IMM 1 Logic

1.1 Calculate, according to truth - tables:
a) $(\mathbf{0} \Rightarrow \mathbf{1}) \Rightarrow [\mathbf{0} \wedge \sim (\mathbf{0} \vee \mathbf{1})]$,
b) $(\mathbf{0} \Rightarrow \mathbf{0}) \Rightarrow [\mathbf{0} \Rightarrow (\mathbf{0} \Rightarrow \mathbf{0})]$,
c) $[(\mathbf{1} \Rightarrow \mathbf{0}) \Rightarrow \mathbf{1}] \Rightarrow (\mathbf{1} \Rightarrow \mathbf{0})$,
d) $[(\mathbf{0} \vee \mathbf{1}) \wedge \mathbf{0}] \vee \mathbf{0}$,
e) $(\mathbf{1} \Rightarrow \mathbf{0}) \Leftrightarrow [(\mathbf{0} \Rightarrow \mathbf{1}) \vee \mathbf{0}]$.

1.2 Let $x, y, z, t \in \{\mathbf{0}, \mathbf{1}\}$. Solve the following equations:
a) $(x \wedge y) \Rightarrow (z \vee t) = \mathbf{0}$
b) $\sim (x \Rightarrow y) \wedge (x \vee y) = \mathbf{1}$
c) $\sim [(x \vee y \vee t) \Rightarrow (\sim x \vee z \vee \sim t)] = \mathbf{0}$
d) $(x \Leftrightarrow y) \wedge (y \Leftrightarrow \sim z) \wedge (z \Leftrightarrow \sim t) = \mathbf{1}$,
e) $(x \Rightarrow y) \vee (y \Rightarrow z) \vee (z \Rightarrow t) \vee (t \Rightarrow x) = \mathbf{0}$.

1.3 Establish whether the following statements are tautologies:
a) $[p \vee (q \Rightarrow r)] \wedge [\sim r \vee (q \Rightarrow p)]$
b) $\sim [p \wedge (r \Rightarrow q)] \vee [q \Rightarrow (p \wedge r)]$
c) $p \Rightarrow [\sim p \Rightarrow (p \Rightarrow \sim p)]$
d) $[(p \Rightarrow \sim p) \Rightarrow p] \Rightarrow \sim p$
e) $(p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee q) \wedge (\sim p \vee \sim q)$
f) $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$
g) $[(p \Rightarrow q) \Rightarrow r] \Rightarrow [(p \Rightarrow r) \Rightarrow (q \Rightarrow r)]$
h) $[(p \Rightarrow q) \Rightarrow (r \Rightarrow q] \Rightarrow [(p \Rightarrow r) \Rightarrow q)]$
i) $[(p \Rightarrow q) \Rightarrow \sim (p \Rightarrow r)] \Rightarrow [\sim (p \Rightarrow r) \Rightarrow (p \Rightarrow q)]$
j) $[(p \Rightarrow \sim q) \Rightarrow \sim p] \Rightarrow [p \Rightarrow (q \Rightarrow p)]$
k) $\sim [(p \vee q \vee r) \Rightarrow (\sim p \wedge \sim q \wedge \sim r)]$
l) $(p \wedge q \wedge \sim r) \Leftrightarrow [(\sim p \Rightarrow \sim q) \vee r]$
m) $(p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim r \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r)$
n) $[(p \Rightarrow q) \Rightarrow r] \vee [(\sim r \vee q) \wedge r]$,
o) $(p \Rightarrow q) \Rightarrow [(q \Rightarrow r) \Rightarrow (p \Rightarrow r)]$.

1.4 Reduce formulas in Exercise 3 to the disjunctive - conjunctive normal form. A formula is in the disjunctive - conjunctive normal form if it is in the form $\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$, where each α_i is in the form $\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_{k_i}$ and each β_j is either a variable or a negation of a variable.

1.5 Define the connectives: negation, disjunction, conjunction and implication in terms of
a) Sheffer's functor $|$, where $p \mid q \Leftrightarrow \sim p \vee \sim q$
b) Peirce's functor $\downarrow$, where $p \downarrow q \Leftrightarrow \sim p \wedge \sim q$.

1.6 Are these statements true?
a) A is not a triangle if and only if: A is a triangle if B is a square or A is not a triangle when B is not a square,
b) If Joan is a student then if Susan is not a student and Joan is not a student then Joan is a student,
c) If 2+2=5 or each cat is an animal then $2 + 2 \neq 4$ implies that it there exist cats which are not animals,
d) John is bald or John is tall if the fact that John isn't tall is implied by the fact that John is not bald.

