IMM 7 Equipollence relation and cardinals

7.1 Prove by definition, that the following pairs consist of mutually equipollent sets

a) the set of all even integers and the set of all odd integers,

b) $\mathbb{N}$ and $\mathbb{N} \cup \{a_1, a_2, \ldots, a_n\}$ where all $a_i \notin \mathbb{N}$,

c) two open segments

d) a circle and a nonempty closed - open segment

e) two circles,

f) two disks,

g) line and open segment.

h) a line and an open half-line

i) an open half-line and closed half-line

j) open segment and closed segment

k) $\mathcal{P}(X)$, $\{0,1\}^X$ for any set X,

l) the family of all nonempty finite subsets of $\mathbb{N}$ and the set of all finite strictly increasing sequences of natural numbers.

7.2 Let $X \sim Y$. Prove that $\mathcal{P}(X) \sim \mathcal{P}(Y)$.

7.3 Prove that for any sets A_1, A_2, B_1 i B_2: if $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 \times A_2 \sim B_1 \times B_2$.

7.4 What is the cardinality of the following sets (and why)?

a) $\{x \in \mathbb{N} : 10|x\}$

b) $\{x \in \mathbb{N} : \exists_{y \in \mathbb{R}} x = \sin y\}$

c) $\{x \in \mathbb{R} : \exists_{y \in \mathbb{N}} x = \ln y\}$

d) $\{x \in \mathbb{N} : \exists_{y \in \mathbb{R}} x = \operatorname{tg} y\}$

e) $\{(x, y) \in \mathbb{R}^2 : y = 3x - 4\}$,

f) $\{x \in \mathbb{R} : (\exists n \in \mathbb{N}) x^n \in \mathbb{Q}\}$,

g) $\{x \in \mathbb{R} : (\exists n \in \mathbb{N}) \sin^n x \in \mathbb{Q}\}$.

7.5 Prove that if A and B are countable then $A \cup B$, $A \cap B$, $A \setminus B$, $A \div B$, $A \times B$ are countable.

7.6 Prove that if $\{A_n\}_{n \in \mathbb{N}}$ ia a countable family of countable sets then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.

7.7 What is the cardinality of the following sets (and why)?

a) the family of all polynomials with rational coefficients,

b) the family of all such infinite sequences of rationales which are constant from some place,

c) the family of all finite subsets of $\mathbb{Q}$

d) the family of all strictly increasing sequences of natural numbers.

7.8 Prove that if $|X| \geqslant \aleph_0$ then there exists a set $Y \subset X$ such that $|Y| = \aleph_0$ and $|X \setminus Y| = |X|$.

7.9 Is the number of partial orders on a countable set countable? By the number of partial orders on a set we mean the cardinality of the set of all partial orders that can be defined on the set.

7.10 Is the set of all functions from $\mathbb{N}$ into $\mathbb{N}$ countable?

7.11 Let X be a finite set. Is the number of all functions from $\mathbb{N}$ into X countable?

7.12 Is the set of all finite subsets of a countable set countable?

7.13 Is the set of all infinite subsets of a countable set countable?

7.14 Prove that the set of all complex roots of 1 of all possible (natural) degrees is countable.

7.15 Prove that every set of disjoint open discs on the plane is countable.