Lecture 8

Renewal Theory

8.1 Introduction

We have seen that a Poisson process is a counting process for which the times between successive events are independent and identically distributed exponential random variables. One possible generalization is to consider a counting process for which the times between successive events are independent and identical distributed with an arbitrary distribution. Such a counting process is called a *renewal process.*

Let $\{N(t), t \geq 0\}$ be a counting process and let X_n denote the time between the $(n-1)$st and the nth event of this process, $n \geq 1$.

If a sequence of nonnegative random variables $\{X_1, X_2, \ldots\}$ is independent and identically distributed, then the counting process $\{N(t), t \geq 0\}$ is said to be a *renewal process.*

Thus, a renewal process is a counting process such that the time until the first event occurs, has some distribution F, the time between the first and the second event has, independently of the time of the first event, the same distribution F and so on. When an event occurs, we say that a renewal has taken place.

For an example of a renewal process, suppose that we have an infinite supply of light bulbs whose lifetimes are independent and identically distributed. Suppose also that we use a single light bulb at a time, and when it fails we immediately replace it with a new one. Under these conditions $\{N(t), t \geq 0\}$ is a renewal process when $N(t)$ represents the number of light bulbs that have failed by time t.

For a renewal process having inter-arrival times $X_1, X_2, \ldots$, let:

$$S_0 = 0, S_n = \sum_{i=1}^{n} X_i; n \geq 1.$$

That is, $S_1 = X_1$ is the time of the first renewal; $S_2 = X_1 + X_2$ is the time until the first renewal plus the time between the first and second renewal, that is, S_2 is the time of the second renewal. In general, S_n denotes the time of the nth renewal.

We shall let F denote the inter-arrival distribution and to avoid trivialities, we assume that $F(0) = P\{X_n = 0\} < 1$. Furthermore, we let

$$\mu = EX_n; n \geq 1$$

be the mean time between successive renewals. It follows from the non-negativity of X_n and the fact that X_n is not identically 0 that $\mu > 0$.

The first question we shall attempt to answer is whether an infinite number of renewals can occur in a finite amount of time. That is, can $N(t)$ be infinite for some (finite) value of t? To show that this cannot occur, we first note that, as S_n is the time of the nth renewal, $N(t)$ may be written as

$$N(t) = \max\{n : S_n \leq t\}. \tag{8.1}$$

To understand why the above equation is valid, suppose, for instance, that $S_4 < t$ but $S_5 > t$. Hence, the fourth renewal had occurred by time t but the fifth renewal occurred after time t; or in other words, $N(t)$, the number of renewals that occurred by time t, must be equal 4. Now by the strong law of large numbers it follows that, with probability 1,

$$\frac{S_n}{n} \to \mu$$

as $n \to \infty$.

But since $\mu > 0$ this means that S_n must be going to infinity as n goes to infinity. Thus, S_n can be less than or equal to t for at most a finite number of values of n, and hence by Equation (8.1), $N(t)$ must be finite.

However, though $N(t) < \infty$ for each t, it is true that, with probability 1,

$$N(\infty) = \lim_{t \to \infty} N(t) = \infty$$

This follows since the only way in which $N(\infty)$, the total number of renewals that occur, can be finite is for one of the inter-arrival times to be infinite. Therefore,

$$\begin{aligned} P\{N(\infty) < \infty\} &= P(X_n = \infty, \ for\ some\ n) \\ &= P\left(\bigcup_{n=1}^{\infty} \{X_n = \infty\}\right) \leq \sum_{n=1}^{\infty} P(X_n = \infty) = 0. \end{aligned}$$

8.1.1 Distribution of N(t)

The distribution of $N(t)$ can be obtained, at least in theory, by first noting the important relationship that the number of renewals by time t is greater than or equal to n if and only if the nth renewal occurs before or at time t. That is,

$$N(t) \geq n \Longleftrightarrow S_n \leq t. \tag{8.2}$$

From Equation (8.2) we obtain

$$\begin{aligned} P\{N(t)=n\} &= P\{N(t)\geq n\}-P\{N(t)\geq n+1\} \\ &= P\{S_n\leq t\}-P\{S_{n+1}\leq t\}. \end{aligned} \tag{8.3}$$

Now since the random variables $X_i, i \geq 1$, are independent and have a common distribution F, it follows that $S_n = \sum_{i=1}^{n} X_i$ is distributed as F_n, the n-fold convolution of F with itself. Therefore, from Equation (8.3) we obtain

$$P\{N(t)=n\} = F_n(t) - F_{n+1}(t). \tag{8.4}$$

Example 30 *Suppose that* $P\{X_n = i\} = p(1-p)^{i-1}, i \geq 1$*. That is, suppose that the inter-arrival distribution is geometric. Now* $S_1 = X_1$ *may be interpreted as the number of trials necessary to get a single success when each trial is independent and has a probability* p *of being a success. Similarly,* S_n *may be interpreted as the number of trials necessary to attain* n *successes, and hence has the negative binomial distribution*

$$P(S_n = k) = \begin{cases} \binom{k-1}{n-1} p^n (1-p)^{k-n} & for \quad k \geq n \\ 0 & for \quad k < n \end{cases}$$

Thus, from Equation (8.4) we have that

$$\begin{aligned} P(N(t)=n) &= \sum_{k=n}^{[t]} \binom{k-1}{n-1} p^n (1-p)^{k-n} - \sum_{k=n+1}^{[t]} \binom{k-1}{n} p^{n+1} (1-p)^{k-n-1} \\ &= \binom{[t]}{n} p^n (1-p)^{[t]-n}. \end{aligned}$$

8.1.2 Renewal function

Function $m(t) = EN(t)$ is called a *renewal function.* Let us calculate it. First lest us notice the following observation.