1.7 Which of the following statements are propositions?
a) $x + y = y + x$,
b) $(\forall x) xy = 7$,
c) $(\exists y) y^2 < 0$,
d) $(\forall a)(\exists x) ax^2 + bx + c = 0$.

1.8 Write the mathematical formulas corresponding to the following statements with the help of the following signs only: propositional connectives, quantifiers, variables varied through set $\mathbb{N}$ and symbols indicated in brackets (define auxiliary symbols if necessary)

a) *x is a divisor of y* (symbols: $=, <, \cdot$),

b) *x is a prime number* $(=, <, \cdot, 1)$,

c) *any two numbers have the least common multiple* $(=, <, \cdot)$,

c) *there doesn't exist the greatest number* $(\leqslant)$,

e) *every even number is a sum of two squares,* $(=, \cdot, +)$,

f) *there exists a number with three divisors only* $(=, \cdot)$,

g) *there exists one and only one prime number between 10 and 20* $(=, <, \cdot, 1)$,

h) *there are no numbers with at most two multiples being squares* $(=, <, \cdot)$,

i) *there exists a number which is not a square of an odd number* $(=, \cdot, +)$,

j) *for any odd number there exists a greater even number* $(<, +)$,

k) *the product of prime numbers is a sum of three prime numbers* $(=, <, \cdot, +, 1)$,

l) *not every real is a value of a quadratic polynomial with positive coefficients* $(=, +, \cdot, 0, >)$.

1.9 The same claim with variables varied through set $\mathbb{R}$:

a) *there are no negative squares* $(<, \cdot, 0)$,

b) *the product of two numbers with different signs is a positive number* $(<, \cdot,)$,

c) *each positive number has a square root* $(=, <, \cdot)$,

d) *each linear equation has a solution* $(=, 0, \cdot, +)$,

e) *each linear equation has a unique solution* $(=, 0, \cdot, +)$,

f) *there exist a quadratic polynomial with exactly two solutions* $(=, 0, \cdot, +)$,

g) *each system of two linear equations with two variables has a unique solution* $(=, 0, \cdot, +)$,

h) *the set of all positive reals is closed with respect to the operation of division* $(=, >, \cdot, 0)$.

1.10 Which of the following formulas are true? For the false formulas show counterexamples.

a) $[(\exists x)\phi(x) \wedge (\exists x)\psi(x)] \Rightarrow (\exists x)(\phi(x) \wedge \psi(x))$,

b) $(\exists x)[\phi(x) \wedge \psi(x)] \Rightarrow [(\exists x)\phi(x) \wedge (\exists x)\psi(x)]$,

c) $[(\forall x)\phi(x) \vee (\forall x)\psi(x)] \Rightarrow (\forall x)(\phi(x) \vee \psi(x))$,

d) $(\forall x)(\phi(x) \vee \psi(x)) \Rightarrow [(\forall x)\phi(x) \wedge (\forall x)\psi(x)]$,

e) $(\forall x)(\exists y)\phi(x, y) \Rightarrow (\exists y)(\forall x)\phi(x, y)$,

f) $(\exists y)(\forall x)\phi(x, y) \Rightarrow (\forall x)(\exists y)\phi(x, y)$,

g) $[(\exists x)(\phi(x) \Rightarrow \psi(x))] \Rightarrow [(\exists x)\phi(x) \Rightarrow (\exists x)\psi(x)]$.

1.11 Reduce the formulas in Exercise 1.10 to the form where all quantifiers are in front of the formula.

1.12 Prove the following rules for formulas with bounded quantifiers (derive them from suitable logical rules for unbounded quatifiers):

a) $[(\forall x)_{\alpha(x)}\phi(x) \wedge (\forall x)_{\alpha(x)}\psi(x)] \Leftrightarrow (\forall x)_{\alpha(x)}(\phi(x) \wedge \psi(x))$

b) $\sim (\exists x)_{\alpha(x)}\phi(x) \Leftrightarrow (\forall x)_{\alpha(x)} \sim \phi(x)$

1.13 Interpret symbols $\alpha, *$ occurring in following formulas to make them (1) true, (2) false.

a) $(\exists x)(\forall y)\alpha(x, y)$

b) $(\forall x, y, z)[\alpha(x, y) \wedge \alpha(y, z) \Rightarrow \alpha(x, z)]$

c) $(\forall x, y)\alpha(x, y) \Leftrightarrow (\forall x)\alpha(x, x)$

d) $(\exists x)(\forall y)x * y = y$