# Lecture 6

# Introduction to queuing systems

## 6.1 Kendall's classification

D.G. Kendall (1951) has introduced the following classification of the problems at hand. Three symbols are given, separated by two oblique strokes:-/-/-. The last symbol specifies the number of servers. The first and the second denote the type of distribution of inter-arrival time and of the holding time (service time) , respectively. An exponential distribution is indicated by the symbol $M$ (called after the "Markovian property", which essentially is the property of forgetfulness). Constant intervals are characterized by the symbol $D$ (deterministic). Arbitrary distribution are given the symbol $G$ ('general"). If the later case refers to inter-arrival times and if, moreover, successive inter-arrivals are supposed to be independent, this is indicated by the symbol $GI$ (general independent"). Some examples:

- $M/M/c$ means: Poisson arrival process, exponential distribution of holding time, $c$ servers,
- $M/G/1$ (sometimes we write also $M/GI/1$where $GI$ stands for general independent) means Poisson arrival process, arbitrary distribution of holding times one server,
- $GI/D/1$ means inter-arrival times 'general independent' , constant holding time, one server.

In the cases to be dealt with, the first symbol will nearly always be $M$ (Poisson). Blocking and delay systems will be distinguished by simply writing ' blocking' or 'delay' behind classification.

## 6.2 Standard systems

By “standard cases“ it is meant the simple steady-state cases M/M/c-blocking and M/M/c-delay. The arrival process is Poissonian in both cases and the distribution of holding-times is exponential. Moreover, it is assumed that the order of serving delayed demands is not dependent on a prior knowledge of holding-times.

### 6.2.1 Probability of blocking for the system M/M/c-blocking

The arrival process is Poissonian with arrival (or birth) rate $\rho$. There are $c$ servers. The holding-time is exponential (1). There is no waiting facility. If all servers are occupied, arriving demands are discarded.

When the number of busy servers is $r$ $(r = 0, 1, ..., c)$ the system will be said to be “in state $[r]$“. It is assumed that stationarity exists. This means that the probability of the system being in $[r]$ is time-independent:

$$p_r := Pr\{[r]\, at\, time\, \tau\} = const.$$

At an arrival (rate=$\rho$) the index $r$ of the state of the system is increased by as long as $r < c$. If $r = c$ an arrival is ineffective. When the system is in $[r]$ the arrival rate remains $\rho$, as for the Poisson process the inferential knowledge conveyed to us by the specification of the sta5te is irrelevant. The total termination rate is $r$; possible inferential knowledge about the ages of the $r$ occupations is irrelevant as the holding-time is exponential. The probability of the state and rates of arrival and termination (collectively called transitions) may be summarized in the following scheme:

$$[0]\underset{1}{\overset{\rho}{\rightleftarrows}}[1]\underset{2}{\overset{\rho}{\rightleftarrows}}\dots[r-1]\underset{r}{\overset{\rho}{\rightleftarrows}}[r]\underset{r+1}{\overset{\rho}{\rightleftarrows}}[r+1]\dots[c-1]\underset{c}{\overset{\rho}{\rightleftarrows}}[c],$$

probability of the states are respectively $p_0, p_1, \dots, p_{r-1}, p_r, \dots, p_{c-1}, p_c$.

Now, stationarity requires that the probability of the formation of $[r]$ in $\Delta\tau$ should equal the probability of a departure from $[r]$ during $\Delta\tau$. The formation is possible from state $[r-1]$, which has probability $p_{r-1}$, by an arrival, which has probability $\rho\Delta\tau$. The joint probability of these events is $p_{r-1}\rho\Delta\tau$. When $r < c$, another possibility is from probability $[r+1]$ by "a death of one out of $r+1$" (conditional probability $(r+1)\Delta\tau$) : joint probability $p_{r+1}$ $\blacksquare(r+1)\Delta\tau$. Formations by multiple transitions have probabilities of order $O(\Delta\tau^2)$ and are discarded. When the system
is in $[r]$, where $r < c$, both arrival and termination would mean departure from $[r]$; hence the probability of a departure from $[r]$ in $\Delta\tau$ is: $p_r\,(\rho + r)\,\Delta\tau$. So

$$p_{r-1}\rho\Delta\tau + p_{r+1}\,(r+1)\,\Delta\tau = p_r\,(\rho + r)\,\Delta\tau.$$

or

$$\rho p_{r-1} - (\rho + r)\,p_r + (r+1)\,p_{r+1} = 0,$$

$r = 0, 1, \ldots, c-1$.

For this equation to hold for $r = 0$ too, $p_{-1}$ should be taken equal to 0. For the state $[c]$ it should be observed that no transitions to and from a state $[c+1]$ are possible. Hence,

$$\rho p_{c-1} - c p_c = 0.$$

The above mentioned set of equations are called *birth-and death equations.* It is a set of $c+1$ homogeneous linear equations in $c+1$ unknowns. Sop, for a nontrivial solution to exist, the equations must be linearly independent. By adding these equations for $0, 1, \ldots, r$ one obtains

$$-\rho p_r + (r+1)\, p_{r+1} = 0. \tag{6.1}$$

the last of which is identical to previous equation, showing the dependency.

The set of equations. (6.1) should be supplemented by the statement that the defined states form a complete set of mutually exclusive events, and hence,

$$\sum_{r=0}^{c} p_r = 1 \tag{6.2}$$

The equations. (6.1) taken consecutively for $r = 0, 1, 2, \ldots$ enable one to express $p_1$ in terms of $p_0$, $p_2$ in terms of $p_1, \ldots, p_c$ in terms of $p_{c-1}$. Hence, all unknowns can be expressed in terms of $p_0$:

$$p_r = \frac{\rho^r}{r!} p_0$$

$r = 1, 2, \ldots, c$.

By insertion into (6.2) one obtains

$$p_0 = 1 / \sum_{i=0}^{c} \frac{\rho^i}{i!}$$

and hence,

$$p_r = \frac{\rho^r}{r!} / \sum_{i=0}^{c} \frac{\rho^i}{i!}$$

The probability of blocking $B$ is the probability of finding the system in $[c]$; hence,

$$p_c = \frac{\rho^c}{c!} / \sum_{i=0}^{c} \frac{\rho^i}{i!} (= E_{1,c}(\rho)) \tag{6.3}$$

This is the famous *Erlang B-formula.* The function $E_{1,c}$ is called the first Erlang function.

When $c - \rho > 2\sqrt{\rho}$ the denominator in (6.3) is practically equal to $e^\rho$ and $B$ is nearly equal to the Poisson expression, to be denoted by $\phi$ :

$$B \cong \phi_c = e^{-\rho} \frac{\rho^c}{c!}.$$