IMM 6 Functions

6.1 Which of the following relations are functions? Find their domains and sets of values. Which of them are injections?
a) for $x, y \in \mathbb{R}$, $xRy \Leftrightarrow x^3 = y^4$,
b) for $x, y \in \mathbb{R}$, $xRy \Leftrightarrow \frac{y-1}{x} = 1$,
c) for $x, y, z \in \mathbb{R}$, $(x, y)Sz \Leftrightarrow x + y + z^2 = 1$
d) for $x, y \in \mathbb{N}$, $xUy \Leftrightarrow x$ is the greatest prime divisor of y,
f) for polynomial p and $x \in \mathbb{R}$, $pTx \Leftrightarrow p(x) = 0$,
g) for $A, B \subseteq X$, $A\mathcal{F}B \Leftrightarrow A \cup B = X \textit{ and } A \cap B = \emptyset$,
h) for $A \subseteq \mathbb{N}, x \in \mathbb{N}$, $A\mathcal{G}x \Leftrightarrow x$ is a product of all elements of A,
i) for a quadratic polynomial f (with real coefficients) and $A \subseteq \mathbb{R}$, $f\mathcal{V}A \Leftrightarrow (\forall x \in \mathbb{R}) x \in A \Leftrightarrow f(x) = 1$,
j) for $x, y \in \mathbb{R}$, $xVy \Leftrightarrow (\exists z \in \mathbb{R}) y = \frac{x+z}{2}$

6.2 Let X be a finite set and $f : X \to X$. Prove that f is one-to-one if and only if f is "onto".

6.3 Let $A \subseteq X$ and $\chi_A : X \to \{0, 1\}$ be a characteristic function of a set A i.e. for $x \in X$, $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$. Prove that the function F defined $F(A) = \chi_A$ is a on-to-on mapping of the powerset of X onto set $\{0, 1\}^X$.

6.4 Let $A, B \subseteq X$ and $x \in X$. Prove
a) $\chi_{-A}(x) = 1 - \chi_A(x)$,
b) $\chi_{A \cap B}(x) = \chi_A(x) \cdot \chi_B(x)$.

6.5 For a function f and a subset A of its domain find $f(A)$ and $f^{-1}(f(A))$
a) $f(x) = x^2$ for $x \in \mathbb{R}$, $A = [-2; 3)$,
b) $f(x, y) = x + y$ for $x, y \in \mathbb{R}$, $A = \{(0, 0), (0, 1)\}$.
c) $f(x, y) = \frac{x}{y+1}$ for $x, y \in \mathbb{N}$, $A = \{(1, 2)\}$.
d) $f(p) = p(1)$ for a polynomial p with real coefficients, A is the family of all linear functions of the form $ax - a$.
e) $f(x, y) = (x + y, x - y)$ for $x, y \in \mathbb{R}$, $A = \{(x, y) \in \mathbb{R}^2 : x = y\}$.
f) $f(x, y) = x^2 + y^2$ for $x, y \in \mathbb{R}$, $A = \{(x, y) \in \mathbb{R}^2 : 1 < x < 2 \wedge 0 < y < 1\}$.
g) $f(x, y) = max(x, y)$ for $x, y \in \mathbb{R}$, $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
h) $f(n) =$ sum of all prime divisors of n for $n \in \mathbb{N}$, $A = \{4, 6\}$,
i) $f(X) = X \times X$ for $X \subseteq \mathbb{R}$, $A = \{[-x; x] : x \in \mathbb{R}\}$,
j) $f(X) = \{x \in \mathbb{R} : (\exists y)(x, y) \in X\}$ for $X \subseteq \mathbb{R}^2$, A is the family of all singletons.

6.6 Which inclusion is true? $A \subseteq f^{-1}(f(A))$ or $f^{-1}(f(A) \subseteq A$? Prove the true one or find a counterexample for the false one. Prove that if f is a bijection then both are true.

6.7 Which inclusion is true? $f(A) \cap f(B) \subseteq f(A \cap B)$ or $f(A \cap B) \subseteq f(A) \cap f(B)$? Prove the true one or find a counterexample for the false one. Prove that if f is a bijection then both are true.

6.8 Which inclusion is true? $f(A) - f(B) \subseteq f(A - B)$ or $f(A - B) \subseteq f(A) - f(B)$? Prove the true one or find a counterexample for the false one. Prove that if f is a bijection then both are true.

6.9 Let X be a finite set and $f : X \to X$. Prove that there exists $A \subseteq X$ such that $f(A) = A$.