Lecture 5

Further properties of Poisson processes

5.1 Conditional distribution of arrival times

Let us start from the moment of arrival of the first call under condition, that on the interval $[0,t]$ there was one call. Another words we will find $P(T_1 < s|N_t = 1)$. We get then:

$$\begin{aligned} P(T_1 < s|N_t = 1) &= \frac{P(T_1 < s, N_t = 1)}{P(N_t = 1)} \\ &= \frac{P(1 \text{ call on } (0,s) \text{ and } 0 \text{ calls on } [s,t])}{P(N_t = 1)} \\ &= \frac{P(1 \text{ call on } (0,s))\, P(\ 0 \text{ calls on } [s,t])}{P(N_t = 1)} \\ &= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t}. \end{aligned}$$

Hence this distribution turns out to be *uniform* on the interval $< 0, t >$. This result can be generalized. However in order to do this one needs to introduce a notion of *order statistics.*

Definition 20 *Let $\{X_1, \ldots, X_n\}$ be a simple random sample. Random vector $\{Y_1, \ldots, Y_n\}$ satisfying conditions: $Y_1 \le Y_2 \le, \ldots, \le Y_n$ with probability 1 and defined in the following way :*

$$Y_i = \textit{and -th (with respect to the magnitude) value of } \{X_1, \ldots, X_n\}$$

will be called vector of order statistics *of the vector $\{X_1, \ldots, X_n\}$.*

Remark 21 *Coordinates of the vector $\{Y_1, \ldots, Y_n\}$ will be denoted traditionally in the following way $\{X_{1:n}, \ldots, X_{n:n}\}$.*

We need also the following two simple lemmas:

Lemma 22 *If the random sample* $\{X_1, \ldots, X_n\}$*is simple (i.e. random variables* $\{X_i\}_{i=1}^n$ *are independent with identical distributions) and the density of* X_1 *is equal* $f(x)$ *, then the joint density of order statistics* $\{X_{1:n}, \ldots, X_{n:n}\}$ *is equal:*

$$g(y_1, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i),$$

for $y_1 \leq y_2 \leq \ldots \leq y_n$*.*

Proof. Let us notice, that in order to observe $X_{1:n} = y_1, \ldots, X_{n:n} = y_n$ any of $n!$ permutations $\left\{X_{(1)}, \ldots, X_{(n)}\right\}$ of variables $\{X_1, \ldots, X_n\}$ has to assume values $\{y_1, \ldots, y_n\}$. And moreover

$$\begin{aligned} & P\left(X_{(1)} \in (y_1, y_1 + dy_1), \ldots, X_{(n)} \in (y_n, y_n + dy_n)\right) \\ \simeq & \prod_{i=1}^{n} f(y_i)\, dy_1, \ldots dy_n. \end{aligned}$$

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Lemma 23 *Let random variables* $S_1, \ldots, S_n$ *be i.i.d. and have uniform distribution* $U(0, t)$ *on the interval* $< 0, t >$ *. Then joint distribution of the vector of order statistics* $(S_{1:n}, \ldots, S_{n:n})$ *has the following density:*

$$g(y_1, \ldots, y_n) = \frac{n!}{t^n}, \tag{5.1}$$

for $y_1 \leq y_2 \leq \ldots \leq y_n$*.*

Theorem 24 *Under condition that there were exactly* n *call on the interval* $< 0, t >$ *moments of successive calls* $S_1, \ldots, S_n$ *are distributed as the order statistics of* n *i.i.d. random variables drawn from uniform distribution on*$<0, t>$ *, i.e.*

$$f_{S_1, \ldots, S_n}(s_1, \ldots, s_n) = \frac{n!}{t^n}; \textit{for } 0 < s_1 < \ldots, < s_n < t.$$

Remark 25 *This result can be expressed also in the following way: Given that there were* n *calls, moments of calls considered as unordered random variables are independent and have the same uniform distributions on the interval* $< 0, t >$.

