IMM 4 Relations

4.1 Establish the properties of the following relations writing sign "+" (yes) or "-" (no) in the appropriate places of this table:

	reflexive	symmetric	transitive	antisymmetric	antireflexive
$=$					
$\neq$					
$<$					
$\leqslant$					
$\subseteq$					
$\mid$					
$\parallel$					
$\perp$					
$\emptyset$					
F					
S					
D					
G					
A					
B					
C					
M					
D					
E					

where the field of $=, \neq, <, \leqslant$ is $\mathbb{N}$,

the field of $\subseteq$ is the powerset of $\mathbb{N}$,

$\mid$ is a relation of divisibility in $\mathbb{N} - \{0\}$.

$\perp$ and $\parallel$ stand for perpendicularity and parallelity between straight lines on the plane, respectively.

$\emptyset$ is an empty relation,

F is a full relation,

R, S, U stand for the following relations in the set of all people.

$xSy \Leftrightarrow$ *x is a son of y*, $xDy \Leftrightarrow$ *x is a descendant of y*, $xGy \Leftrightarrow$ *x and y have a common grandmather*,

$xAy \Leftrightarrow 2|x+y$ where $x, y \in \mathbb{Z}$,

$xBy \Leftrightarrow 3|x+y$ where $x, y \in \mathbb{Z}$

$xCy \Leftrightarrow 3|x-y$ where $x, y \in \mathbb{Z}$

$xMy \Leftrightarrow n|x-y$ where $x, y \in \mathbb{Z}$ and $n \in \mathbb{N}$ is fixed

$xDy \Leftrightarrow xy = 4$ where $x, y \in \mathbb{R}$

$xEy \Leftrightarrow \lfloor x \rfloor = \lfloor y \rfloor$ where $x, y \in \mathbb{R}$

4.2 Let $A = \{a, b, c, d, e\}$ and $R = \{(a, a), (a, b), (b, c), (b, d), (a, d), (c, d), (e, e), (a, c), (e, d)\}$. Draw a graph of R. What are the properties of R. How to modify R to make it reflexive, transitive...

4.3 Determine which of the following relations are equivalence relations. Find equivalence classes for those who are.

a) for $x, y \in \mathbb{Z}$, $x \sim y \Leftrightarrow x + y$ is odd,

b) for $x, y \in \mathbb{Z}$, $x \sim y \Leftrightarrow xy$ is (i) even (ii) odd,

c) for $x, y \in \mathbb{N}$, $x, y > 1$ $x \sim y \Leftrightarrow gcd(x, y) = 1$

d) for $x, y \in \mathbb{N}$, $x, y > 1$ $x \sim y \Leftrightarrow gcd(x, y) > 1$

e) for $w(x), u(x) \in \mathbb{R}[x]$, $w(x) \equiv u(x) \Leftrightarrow u(x) \cdot w(x)$ has an even degree.

f) for $x, y \in \mathbb{Z}$ and for $p \in \mathbb{N}$, $x \sim_p y \Leftrightarrow p \mid x + y$. Consider (i) $p = 1$, (ii) $p = 2$ (iii) $p > 2$.

g) for $x, y \in \mathbb{R}$, $x \sim y \Leftrightarrow x - y$ is of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$,

h) for $A, B \subseteq \mathbb{Z}$, $A \sim B \Leftrightarrow A \div B$ is a finite set

i) for $A, B \subseteq \mathbb{Z}$, $A \sim B \Leftrightarrow A \cap B = \emptyset$

j) for $x, y \in \mathbb{R}$, $x \sim y \Leftrightarrow \mid x - y \mid < 1$

k) for $A, B \subseteq X$, $A \sim B \Leftrightarrow A \cup -B = X$,

l) for $A, B, C \subseteq X$, $A \sim B \Leftrightarrow A \cap B \supset C$,

m) for $A, B \subseteq \mathbb{N}$, $A \sim B \Leftrightarrow$ there exists a bijection $f : A \to B$,

n) for a, b $0 - 1$ sequences of length 100 $a \sim b \Leftrightarrow |\{i : a(i) = b(i)\}|$ is even.

4.4 Let R be an equivalence relation. Is R^{-1} equivalence relation too?

4.5 Let R and S be equivalence relations in a set X. Consider $R \cup S$, $R \cap S$ and $R \setminus S$. Are they equivalence relations? If so what the connection between equivalence classes of R, S and those of newly defined relation? Is $R \times S$ an equivalent relation on $X \times X$?

4.6 Show that for every partition π of the set X there exists an equivalence relation on X whose equivalence classes are exactly elements of π.

4.7 Let $\mathcal{X} = \{[n; n + 1) : n \in \mathbb{Z}\}$. Define an equivalence relation $\sim$ such that $\mathbf{R}/\sim = \mathcal{X}$

4.8 Define an equivalence relation R in plane $\mathbb{R}^2$ such that $\mathbb{R}^2/R$ is the family of all circles with a center in the origin.