IFLST 6 Ordered sets

6.1 Draw Hasse diagrams for the following partial orders and show their minimal (maximal, the least, the greatest) elements. Show an example of a maximal chain and a maximal antichain.
a) $|$ (the divisibility) in the set $\{p \in \mathbb{N}_+ : p|96\}$,
b) $\leqslant$ in the set $\{0, 1, 2, 3\}^2$, where

$$(x_1, y_1) \leqslant (x_2, y_2) \ \Leftrightarrow\ x_1 \leqslant x_2 \wedge y_1 \leqslant y_2 \tag{1}$$

c) $\subseteq$ in the family $\{D(A_i, r_i) : i = 0, 1, \ldots, n\}$ of open plane discs (A_i is a center and r_i is a radius of $D(A_i, r_i)$ where $A_i = (i, i)$, $r_i = 2\sqrt{2}$ for even i and $r_i = \sqrt{2}$ for odd i,
d) $\preceq$ in the set of all sequences of length 3 with elements of the set $\{0, 1, 2\}$ where

$$a_1b_1c_1 \preceq a_2b_2c_2 \ \Leftrightarrow\ a_1 \leqslant a_2 \wedge b_1 \leqslant b_2 \wedge c_1 \leqslant c_2$$

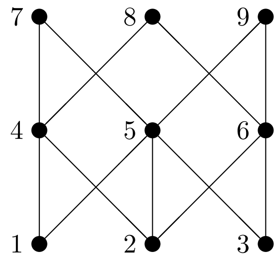
6.2 Let P and Q be ordered sets and let $\leqslant$ be an order in $P \times Q$ defined as in 6.1b).
a) Draw a Hasse diagram of the set $X \times Y$ where $X = \{a, b, c\}$, $= \{x, y, z\}$ and $a < b > c$, $x > y < z$,
b) What is the relation between minimal (maximal) elements in $P \times Q$ and minimal (maximal) elements in P and in Q?

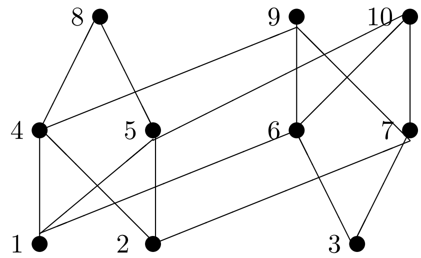
6.3 For $x, y \in \mathbb{N}$ $x \preceq y$ iff $x = y \vee 2x \leqslant y$. Prove that $\preceq$ is a partial order. Draw the Hasse diagram for $(\{1, .., 9\}, \preceq)$.

6.4 For $x, y \in \mathbb{N}$ $x \preceq y$ iff $x = y \vee x^2 \leqslant y$. Prove that $\preceq$ is a partial order. Draw the Hasse diagram for $(\{1, .., 9\}, \preceq)$.

6.5 For $x, y, z, t \in \mathbb{N}$ $(x, y) \preceq (z, t)$ iff $x \leqslant z \wedge x \cdot y \leqslant z \cdot t$. Prove that $\preceq$ is a partial order. Draw the Hasse diagram for $(\{(x, y) : x, y \in \{1, 2, 3\}\}, \preceq)$. Find the smallest, largest, all minimal, all maximal elements. Give an example of a maximal chain and antichain in $(\{(x, y) : x, y \in \{1, 2, 3\}\}, \preceq)$.

6.6 Find $\sup(x, y)$ and $\inf(x, y)$ (if they exist) for each $x, y \in P$ in the following ordered sets:





6.7 Let $P = \{2n : n \in \mathbb{N}_+\} \cup \{3\}$ and let $|$ be the divisibility relation. Find all maximal and all minimal elements in the poset $(P, |)$.

6.8 Is it true that in a finite poset the only maximal element is the largest element (the only minimal element is the least element)?

6.9 Prove that if each two element subset of an ordered set has a least upper bound (a greatest lower bound) then every nonempty finite subset has a least upper bound (the greatest lower bound).

6.10 For an ordered set P and its subset P let X^*, X_* stand for sets $\{p \in P : (\forall x \in X) x \leqslant p\}$ and $\{p \in P : (\forall x \in X) p \leqslant x\}$, respectively.
Find X^* and X_* where
(i) $P = \mathbf{N}$, $x \preceq y \Leftrightarrow x \mid y$, $X = \{12, 16, 24\}$,
(ii) $P = \mathcal{P}(A)$ ordered by $\subseteq$, $X = \{B, C, D\}$, where $B, C, D \subseteq A$,

6.11 Find $\emptyset^*$ and $\emptyset_*$,

6.12 When $\sup \emptyset$ (respectively $\inf \emptyset$) exists in a given ordered set?

6.13 Let $X \subseteq Y \subseteq P$. What are the inclusions between X^*, X_* and Y^*, Y_*?

6.14 What are the inclusions between P and $(X^*)_*$, $(X_*)^*$? .

6.15* For an ordered set P let $A(P)$ stand for a family of all antichains in P. Consider a binary relation $\preceq$ in that family

$$X \preceq Y \;\Leftrightarrow\; (\forall x \in X)(\exists y \in Y) x \leqslant y.$$

Prove that $\preceq$ partially orders $A(P)$.