Proof. In order to get joint density of the vector $(S_1, \ldots, S_n)$ under condition $N(t) = n$ let us notice, that for $0 < s_1 <, \ldots, < s_n < t$ an event $S_1 = s_1$, $S_2 = s_2, \ldots, S_n = s_n$, $N(t) = n$ means, that we have for the first $n+1$ inter-arrival times $T_1 = s_1$, $T_2 = s_2 - s_2, \ldots, T_n = s_n - s_{n-1}$, $T_{n+1} > t - s_n$. Hence making use of independence of inter-arrival times we have:

$$\begin{aligned} f\left(s_1,\ldots,s_n|n\right) &= \frac{P\left(T_1=s_1,\ldots,T_n=s_n-s_{n-1},T_n>t-s_n\right)}{P\left(N\left(t\right)=n\right)} \\ &= \frac{\lambda e^{-\lambda s_1}\lambda e^{-\lambda(s_2-s_1)}\ldots\lambda e^{-\lambda(s_n-s_{n-1})}e^{-\lambda(t-s_n)}}{e^{-\lambda t}\left(\lambda t\right)^n/n!} \\ &= \frac{n!}{t^n}. \end{aligned}$$

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The above mentioned theorem is used to generalize proposition 18. Let us assume now, that upon the arrival every event is classified as being of one of k types, and that the probability that an event is classified as type i event , $i=1,\ldots,k$, depends on the time the event occurs. Specifically, suppose that if an event occurs at time y then it will be classified as type i event, independently of anything that has previously occurred, with probability $P_i\left(y\right)$, $i=1,\ldots,k$ where $\sum_{i=1}^k P_i\left(y\right)=1$. Upon using Theorem 24 we can prove the following useful Theorem:

Theorem 26 *Let $N_t^i; t\geq 0$, $i=1,\ldots,k$ denotes number elements of type i that arrived by the time t i.e. that occurred in $<0,t>$. Then N_t^i are independent Poisson random variables with expectations respectively:*

$$EN_t^i=\lambda\int_0^t P_i\left(s\right)ds.$$

Proof. Let us compute joint probability $P\left(N_t^1=n_1,\ldots,N_t^k=n_k\right)$. To do so first note that in order for there to have been n_i type i events for $i=1,\ldots,k$ there must have been a total of $\sum_{i=1}^k n_i$ events. Hence, conditioning on $N\left(t\right)$ yields

$$\begin{aligned} & P\left(N_t^1=n_1,\ldots,N_t^k=n_k\right) \\ = & P\left(N_t^1=n_1,\ldots,N_t^k=n_k|N_t=\sum_{i=1}^k n_i\right)\times P\left(N_t=\sum_{i=1}^k n_i\right). \end{aligned}$$

Now consider an arbitrary call, that occurred in the interval $<0,t>$. If it had occurred at time s, then the probability that it would be a type i event would be $P_i\left(s\right)$. Hence since by Theorem 26 this event will have occurred at some time uniformly distributed on $<0,t>$, it follows that the probability that this event will be a type i event is

$$p_i=\frac{1}{t}\int_0^t P_i\left(s\right)ds,$$

independently of the other events. Hence

$$P\left(N_t^1=n_1,\ldots,N_t^k=n_k|N_t=\sum_{i=1}^k n_i\right)$$

will just equal the multinomial probability of n_i type i outcomes for $i = 1, \dots, k$ when each of $\sum_{i=1}^k n_i$ independent trials results in outcome i with probability $p_1, \dots, p_k,$. That is

$$P\left(N_t^1 = n_1, \dots, N_t^k = n_k | N_t = \sum_{i=1}^k n_i\right) = \frac{\left(\sum_{i=1}^k n_i\right)!}{\prod_{i=1}^k n_i!} \prod_{i=1}^k p_i^{n_i}$$

Consequently

$$\begin{aligned} & P\left(N_t^1 = n_1, \dots, N_t^k = n_k\right) \\ = & \frac{\left(\sum_{i=1}^k n_i\right)!}{\prod_{i=1}^k n_i!} \prod_{i=1}^k p_i^{n_i} e^{-\lambda t} \frac{(\lambda t)^{\sum_{i=1}^k n_i}}{\left(\sum_{i=1}^k n_i\right)!} \\ = & \prod_{i=1}^k \left[e^{-\lambda t p_i} \frac{(\lambda t p_i)^{n_i}}{n_i!}\right]. \end{aligned}$$

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Example 27 *(Minimizing the Number of Encounters): Suppose that cars enter a one-way highway in accordance with a Poisson process with rate* λ *The cars enter at point* a *and depart at point* b*. Each car travels at a constant speed that is randomly determined, independently from car to car, from the distribution* G*. When a faster car encounters a slower one, it passes it with no time being lost. If your car enters the highway at time* s *and you are able to choose your speed, what speed minimizes the expected number of encounters you will have with other cars, where we say that an encounter occurs each time your car either passes or is passed by another car?*

