DM 7 Graphs, Trees, Connectivity

7.1 Draw a graph with the following degree sequence or proof there is no such graph:
a) (4,2,2,2,2), b) (3,3,3,3,1), c) (5,5,5,5,3,3),
d) (5,5,3,3,2,2), e) (2,2,1,1,1,1), f) (7,6,5,4,3,3,2)
g) (5,4,4,3,3,2).

7.2 Find a pair of nonisomorphic graphs with the same degree sequence.

7.3 Find all nonisomorphic graphs on 4 vertices.

7.4 Show, that in any group of two or more people, there are always at least two with exactly the same number of friends inside the group. (relation of being a friend is symmetric).

7.5 Show that if G is simple, then $e(G) \leq \binom{|G|}{2}$.

7.6 Show that if $e(G) > \binom{|G|-1}{2}$, then G is connected.

7.7 Show that $\delta(G) \leq \frac{2e(G)}{|G|} \leq \Delta(G)$.

7.8 Show that if $\delta(G) \geq 2$, then G contains a cycle.

7.9 Show that if $\delta(G) > \lfloor \frac{|G|}{2} \rfloor - 1$, then G is connected.

7.10 *The complement* $\overline{G}$ of a simple graph G is the graph with vertex set $V(G)$ and edge set $\mathcal{P}_2(V(G)) \setminus E(G)$. Show that, for any graph G, G is connected or $\overline{G}$ is connected.

7.11 A simple graph G is *self-complementary* if $G \cong \overline{G}$. Show that if G is self-complementary, then $|G| = 0 \pmod 4$ or $|G| = 1 \pmod 4$.

7.12 Show that every graph with n vertices and at least n edges contains a cycle.

7.13 Find all nonisomorphic trees on 6 vertices.

7.14 Let G be a connected graph and let $e \in E(G)$. Show that: e is in every spanning tree of $G \Leftrightarrow e$ is a cut edge of G.

7.15 Show that if G is a tree with $\Delta(G) \geq k$, then G has at least k vertices of degree one.

7.16 Let G be a graph such that $e(G) = |G| - 1$. Show that the following statements are equivalent:
(i) G is connected,
(ii) G is acyclic
(iii) G is a tree.

7.17 Show that every tree with exactly two vertices of degree one is a path.

7.18 Let G be a graph and u, v be any two vertices, which are connected by a path in G. The *distance* between u and v, denoted by $\mathrm{dist}_G(u, v)$ is the minimum number of edges, which form an (u, v)-path in G. If there is no path connecting u and v define $\mathrm{dist}_G(u, v)$ to be infinite. A *center* of G is a vertex u such that $\max_{v \in V} \mathrm{dist}_G(u, v)$ is as small as possible. Show that a tree has either exactly one center or two adjacent centers.

7.19 The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between two vertices of G. Show that if $G \cong \overline{G}$, then either $\mathrm{diam}(G) = 2$ or $\mathrm{diam}(G) = 3$.

7.20 Prove or disprove: If $\mathrm{diam}(G) = 2$, then G contains a spanning tree T, such that $\mathrm{diam}(T) = 2$.

7.21 By $\omega(G)$ we denote the number of components in G. Show that $\omega(G) + e(G) \geq |G|$.

7.22 Show that each graph contains vertices which are not cut-vertices.

7.23 Show that if each vertex in G is on even degree, then G has no bridge.

7.24 Show that if G is k-edge connected, then $e(G) \geq \frac{k}{2}|G|$.