Hence we see that g_1 satisfies renewal equation, thus, by the uniqueness we see that $g_1(t) = m(t)$, or equivalently:

$$E\left(S_{N(t)+1}\right) = \mu\left(m(t)+1\right).$$

■

We start with the following Elementary theorem

Theorem 36 *For a simple renewal stream we have:*

$$\frac{t}{\mu} - 1 \leq m(t), \tag{8.8}$$

If additionally

$$F(x) \leq \frac{1}{\mu}\int_0^x (1-F(u))\,du,$$

then

$$m(t) \leq \frac{t}{\mu}.$$

Proof. It is easy to note, that $S_{N(t)+1} - t = \gamma(t) \geq 0$, so $0 \leq E(\gamma(t)) = \mu(m(t)+1) - t$. Hence we get (8.8). Proof of the second part will be skipped. ■

Proposition 37 *If $EX_i = \mu$, then*

$$\lim_{t\to\infty} \frac{N(t)}{t} = \mu^{-1}.$$

Proof. We have

$$S_{N(t)} \leq t < S_{N(t)+1}.$$

Now

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}. \tag{8.9}$$

Moreover since $\frac{S_{N(t)}}{N(t)} = \sum_{i=1}^{N(t)} X_i/N(t)$ is the average of $N(t)$ independent and identically distributed random variables, it follows by the strong law of large numbers that $S_{N(t)}/N(t) \to \mu$ as $N(t) \to \infty$. But since we know that $N(t) \to \infty$ as $t \to \infty$, we obtain

$$\frac{S_{N(t)}}{N(t)} \to \mu$$

as $t \to \infty$. Furthermore, writing

$$\frac{S_{N(t)+1}}{N(t)} = \left(\frac{S_{N(t)+1}}{N(t)+1}\right)\left(\frac{N(t)+1}{N(t)}\right)$$

we have: $\frac{S_{N(t)+1}}{N(t)+1} \to \mu$ by the same reasoning as before and

$$\frac{N(t)+1}{N(t)} \to 1$$

as $t \to \infty$. Hence

$$\frac{S_{N(t)+1}}{N(t)} \to \mu$$

as $t \to \infty$. The result follows now by (8.9) since $t/N(t)$ is between two random variables each of which converges to μ as $t \to \infty$. ■

Theorem 38 (Elemetary Renewal Theorem) *If $EX_i = \mu$, then*

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu},$$

with the understanding, that if $\mu = \infty$ then the limit is 0.

Theorem 39 (Blackwell) *If renewal stream is not periodic, then for every $h > 0$ we have:*

$$\lim_{t\to\infty} (m(t+h) - m(t)) = \frac{h}{\mu}.$$

where μ is a mean life time of renewed elements. If $\mu = 0$ then this limit is equal 0.

Theorem 40 (basic renewal theorem) *If renewal stream is not periodic and g is a real function with bounded variation, integrable over $< 0, \infty)$, then*

$$\lim_{t\to\infty} \int_0^t g(t-u)\, dm(u) = \frac{1}{\mu} \int_0^\infty g(u)\, du,$$

where μ is a mean life time of renewed elements. If $\mu = 0$ then this limit is equal 0.

Definition 41 *Quantities $\gamma(t) = S_{N(t)+1} - t$ and $\gamma_1(t) = t - S_{N(t)}$ denote respectively* residual *(or excess) life at t and* life time *at t .*

We have the following fact:

Lemma 42 *If life time of an element is non-periodic, then*

$$\lim_{t\to\infty} P(\gamma_1(t) \geq x, \gamma(t) \geq y) = \frac{1}{\mu} \int_0^\infty (1 - F(u+x+y))\, du. \tag{8.10}$$

Proof. We will derive this fact from basic renewal theorem. We have

$$\begin{aligned}
P\left(\gamma_1(t) \geq x, \gamma(t) \geq y\right) &= \sum_{n=0}^{\infty} P\left(\gamma_1(t) \geq x, \gamma(t) \geq y, N(t) = n\right) = \\
&= P\left(X_1 \geq t+y\right) + \sum_{n=1}^{\infty} P\left(S_n < t-x\right), S_n + X_{n+1} \geq t+y) \\
&= 1 - F(t+x) + \sum_{n=1}^{\infty} \int_0^{t-x} P\left(X_{n+1} \geq t+y-u\right) dF_n(u) \\
&= 1 - F(t+x) + \int_0^{t-x} P\left(X_{n+1} \geq t+y-u\right) dm(u)
\end{aligned}$$

Now we apply basic Theorem with $g(u) = 1 - F(u+x+y)$, since $g(t-x-u) = 1 - F(t-x-u+x+y) = 1 - F(t+y-u)$. ∎