Solution: We will show that for large s *the speed that minimizes the expected number of encounters is the median of the speed distribution* G*. To see this, suppose that the speed* x *is chosen. Let* $d = b - a$ *denote the length of the road. Upon choosing the speed* x*, it follows that your car will enter the road at time* s *and will depart at time* $s + t_0$*, where* $t_0 = d/x$ *is the travel time.*
Now, the other cars enter the road according to a Poisson process with rate λ*. Each of them chooses a speed* X *according to the distribution* G*, and this results in a travel time* $T = d/X$*. Let* F *denote the distribution of travel time* T*. That is,*

$$F(t) = P(T < t) = P(d/X < t) = P(X > d/t) = \bar{G}(d/t).$$

(where we denoted by $\bar{F}(t) = 1 - F(t)$ *a tail of a distribution* F*)*
Let us say that an event occurs at time t *if a car enters the highway at time* t*. Also, say that the event is a type* 1 *event if it results in an encounter with your car. Now, your car will enter the road at time* s *and will exit at time* $s + t_0$*. Hence, a car will encounter your car if it enters before* s *and exits after* $s + t_0$ *(in which case your car will pass it on the road) or if it enters after* s *but exits*

before $s+t_0$ *(in which case it will pass your car). As a result, a car that enters the road at time* t *will encounter your car if its travel time* T *is such that*

$$\begin{array}{lll} t+T>s+t_0 & if & t<s \\ t+T<s+t_0 & if & s<t<s+t_0 \end{array}$$

From the preceding, we see that an event at time t *will, independently of other events, be a type* 1 *event with probability* $p(t)$ *given by*

$$p(t)=\begin{cases} P(t+T>s+t_0)=\bar{F}(s+t_0-t) & if \quad t<s \\ P(t+T<s+t_0)=F(s+t_0-t) & if \quad s<t<s+t_0 \\ 0 & if \quad t>s>t_0 \end{cases}$$

Since events (that is, cars entering the road) are occurring according to Poisson process it follows, upon applying Theorem 26, that the total number of type 1 *events that ever occurs is Poisson with mean*

$$\begin{aligned} \lambda\int_0^\infty p(t)\,dt &= \lambda\int_0^s \bar{F}(s+t_0-t)\,dt+\lambda\int_0^{s+t_0} F(s+t_0-t)\,dt \\ &= \lambda\int_{t_0}^{s+t_0}\bar{F}(y)\,dy+\lambda\int_0^{t_0}F(y)\,dy \end{aligned}$$

To choose the value of t_0 *that minimizes the preceding quantity, we differentiate. This gives*

$$\frac{d}{dt_0}\left(\lambda\int_{t_0}^{s+t_0}\bar{F}(y)\,dy+\lambda\int_0^{t_0}F(y)\,dy\right)=\lambda\left(\bar{F}(s+t_0)-\bar{F}(t_0)+F(t_0)\right).$$

Setting this equal to 0, *and using that* $\bar{F}(s+t_0)\equiv 0$ *when* s *is large, we see that the optimal travel time* t_0 *is such that*

$$F(t_0)=\bar{F}(t_0)=1-F(t_0)$$

or

$$F(t_0)=\frac{1}{2}.$$

That is the optimal travel time is the median of the travel time distribution Since the speed X *is equal to the distance* d *divided by the travel time* T, *it follows that the optimal speed* $x_0=d/t_0$ *is such* $F(d/x0)=\frac{1}{2}$ *Since*

$$F(d/x_0)=\bar{G}(x_0)$$

we see that $G(x_0)=\frac{1}{2}$, *and so the optimal speed is the median of the distribution of speeds.*

Summing up, we have shown that for any speed x *the number of encounters with other cars will be a Poisson random variable, and the mean of this Poisson will be smallest when the speed* s *is taken to be the median of the distribution* G.

5.2 Generalizations of Poisson process

5.2.1 Nonhomogeneous process

Definition 28 *A counting process $\mathcal{N} = \{N_t; t \geq 0\}$ will be called nonhomogeneous Poisson process with intensity function $\lambda(t), t \geq 0$, if*
i) $N_0 = 0$,
ii) $\mathcal{N}$ *has independent increments*
iii) $P(N_{t+h} - N_t \geq 2) = o_t(h)$
iv) $P(N_{t+h} - N_t = 1) = \lambda(t)h + o_1(h)$.

