IMM 3 Unions and intersections of families of sets

3.1 Find $\bigcap_{i\in\mathbb{N}} A_i$ for:

a) $A_i = (0; \frac{1}{i+1})$, b) $A_i = (0; \frac{1}{i+1}]$, c) $A_i = [0; \frac{1}{i+1}]$, d) $A_i = (i; \infty)$

e) $A_i = \{(x,y) \in \mathbb{R}^2 : x + y \leqslant \frac{1}{i+1}\}$ f) $A_i = \{(x,y) \in \mathbb{R}^2 : x, y \geqslant 0, y \leqslant x^i\}$.

3.2 Find $\bigcup_{i\in\mathbb{N}} A_i$ for:

a) $A_i = (i; \infty)$, b) $A_i = (\frac{1}{i+1}; 1 - \frac{1}{i+1}]$; , c) $A_i = (-i; i)$, d) $A_i = (i; \infty)$,

e) $A_i = \{(x,y) \in \mathbb{R}^2 : x + y \leqslant i\}$, f) $A_i = \{(x,y) \in \mathbb{R}^2 : x + y \leqslant \frac{1}{i+1}\}$,

h) $A_i = \{(x,y) \in \mathbb{R}^2 : x, y \geqslant 0, y \leqslant x^i\}$.

3.3 Find $\bigcup_a \bigcap_b X_{a,b}$, $\bigcap_a \bigcup_b X_{a,b}$, $\bigcap_b \bigcup_a X_{a,b}$, $\bigcup_b \bigcap_a X_{a,b}$ for:

a) $X_{a,b} = \{(x,y) \in \mathbb{R}^2 : 0 < x \ \wedge 0 \leqslant y \ \wedge \frac{x}{a} + \frac{y}{b} \leqslant 1\}$, $a, b \in \mathbb{R}$, $\ a, b > 0$,

b) $X_{a,b} = \{(x,y) \in \mathbb{R}^2 : y \leqslant ax + b\}$, $a, b \in \mathbb{R}$,

c) $X_{a,b} = \{(x,y) \in \mathbb{R}^2 : y \geqslant \frac{a}{b^2} x(x - 2b)\}$

d) $X_{a,b} = \{(x,y) \in \mathbb{R}^2 : y \leqslant ax(x - b)\}$, $a, b \in \mathbb{R}$, $a, b > 0$,

e) $X_{a,b} = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \ \wedge 0 < y \ \wedge \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$, $a, b \in \mathbb{R}$, $\ a, b > 0$,

f) $X_{a,b} = \{x \in \mathbb{R} : a - \frac{1}{b} \leqslant x < a + \frac{1}{b}\}$, $a \in \mathbb{Z}$, $\ b \in \mathbb{N} - \{0\})$,

g) $X_{a,b} = \{(x,y) \in \mathbb{R}^2 : ax^2 < y \leqslant b\}$, $a, b \in \mathbb{R}$, $\ a, b > 0$,

h) $X_{a,b} = \{(x,y) \in \mathbb{R}^2 : 0 < y \leqslant ax^2 + b\}$, $a, b \in \mathbb{R}$, $\ a < 0, \ b > 0$,

3.4 Find $\bigcup_{i\in\emptyset} A_i$ and $\bigcap_{i\in\emptyset} A_i$.

3.5 What relations of inclusions hold? Find counterexamples for false ones.

a) $\bigcup_{i\in I} A_i \cap \bigcup_{i\in I} B_i$ and $\bigcup_{i\in I}(A_i \cap B_i)$

b) $\bigcap_{i\in I} A_i \cup \bigcap_{i\in I} B_i$ and $\bigcap_{i\in I}(A_i \cup B_i)$

c) $\bigcup_{i\in I}(A_i - B_i)$ and $\bigcup_{i\in I} A_i - \bigcap_{i\in I} B_i$

d) $\bigcap_{i\in I}(A_i \cup -B_i)$ and $\bigcap_{i\in I} A_i \cup -\bigcup_{i\in I} B_i$

3.6 Prove:

a) $(\bigcup_{i\in I} A_i) \times (\bigcup_{j\in J} B_j) = \bigcup_{i\in I} \bigcup_{j\in J}(A_i \times B_j)$,

b) $(\bigcap_{i\in I} A_i) \times (\bigcap_{j\in J} B_j) = \bigcap_{i\in I} \bigcap_{j\in J}(A_i \times B_j)$,

3.7 Find infinite family $\mathcal{X}$ of subsets of $\mathbb{N}$ such, that $\bigcap \mathcal{X} = \emptyset$ and $\bigcap \mathcal{Y} \neq \emptyset$ for every proper subfamily $\mathcal{Y}$ of the family $\mathcal{X}$.