) = \exp(-\lambda t).$$

Similarly we have: $P(T_2 > t) = E(P(T_2 > t|T_1))$. But

$$\begin{aligned} P(T_2 > t|T_1 = s) &= P(\text{lack of events in time interval } (s, s+t > |T_1 = s) \\ &= P(\text{lack of events in time interval } (s, s+t >) = \exp(-\lambda t). \end{aligned}$$

In the last equalities we used independence and satisfactorily of increments. Thus we have:

Proposition 17 *Sequence $\{T_n\}_{n\geq 1}$ of inter-arrival times consists of i.i.d. random variables having exponential distributions with parameter λ.*

Further let us denote by S_n moment of arrival of the $n-$ th call or otherwise a waiting time for the $n-$ th call. It is easy to deduce that

$$S_n = \sum_{i=1}^{n} T_i,$$

hence S_n has gamma distribution with parameters n and λ. Other words , that distribution of S_n has density equal to:

$$f_{S_n}(t) = \lambda \exp(-\lambda t) \frac{(\lambda t)^{n-1}}{(n-1)!}; t \geq 0.$$

The above mentioned density could have been obtained noticing that $n-$ th call happens before t if and only if, when the number of events before the moment t was at least n. i.e.

$$N_t \geq n \Longleftrightarrow S_n \leq t,$$

hence

$$F_{S_n}(t^+) = P(S_n \leq t) = P(N_t \geq n) = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}.$$

After differentiating with respect to t, we get:

$$\begin{aligned} f_{S_n}(t) &= -\lambda \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} + \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}. \end{aligned}$$

4.3 Summing of Poisson processes

Let be given Poisson process $\{N_t; t \geq 0\}$ with intensity λ. Let us suppose, that every event (call) is classified as of I or II type. Further let us assume , that call is classified as of type I or II with probabilities respectively p and $1-p$ independently of other events (calls). (For example let us assume, that clients arrive to the shop and every one of them turns out to be a man with probability 1.2 or a women with probability 1/2.

Let $N_t^{(1)}$ and $N_t^{(2)}$denote respectively numbers of events of type I or II during time interval $[0,t]$. Let us notice that $N_t = N_t^{(1)} + N_t^{(2)}$. We have the following proposition:

Proposition 18 *Both $N_t^{(1)}$ and $N_t^{(2)}$ are Poisson processes with intensities respectively λp and $\lambda(1-p)$. Moreover these processes are independent.*

Proof. Let us calculate joint probability:

$$\begin{aligned} & P\left(N_t^{(1)} = n, N_t^{(2)} = m\right) \\ = & \sum_{k=0}^{\infty} P\left(N_t^{(1)} = n, N_t^{(2)} = m | N_t = k\right) P\left(N_t = k\right). \end{aligned}$$

Let us notice that firstly in order to have n events of type I and m events of type II we should have jointly $n+m$ events, hence

$$\begin{aligned} & P\left(N_t^{(1)} = n, N_t^{(2)} = m\right) \\ = & P\left(N_t^{(1)} = n, N_t^{(2)} = m | N_t = n+m\right) e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!}. \end{aligned}$$

Now let us notice, that if there were $n+m$ events then number of events of type I has binomial distribution (number of successes among $n+m$ Bernoulli experiments). Thus

$$P\left(N_t^{(1)} = n, N_t^{(2)} = m | N_t = n+m\right) = \binom{n+m}{n} p^n (1-p)^m,$$

so

$$\begin{aligned} P\left(N_t^{(1)} = n, N_t^{(2)} = m\right) &= \binom{n+m}{n} p^n (1-p)^m e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!} \\ &= e^{-\lambda p t} \frac{(\lambda p t)^n}{n!} e^{-\lambda(1-p)t} \frac{(\lambda(1-p)t)^m}{m!}. \end{aligned}$$

Moreover we have

$$
\begin{aligned}
& P\left(N_t^{(1)}=n\right) \\
= & \sum_{m=0}^{\infty} P\left(N_t^{(1)}=n, N_t^{(2)}=m\right) \\
= & e^{-\lambda p t} \frac{(\lambda p t)^n}{n!} \sum_{m=0}^{\infty} e^{-\lambda(1-p) t} \frac{(\lambda(1-p) t)^m}{m!} \\
= & e^{-\lambda p t} \frac{(\lambda p t)^n}{n!} .
\end{aligned}
$$

Hence it can be seen, that $N_t^{(1)}$ and $N_t^{(2)}$ are independent Poisson processes. ■

Remark 19 *The fact that processes N^1 and N^2 are Poissonian is not very surprising. One could have expect it. Unexpected seems to be the fact that these processes are independent.*