

# DM 4 GENERATING FUNCTIONS

4.1 Find generating functions of the following sequences:

a)  $a_n = \alpha^n, n = 0, 1, 2, \dots, \alpha \in R,$

b)  $a_n = \begin{cases} 1, & n = 0, 1, \dots, N, \\ 0, & n > N \end{cases},$

c)  $a_n = \begin{cases} n + 1, & n = 0, 1, \dots, N, \\ 0, & n > N \end{cases},$

d)  $a_n = \alpha n, n = 0, 1, 2, \dots, \alpha \in R,$

e)  $a_n = n^2, n = 0, 1, 2, \dots,$

f)  $a_n = n\alpha^n, n = 0, 1, 2, \dots, \alpha \in R.$

4.2 Find a generating function  $F(x)$  for the sequence  $A_n$  if the generating function  $f(x)$  for  $a_n$  is given and:

a)  $A_n = a_{n+1}, n = 0, 1, 2, \dots,$

b)  $A_n = a_{n+k}, n = 0, 1, 2, \dots, k$  is a fixed positive integer,

c)  $A_n = a_{n+1} - a_n, n = 0, 1, 2, \dots,$

d)  $A_n = n \cdot a_n, n = 0, 1, 2, \dots,$

e)  $A_n = \begin{cases} a_{n-1}, & n = 1, \dots, \\ 0, & n = 0 \end{cases}$

4.3 Find a coefficient of  $x^{12}$  in

a)  $(1 + x^3 + x^6 + x^9 + \dots)^7,$

b)  $(x + x^2 + x^3 + x^4)^5,$

c)  $x^2(1 - x)^{12}.$

4.4 Find a coefficient of  $x^{20}$  in  $(x + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + \dots)^5.$

4.5 In how many ways can 500\$ be paid if there are note of

a) 10\$ and 20\$,

b) 10\$, 20\$ and 40\$

c) 10\$, 20\$ and 40\$, but there are only 10 notes of 40\$.

4.6 Use generating functions to find the number of ways to select 10 balls from a large pile of red, white and blue balls if:

a) the selection has at least 2 balls of each color,

b) the selection has at least 2 red balls.

4.7 Find the number of ways to select

1, 2, 3, 4, 5, 6, 7, 8, 9, 10 fruits from a pile of 3 apples, 5 oranges and 2 bananas.

4.8 How many ways are there to divide 2 blue, 5 red and 9 white balls into equal unordered piles?

4.9 Use generating functions to find  $a_n$  if:

a)  $a_n = 6n + a_{n-1},$  for  $n \geq 1$  and  $a_0 = 0,$

b)  $a_{n+2} = 2a_{n+1} + 3a_n$   
for  $n \geq 0$  and  $a_0 = 1, a_1 = 2,$

c)  $a_n = -a_{n-1} + 2a_{n-2}$   
for  $n \geq 2$  and  $a_0 = 1, a_1 = 2,$

d)  $a_n = 2a_{n-1} + 8a_{n-2}$   
for  $n \geq 2, a_0 = 0, a_1 = 2,$

e)  $a_n = 5a_{n-1} - 6a_{n-2}$   
for  $n \geq 2, a_0 = 2, a_1 = 5,$

f)  $a_n = 6a_{n-1} - 9a_{n-2} - 4, a_0 = 0, a_1 = 3,$

g)  $a_n = 5a_{n-1} - 6a_{n-2} - 2, a_0 = 0, a_1 = 3,$

h)  $a_n = a_{n-1} - 3a_{n-2} + 3^n - 5 \cdot 2^{n-2}$   
for  $n \geq 2, a_0 = 0, a_1 = 1,$

i)  $a_n = 2a_{n-1} - a_{n-2} + (n + 3)2^{n-2}$   
for  $n \geq 2, a_0 = 1, a_1 = 4,$

j)\*  $a_n = 3a_{n-1} - 2a_{n-2} - 1$   
for  $n \geq 2, a_0 = 1, a_1 = 3,$

k)\*  $a_n = 3a_{n-1} - 2a_{n-2} - 2n$   
for  $n \geq 2, a_0 = 10, a_1 = 20,$

l)\*  $a_n = 4a_{n-1} - 4a_{n-2} + 1$   
for  $n \geq 2, a_0 = 3, a_1 = 7,$

m)\*  $a_n = 3a_{n-1} - 2a_{n-2} - 2n + 5$   
for  $n \geq 2, a_0 = -3, a_1 = -4,$

n)\*  $a_n = a_{n-1} + 2a_{n-2} + 2n - 5$   
for  $n \geq 2, a_0 = 1, a_1 = 4.$