Lemma 31 *Let* $X \geq 0$ *be integer valued random variable. We have then*

$$X = \sum_{i \geq 1} I(X \geq i).$$

If additionally $EX < \infty$*, then we have*

$$\mathbb{E}X = \sum_{i \geq 1} P(X \geq i).$$

Proof. On the set $\{X = j\}$ we have on the left hand side j and on the right hand side we have j non zero values of functions $I(X \geq i)$, for $i = 1, 2, \ldots, j$. ■

Remark 32 *Formula* $\mathbb{E}X = \sum_{i\geq 1} P(X \geq i)$ *can be generalized to* $\mathbb{EX} = \int_0^\infty P(X \geq t)\, dt$ *valid for all nonnegative random variables* X.

Now using this identity we have

$$\begin{aligned} m(t) &= EN(t) = \sum_{i\geq 1} P(N(t) \geq i) \\ &= \sum_{i\geq 1} P(S_i \leq t) = \sum_{i\geq 1} F_i(t). \end{aligned}$$

It turns out that renewal function uniquely determines renewal process in the sense that it specifies c.d.f of inter-arrival times.

8.1.3 Renewal equation

Let us assume that c.d.f F has density f. Let us condition upon the moment of the first arrival X_1. We have

$$m(t) = EN(t) = \int_0^\infty E[N(t)\,|X_1 = x] f(x)dx \tag{8.5}$$

Now notice that if only x is less that t then using the fact that renewal process starts over when a renewal occurs. Thus if $x < t$ number of arrivals by time t has the same distribution as 1 plus the number of renewals in the first $t - x$ time units. Thus

$$E[N(t)\,|X_1 = x] = 1 + EN(t - x)$$

when $x < t$ and

$$E[N(t)\,|X_1 = x] = 0$$

if $x > t$. Hence we have

$$\begin{aligned} m(t) &= \int_0^t [1 + m(t - x)] f(x)dx \qquad (8.6) \\ &= F(t) + \int_0^t m(t - x) f(x)dx. \end{aligned}$$

Example 33 *Suppose* $X_1 \sim U(0, 1)$. *Find renewal function of renewal process generated by these inter-arrival times.*
Renewal equation has the following form in this case

$$m(t) = t + \int_0^t m(t - x)dx = t + \int_0^t m(y)dy.$$

for $0 < t < 1$. *And further* $m(t) = exp(t) - 1$.

Example 34 *In the second example let us suppose that* $X_1 \sim Exp(\lambda)$. *Find renewal function. We have*

$$\begin{aligned} m(t) &= (1-\exp(-\lambda t)) + \int_0^t m(t-x)\lambda \exp(-\lambda x)dx \qquad (8.7) \\ &= (1-\exp(-\lambda t)) + \lambda \exp(-\lambda t)\int_0^t m(y)\exp(\lambda y)dy. \end{aligned}$$

Note that from the above equality it follows that m *is differentiable. Thus differentiating both sides we get*

$$m'(t) = -\lambda \exp(-\lambda t) - \lambda^2 \exp(-\lambda t)\int_0^t m(y)\exp(\lambda y)dy + \lambda m(t).$$

From (8.7) we get $\lambda \exp(-\lambda t)\int_0^t m(y)\exp(\lambda y)dy = m(t) - 1 + \exp(-\lambda t)$, *hence*

$$\begin{aligned} m'(t) &= -\lambda \exp(-\lambda t) - \lambda^2 \exp(-\lambda t)\int_0^t m(y)\exp(\lambda y)dy + \lambda m(t) \\ &= -\lambda \exp(-\lambda t) - \lambda(m(t) - 1 + \exp(-\lambda t)) + \lambda m(t) = \lambda \end{aligned}$$

Taking into account initial condition $m(0) = 0$ *we see that* $m(t) = \lambda t$.

8.2 Elementary Renewal Theorem

We start with Wald's identity.

Lemma 35 $E\left(S_{N(t)+1}\right) = \mu(m(t)+1)$.

Proof. Notice that $S_{N(t)+1}$ is the time of the first renewal after t. Consider function $g(t) = E\left(S_{N(t)+1}\right) = \int_0^\infty E\left(S_{N(t)+1}|X_1 = x\right) f(x)dx$. We have

$$E\left(S_{N(t)+1}|X_1 = x\right) = \begin{cases} g(t-x)+x & for \quad x < t \\ x & for \quad x \geq t \end{cases}$$

Since if $x \geq t$ then first time of the first renewal after t is x, if $x < t$ then treating x as new origin we see that expected time of the first renewal after t is $g(t-x)+x$. From this it follows that

$$g(t) = \int_0^t (g(t-x)+x) f(x)\,dx + \int_t^\infty x f(x)\,dx = \mu + \int_0^t g(t-x) f(x)\,dx$$

Now let $g_1(t) = (g(t) - \mu)/\mu$. We have:

$$g_1(t) + 1 = 1 + \int_0^t (g_1(t-x)+1) f(x)\,dx = 1 + F(t) + \int_0^t g_1(t-x) f(x)\,dx.$$