If we let $m_t = \int_0^t \lambda(y)\,dy$, then we will show that

$$P(N_{t+s} - N_t = n) = e^{-(m_{t+s} - m_t)} \frac{(m_{t+s} - m_t)^n}{n!}, \; n \geq 0.$$

Let us denote

$$g(t) = Exp(-v(N_{t+s} - N_s)),$$

for fixed s and $v \geq 0$. We have

$$\begin{aligned} g(t+h) &= Exp(-v(N_{t+s+h} - N_s)) \\ &= Exp(-v(N_{t+s+h} - N_{t+s}) - v(N_{t+s} - N_s)) \\ &= g(t)\,Exp(-v(N_{t+s+h} - N_{t+s})) \\ &= g(t)\left(1 - \lambda(t+s)h + e^{-v}\lambda(s+t)h + o(h)\right). \end{aligned}$$

Hence:

$$\frac{g(t+h) - g(t)}{hg(t)} = \left(e^{-v} - 1\right)\lambda(s+t) + \frac{o(h)}{h}.$$

Passing with h to 0^+ yields:

$$\frac{g'(t)}{g(t)} = \left(e^{-v} - 1\right)\lambda(s+t).$$

Thus

$$g(t) = Exp\left(\left(e^{-v} - 1\right)\int_s^{t+s} \lambda(\tau)\,d\tau\right).$$

But this is a p.g.f. of Poisson r.v. with parameter $\int_s^{t+s} \lambda(\tau)$.

5.2.2 Compound Poisson process

Let $\{Y_i\}_{s\geq 1}$ be a sequence of independent random variables with identical distributions (a i.i.d. sequence) . A sequence $\{Y_i\}_{s\geq 1}$ will be called *a sequence of losses* . Let further $\{N_t; t \geq 0\}$ be independent of $\{Y_i\}$ Poisson process with intensity λ. *A compound Poisson process* will be called the following process:

$$X_t = \sum_{i=1}^{N_t} Y_i.$$

Examples of such processes are process of losses of an insurance company (process of events (accidents) is Poissonian, i-th loss Y_i), a process of incomes to supermarket cashier (clients leave at a Poissonian flow and an $i-$ th leaves Y_i \$). We have the following proposition:

Proposition 29 *i)* $EX_t = \lambda t EY_1$
ii) $\mathrm{var}\,(X_t) = \lambda t EY_1^2$.

Proof. Using properties of conditional expectation we have:

$$EX_t = E\left(E\left(X_t|N_t\right)\right).$$

hence $E\left(X_t|N_t = n\right) = E\left(\sum_{i=1}^{N_t} Y_i|N_t = n\right) = E\left(\sum_{i=1}^{n} Y_i|N_t = n\right) = E\left(\sum_{i=1}^{n} Y_i\right)$ $= nEY_1$. Generally we have

$$E\left(X_t|N_t\right) = N_t EY_1,$$

which leads immediately to the first assertion.

To get the second we will use the following property of conditional variance:

$$\mathrm{var}\,(X) = E\left(\mathrm{var}\,(X|\mathcal{A})\right) + \mathrm{var}\,(E\left(X|\mathcal{A}\right))$$

that is true for any random variable X such that, $EX^2 < \infty$ and arbitrary $\sigma-$ field $\mathcal{A}$.

Thus

$$\mathrm{var}\,(X_t) = E\left(\mathrm{var}\,(X_t|N_t)\right) + \mathrm{var}\,(E\left(X_t|N_t\right)).$$

$\mathrm{var}\,(E\left(X_t|N_t = n\right)) = \mathrm{var}\left(\sum_{i=1}^{N_t} Y_i|N_t = n\right) = \mathrm{var}\left(\sum_{i=1}^{n} Y_i|N_t = n\right) = \mathrm{var}\left(\sum_{i=1}^{n} Y_i\right)$ $= n\,\mathrm{var}\,(Y_1)$. Hence $\mathrm{var}\,(X_t|N_t) = N_t\,\mathrm{var}\,(Y_1)$. Summarizing we get:

$$\begin{aligned}\mathrm{var}\,(X_t) &= E\left(N_t\,\mathrm{var}\,(Y_1)\right) + \mathrm{var}\,(N_t EY_1) \\ &= \lambda t\,\mathrm{var}\,(Y_1) + \lambda t\left(EY_1\right)^2 = \lambda t EY_1^2.\end{aligned}$$

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