Corollary 43 $\lim\limits_{t\to\infty} P\left(\gamma_1(t) < x\right) = \lim\limits_{t\to\infty} P\left(\gamma(t) < x\right) = 1 - \frac{1}{\mu}\int_0^\infty \left(1 - F(u+x)\right) du = \frac{1}{\mu}\int_0^x \left(1 - F(u)\right) du.$

and $\lim\limits_{t\to\infty} E\left(\gamma(t)\right) = \lim\limits_{t\to\infty} E\left(\gamma_1(t)\right) = \frac{\sigma^2+\mu^2}{2\mu}$. *In particular:* $\lim\limits_{t\to\infty} E\left(\gamma(t) + \gamma_1(t)\right) = \mu + \frac{\sigma^2}{\mu} > \mu!$

Proof. Formula for $\lim\limits_{t\to\infty} P\left(\gamma_1(t) < x\right)$ is an elementary consequence of (8.10). To prove formula for expectation, let us take $g(u) = \int_u^\infty \left(1 - F(x)\right) dx$ and use the fact that $P(\gamma(t) \geq y) = 1 - F(t+y) + \int_0^t \left(1 - F(t+y-u)\right) dm(u)$. Now

$$\begin{aligned}
E\left(\gamma(t)\right) &= \int_0^\infty P(\gamma(t) \geq y) dy \\
&= \int_0^\infty \left(1 - F(t+y)\right) dy + \int_0^\infty \int_0^t \left(1 - F(t+y-u)\right) dm(u)\, dy \\
&= \int_t^\infty \left(1 - F(x)\right) dx + \int_0^t \int_0^\infty \left(1 - F(t+y-u)\right) dy dm(u) \\
&= g(t) + \int_0^t g(t-u)\, dm(u)
\end{aligned}$$

Hence remembering that $\lim_{t\to\infty} g(t) = 0$, we get from basic renewal theorem :

$$\begin{aligned}
\lim_{t\to\infty} E\left(\gamma(t)\right) &= \frac{1}{\mu}\int_0^\infty \int_t^\infty \left(1 - F(x)\right) dx dt = \\
&= \frac{1}{\mu}\left(t\int_t^\infty \left(1 - F(x)\right) dx \Big|_{t=0}^{\infty} + \int_0^\infty t\left(1 - F(t)\right) dt \right) \\
&= \frac{1}{\mu}\left(\frac{1}{2} t^2 \left(1 - F(t)\right) \Big|_{t=0}^{\infty} + \frac{1}{2}\int_0^\infty t^2 dF(t) \right) = \frac{\sigma^2+\mu^2}{2\mu}.
\end{aligned}$$

∎

Moreover we have:

Corollary 44 *If a renewal stream is non-periodic and lifetime of an element has finite variance, then*

$$m\left(t\right)=\frac{t}{\mu}+\frac{\sigma^{2}+\mu^{2}}{2\mu}-1+o(t)$$

as $t\rightarrow\infty$.

Proof. On one hand we have by Wold's identity $E\left(\gamma\left(t\right)\right)=E\left(S_{N(t)+1}-t\right)=\mu m(t)+\mu-t$. On the other hand we have

$$\lim_{t\rightarrow\infty}E\left(\gamma\left(t\right)\right)=\frac{\sigma^{2}+\mu^{2}}{2\mu}.$$

■

8.2.1 Variance of the number of renewals

Let us denote by $\Psi(t)=E\left(N\left(t\right)\left(N\left(t\right)+1\right)\right)$. We have:

$$\begin{aligned}\Psi\left(t\right) &= \sum_{n=1}^{\infty}n(n+1)P\left(N\left(t\right)=n\right)= \\ &= \sum_{n=1}n\left(n+1\right)\left(F_{n}(t)-F_{n+1}(t)\right)= \\ &= 2\sum_{n=1}^{\infty}nF_{n}\left(t\right)\end{aligned}$$

It can be shown that $\Psi\left(t\right)=2\int_{0}^{t}m\left(t-x\right)dm\left(x\right)+2m\left(t\right)$, thus

$$\begin{aligned}D^{2}\left(N\left(t\right)\right) &= \Psi\left(t\right)-m\left(t\right)-m^{2}\left(t\right)= \\ & 2\int_{0}^{t}m\left(t-x\right)dm\left(x\right)+m\left(t\right)-m^{2}\left(t\right).\end{aligned}$$

It follows from these formulae that

$$D^{2}\left(N\left(t\right)\right)=\frac{\sigma^{2}t}{\mu^{3}}+\left(\frac{1}{12}+\frac{5\sigma^{4}}{4\mu^{4}}-\frac{2\mu_{3}}{3\mu^{3}}\right)+o\left(t\right)$$

Moreover we have

$$\lim_{t\rightarrow\infty}P\left\{\frac{N\left(t\right)-\frac{t}{\mu}}{\sqrt{t\sigma^{2}/\mu^{3}}}<x\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-x^{2}/2\right)dx.$$