IMM 5 Ordered sets

5.1 Draw Hasse diagrams for the following partial orders and show their minimal (maximal, the least, the greatest) elements. Show an example of a maximal chain and a maximal antichain. a) $|$ (the divisibility) in the set $\{2, 3, 4, \ldots, 15, 16\}$,
b) $|$ (the divisibility) in the set $\{p : p|96\}$,
c) $\subseteq$ in the family $\mathcal{P}(\{a, b\})$,
d) $\subseteq$ in the family $\mathcal{P}(\{a, b, c\})$,
e) $\leqslant$ in the set $\{0, 1, 2, 3\}^2$, where

$$(x_1, y_1) \leqslant (x_2, y_2) \ \Leftrightarrow \ x_1 \leqslant x_2 \wedge y_1 \leqslant y_2 \tag{1}$$

f) $\subseteq$ in the family $\{D(A_i, r_i) : i = 0, 1, \ldots, n\}$ of open plane discs (A_i is a center and r_i is a radius of $D(A_i, r_i)$ where $A_i = (i, i)$, $r_i = 2\sqrt{2}$ for even i and $r_i = \sqrt{2}$ for odd i,
g) $\preceq$ in the set of all sequences of length 3 with elements of the set $\{0, 1, 2\}$ where

$$a_1b_1c_1 \preceq a_2b_2c_2 \ \Leftrightarrow \ a_1 \leqslant a_2 \wedge b_1 \leqslant b_2 \wedge c_1 \leqslant c_2$$

5.2 Describe all maximal chains in $(\mathbb{N}, |)$.

5.3 Prove that the largest element in a poset P is the only maximal element in P.

5.4 Prove that the least element in a poset P is the only minimal element in P

5.5. Suppose that x_0 is the only minimal (maximal) element in a poset P. Does this imply that x_0 is the least (largest) element in P?

5.6 For every positive integer n find an example of a poset with precisely n minimal (maximal) elements.

5.7 Can an element of a poset be at the same time minimal and maximal?

5.8 Is there an ordering relation such that every element of the poset is both maximal and minimal? If YES, describe all such relations.

5.9 Is there an ordering relation, which is at the same time an equivalence relation? If YES, describe all such relations.

5.10 Let $P = \{2n : n \in \mathbb{Z}\} \cup \{3\}$ and let $|$ be the divisibility relation. Find all maximal and all minimal elements in the poset $(P, |)$.

5.11 Prove that in a finite poset there exists at least one minimal element and at least one maximal element.

5.12 Is it true that in a finite poset the only maximal element is the largest element (the only minimal element is the least element)?

5.13 Show that in a totally ordered set, every finite subset has the largest element.

5.14 Let P be a finite set ordered by a relation $\leqslant$. Prove that for each $p \in P$ there exist x, y such that x is minimal in P, y is maximal in P and $x \leqslant p \leqslant y$.

5.15 Show an ordered set containing exactly one minimal element and not containing the least element.

5.16 Let S be a maximal antichain in an ordered set P.
a) Show that for each $x \in P$ there is $y \in S$ that x and y are comparable
b) Is it true that every maximal chain in P has a nonempty intersection with S?

5.17 Show an example of an ordered set containing finite antichains of an arbitrary large size but with no infinite ones.

5.18 For an ordered set P let $A(P)$ stand for a family of all antichains in P. Consider a binary relation $\preceq$ in that family

$$X \preceq Y \;\Leftrightarrow\; (\forall x \in X)(\exists y \in Y) x \leqslant y.$$

Prove that $\preceq$ partially orders $A(P)$.

5.19 For an ordered set P and its subset P let X^*, X_* stand for sets $\{p \in P : (\forall x \in X) x \leqslant p\}$ and $\{p \in P : (\forall x \in X) p \leqslant x\}$, respectively.
Find X^* and X_* where
(i) $P = \mathbf{N}$, $x \preceq y \Leftrightarrow x \mid y$, $X = \{12, 16, 24\}$,
(ii) $P = \mathcal{P}(A)$ ordered by $\subseteq$, $X = \{B, C, D\}$, where $B, C, D \subseteq A$,

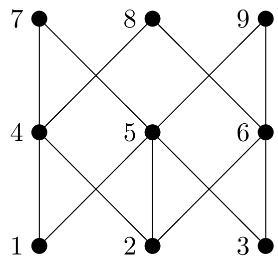
5.20 Find $\emptyset^*$ and $\emptyset_*$,

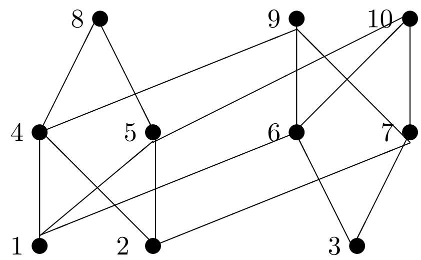
5.21 Let $X \subseteq Y \subseteq P$. What are the inclusions between X^*, X_* and Y^*, Y_*?

5.22 What are the inclusions between P and $(X^*)_*$, $(X_*)^*$? 5. Find $(X^*)_*$ and $(X_*)^*$ for examples in

Exercise

5.23 Find $\sup(x, y)$ and $\inf(x, y)$ (if they exist) for each $x, y \in P$ in the following ordered sets:





5.24 When $\sup \emptyset$ (respectively $\inf \emptyset$) exists in a given ordered set?

5.25 Prove that if each two element subset of an ordered set has a least upper bound (a greatest lower bound) then every nonempty finite subset has a least upper bound (the greatest lower bound).

5.26 Let P and Q be ordered sets and let $\leqslant$ be an order in $P \times Q$ defined as in 5.1e).
a) Draw a Hasse diagram of the set $X \times Y$ where $X = \{a, b, c\}$, $= \{x, y, z\}$ and $a < b > c$, $x > y < z$,
b) What is the relation between minimal (maximal) elements in $P \times Q$ and minimal (maximal) elements in P and in Q?